PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 9, September 1998, Pages 2619–2622
S 0002-9939(98)04409-8

THE SUPREMUM OF THE DIFFERENCE BETWEEN THE BIG AND LITTLE FINITISTIC DIMENSIONS IS INFINITE

SVERRE O. SMALØ

(Communicated by Ken Goodearl)

ABSTRACT. For each natural number n, an example of a finite dimensional algebra Λ_n is given, which has left little finitistic dimension equal to 1 and left big finitistic dimension equal to n.

INTRODUCTION

The objective of this note is to give, for each natural number $n \geq 1$, an example of a finite dimensional algebra Λ_n over a field such that the left little finitistic dimension of Λ_n is 1 and the left big finitistic dimension of Λ_n is n.

Recall that the left little finitistic dimension of an algebra Λ is the supremum of the projective dimensions of the finitely generated left Λ-modules of finite projective dimension, and that the left big finitistic dimension of an algebra Λ is the supremum of the projective dimensions of all Λ-modules having finite projective dimension. The first example of a finite dimensional algebra Λ where the two dimensions do not coincide was given in 1992 by Birge Huisgen-Zimmermann [1]. The examples given there are a family of monomial relation algebras, and they are rather complicated compared to the algebras given in this note. In [1] it was also proved that, for monomial relation algebras, the difference between the two dimensions is at most 1, and in the examples the left little finitistic dimension was at least 2. For monomial relation algebras it was also proved that the two dimensions have to coincide if $\mathfrak{r}^3 = 0$ where $\mathfrak{r}$ denotes the radical of the algebra. However, the algebras given in this note have the property that $\mathfrak{r}^3 = 0$.

An alternative approach to the problem of making the difference between the little and big finitistic dimensions arbitrarily large, which was pointed out to me by Jeremy Rickard, is to use the first example constructed by Birge Huisgen-Zimmermann together with properties of tensor products of algebras. For finite dimensional algebras Λ_1 and Λ_2 over an algebraically closed field k one has that the global dimension of $\Lambda_1 \otimes \Lambda_2$ is the sum of the global dimension of Λ_1 and the global dimension of Λ_2. Letting $\ell\,\mathrm{fin.\,dim}\,\Lambda$ denote the left little finitistic dimension of Λ and $\ell\,\mathrm{Fin.\,dim}\,\Lambda$ denote the left big finitistic dimension of Λ one also gets that $\ell\,\mathrm{fin.\,dim}\,\Lambda_1 \otimes \Lambda_2 = \ell\,\mathrm{fin.\,dim}\,\Lambda_1 + \ell\,\mathrm{fin.\,dim}\,\Lambda_2$ and $\ell\,\mathrm{Fin.\,dim}\,\Lambda_1 \otimes \Lambda_2 = \ell\,\mathrm{Fin.\,dim}\,\Lambda_1 + \ell\,\mathrm{Fin.\,dim}\,\Lambda_2$. From these two equalities one can just take repeated

Received by the editors May 22, 1996 and, in revised form, February 24, 1997.
1991 *Mathematics Subject Classification.* Primary 16E10, 16G10, 16G20,16P10.
Supported by the Norwegian Research Council and by the U.S.-Norway Fulbright Foundation.

tensor products of the algebra mentioned above to obtain arbitrary differences between the two finitistic dimensions. I include the arguments presented to me by Rickard that the two equalities above hold. First observe that if M is a Λ_1-module of projective dimension m and N is a Λ_2-module of projective dimension n, then $M \otimes N$ has projective dimension $m+n$ as a $\Lambda_1 \otimes \Lambda_2$-module. Hence the inequalities $\ell\,\mathrm{fin.dim}\,\Lambda_1 \otimes \Lambda_2 \geq \ell\,\mathrm{fin.dim}\,\Lambda_1 + \ell\,\mathrm{fin.dim}\,\Lambda_2$ and $\ell\,\mathrm{Fin.dim}\,\Lambda_1 \otimes \Lambda_2 \geq \ell\,\mathrm{Fin.dim}\,\Lambda_1 + \ell\,\mathrm{Fin.dim}\,\Lambda_2$ follow readily. We now carry through the argument for the other inequality for the little finitistic dimension. If one lets X be a finite dimensional $\Lambda_1 \otimes \Lambda_2$-module of finite projective dimension, then X as a Λ_1-module has finite projective dimension, since a $\Lambda_1 \otimes \Lambda_2$-projective resolution restricts to a Λ_1-projective resolution. Hence the m-th syzygy, $\Omega^m X$, in the $\Lambda_1 \otimes \Lambda_2$-projective resolution of X is projective as a Λ_1-module when $m = \ell\,\mathrm{fin.dim}\,\Lambda_1$. Next observe that if P is a projective $\Lambda_1 \otimes \Lambda_2$-module then the isomorphism $(P \otimes_{\Lambda_1} M) \otimes_{\Lambda_2} N \simeq P \otimes_{\Lambda_1 \otimes \Lambda_2} (M \otimes N)$ shows that $P \otimes_{\Lambda_1} M$ is a flat and hence projective Λ_2-module for all Λ_1-modules M. Now let S be a simple Λ_1-module. Then $\Omega^m X \otimes_{\Lambda_1} S$ has finite projective dimension as a Λ_2-module since a Λ_2-projective resolution can be obtained by tensoring the $\Lambda_1 \otimes \Lambda_2$-resolution of $\Omega^m X$ by S over Λ_1 and observing that this sequence is exact since $\Omega^m X$ is projective as a Λ_1-module. Now using the isomorphism $\mathrm{Tor}_i^{\Lambda_2}(\Omega^m(X) \otimes_{\Lambda_1} S, T) \simeq \mathrm{Tor}_i^{\Lambda_1 \otimes \Lambda_2}(\Omega^m(X), S \otimes T)$ for any simple Λ_2-module T, and letting n be the finitistic dimension of Λ_2, we obtain that $\mathrm{Tor}_i^{\Lambda_1 \otimes \Lambda_2}(X, S \otimes T) = 0$ for all $i > m+n$. Hence the flat dimension and therefore also the projective dimension of X are at most $m+n$ as a $\Lambda_1 \otimes \Lambda_2$-module.

Observe that using the tensor product construction one also increases the Loewy lengths of the algebras involved.

One can now ask if the two equalities involving the finitistic dimensions are valid also for nonseparable finite dimensional algebras over a field.

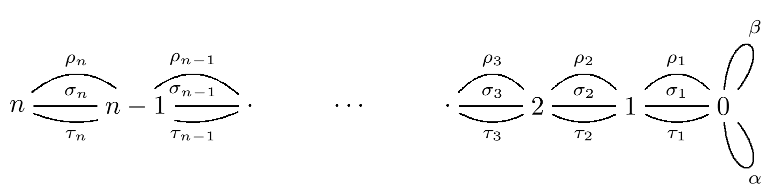
For a historical account of the finitistic dimension conjectures, we refer to the paper [2].

This paper was written while the author was visiting The University of California at Santa Barbara and he would like to take the opportunity to thank the algebra group there for their hospitality.

The example

The algebra Λ_n will be given as a path algebra of a quiver modulo relations and it will then be explained how it is obtained by means of repeated one point extensions using basically injective modules except in the first step. This point of view will then serve as basis for the argument leading to the announced conclusion.

Let Γ_n be the quiver



,

let k be any field, and let Λ be the path algebra of Γ over k modulo the ideal generated by the following relations: $\alpha^2, \beta^2, \alpha\beta, \beta\alpha, \alpha\rho_1, \alpha\sigma_1, \beta\tau_1, x_i y_{i+1}$ for $i = 1, 2, \cdots, n-1$ and $x \neq y; x, y \in \{\rho, \sigma, \tau\}$ and $x_i x_{i+1} - y_i y_{i+1}, x, y \in \{\rho, \sigma, \tau\}, i = 1, \cdots, n-1$.

Theorem. *For each natural number $n \geq 1$, the algebra Λ_n given above has left little finitistic dimension 1 and left big finitistic dimension n.*

Proof. For each vertex $i = 1, 2, \cdots, n$ in the quiver Γ_n, let e_i be the corresponding idempotent in Λ_n, and let $P_i = \Lambda e_i$ and $S_i = P_i/\mathfrak{r}P_i$ be representatives of the indecomposable projective and simple Λ_n-modules respectively. Here $\mathfrak{r}$ denotes the radical of the algebra Λ_n. Since the Loewy length of P_0 is 2 and all the indecomposable projective Λ_n-modules except P_0 have Loewy length 3, there can be no inclusion from a projective P into the radical of another projective Q if P contains an indecomposable direct summand not isomorphic to P_0. So, in order to have an inclusion of a nonzero projective module P into the radical of another projective module Q, the projective P has to be isomorphic to a direct sum of copies of P_0. However, P_0 has nonzero morphisms only to P_1, P_2 and itself. So if P is a finitely generated projective module which is a submodule of the radical of a finitely generated projective module, we have the situation $f : P_0^m \to P_0^{m_0} \amalg P_1^{m_1} \amalg P_2^{m_2}$ where the image of f is in the radical. But then, since the image of $p_0 f$, where p_0 is the projection of $P_0^{m_0} \amalg P_1^{m_1} \amalg P_2^{m_2}$ onto $P_0^{m_0}$, and the image of $p_2 f$, where p_2 is the projection of $P_0^{m_0} \amalg P_1^{m_1} \amalg P_2^{m_2}$ onto $P_2^{m_2}$, are both semisimple, we obtain an inclusion $P_0^m \to P_1^{m_1}$ by composing the given inclusion f with the projection p_1 of $P_0^{m_0} \amalg P_1^{m_1} \amalg P_2^{m_2}$ onto $P_1^{m_1}$. However, it is not hard to see that this is possible only if $m_1 \geq m$. But then $\mathfrak{r}^2 P_1^{m_1}$, which has dimension $3m_1$ as a k-vector space, is not contained in the image of P_0^m which has an intersection with $r^2 P_1^{m_1}$ of dimension $2m$. Therefore the cokernel of any inclusion $P \to Q$ of finitely generated projective Λ_n-modules P and Q with the image in the radical has Loewy length three and can therefore not be embedded in the radical of any projective Λ_n-module. Therefore the syzygy of any finitely generated Λ_n-module cannot have projective dimension 1, which shows that the left little finitistic projective dimension of Λ_n is 1 for all $n \geq 1$.

Next, let $e_{0,i}$ be the coordinates of $P_0^{(\mathbb{N})}$ with e_0 in the i'th place and 0 otherwise, and let analogously $e_{1,i}$ be the coordinates of $P_1^{(\mathbb{N})}$ with e_1 in the i'th place and 0 otherwise. Letting $\phi : P_0^{(\mathbb{N})} \to P_1^{(\mathbb{N})}$ be given by $\phi(e_{0,2i-1}) = \tau_1 e_{1,2i-1} + \sigma_1 e_{1,i}$ and $\phi(e_{0,2i}) = \tau_1 e_{1,2i} + \rho_1 e_{1,i}$, where τ_1, σ_1 and ρ_1 also denote the cosets of τ_1, σ_1 and ρ_1 in Λ_n respectively, it is not hard to verify that ϕ is an inclusion $P_0^{(\mathbb{N})} \to P_1^{(\mathbb{N})}$ such that $\operatorname{coker}\phi$ is annihilated by the residue classes of α and β and has socle equal to $S_0^{(\mathbb{N})}$. If we now go back and look at the definition of the algebra Λ_n, it follows that for $n \geq 3$ one has that Λ_n is a one point extension of Λ_{n-1} by the injective envelope of S_{n-2} considered as a Λ_{n-1}-module. Λ_2 is the one point extension of Λ_1 by the injective $\Lambda_1/(\alpha, \beta)$-envelope of S_0 considered as a Λ_1-module, where we also let α and β denote the residue classes of α and β in Λ_1. Denoting $\operatorname{coker}\phi$ by X_1, we know that we can embed X_1 into $P_2^{(\mathbb{N})}$ in such a way that their socles coincide, yielding a quotient module which we denote by X_2 having Loewy length two with socle isomorphic to $S_1^{(\mathbb{N})}$. This can now be repeated leading to an exact sequence of modules

$$0 \to P_0^{(\mathbb{N})} \to P_1^{(\mathbb{N})} \to P_2^{(\mathbb{N})} \to \cdots \to P_n^{(\mathbb{N})} \to X_n \to 0$$

where each $P_i^{(\mathbb{N})}$ is projective. This shows that the big finitistic dimension of Λ_n is at least n.

That the big finitistic dimension cannot exceed n follows from the fact that the vertices in Γ_n corresponding to all indecomposable projective Λ_n-modules P_i except P_0 do not belong to any oriented cycle in Γ_n . This finishes the proof of the theorem. □

References

1. B. Zimmermann Huisgen, *Homological domino effect and the first finitistic dimension conjecture,* Invent. Math, 108 (1992), 369-383. MR **93i:**16016
2. B. Zimmermann Huisgen, *The Finitistic Dimension Conjectures-A Tale of 3.5 Decades,* in; Abelian Groups and Modules, Proceedings of the Padova Conference 1994, A. Facchini and C. Menini, eds., Kluwer 1995. MR **97a:**16018

Department of Mathematics, Lade, The Norwegian University for Science and Technology, 7034 Trondheim, Norway

E-mail address: sverresm@math.ntnu.no