THE EQUIVALENCE OF SOME BERNOULLI
CONVOLUTIONS TO LEBESGUE MEASURE

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Abstract. Since the 1930’s many authors have studied the distribution \( \nu_\lambda \)
of the random series \( Y_\lambda = \sum \pm \lambda^n \) where the signs are chosen independently
with probability \((1/2, 1/2)\) and \(0 < \lambda < 1\). Solomyak recently proved that
for almost every \( \lambda \in \left[ \frac{1}{2}, 1 \right] \), the distribution \( \nu_\lambda \) is absolutely continuous with
respect to Lebesgue measure. In this paper we prove that \( \nu_\lambda \) is even equivalent
to Lebesgue measure for almost all \( \lambda \in \left[ \frac{1}{2}, 1 \right] \).

1. Introduction

For each \( \lambda \in (0, 1) \) we define the random variable
\[
Y_\lambda = \sum_{n=0}^{\infty} \theta_n \cdot \lambda^n,
\]
where \( \theta_n \) are independent random variables with \( \text{Prob}(\theta_n = -1) = \text{Prob}(\theta_n = 1) = \frac{1}{2} \). The distribution \( \nu_\lambda \) of \( Y_\lambda \) is sometimes called a symmetric infinite Bernoulli
convolution. One can easily see that for \( 0 < \lambda < \frac{1}{2} \) the distribution \( \nu_\lambda \) is supported
on a Cantor set of zero Lebesgue measure. Since the 1930’s a lot of work has been
done to characterize \( \nu_\lambda \) for \( \frac{1}{2} < \lambda \) (for a good survey see e.g. Peres, Solomyak
(1996a) [4]). Among these results the most interesting ones are as follows: P. Erdős
(1939) [1] proved that \( \nu_\lambda \) is singular with respect to Lebesgue measure, if \( \lambda \) is
the reciprocal of a PV number. (An algebraic integer is a PV number provided
all of its conjugates are less than one in modulus.) On the other hand, Wintner
(1935) [7] proved that \( \nu_\lambda \) is absolutely continuous for \( \lambda = 2^{-\frac{k}{2}} \), for each \( k \geq 1 \),
and Garsia (1962) [3] found some other algebraic integers for which \( \nu_\lambda \) is absolutely
continuous. Moreover, Erdős (1940) [2] also proved that there exists \( a < 1 \) such
that the distribution \( \nu_\lambda \) is absolutely continuous with respect to Lebesgue measure
for (Lebesgue) a.e. \( \lambda \in (a, 1) \). Then P. Erdős asked:

Is this statement true with \( a = \frac{1}{2} \)?
This exciting problem remained open for more than fifty years. Then Solomyak (1995) [6] gave a positive answer (see also Peres, Solomyak (1996a) [4] for a shorter proof). Namely,

**Theorem 1** (Solomyak).

\[ \nu_\lambda \ll m \text{ for Lebesgue a.e. } \lambda \in \left( \frac{1}{2}, 1 \right), \]

where \( m \) is Lebesgue measure.

Answering a problem of the first author, posed to the Conference on Fractals and Stochastics (1994, Finsterbergen), we prove that that \( \nu_\lambda \) is even equivalent to Lebesgue measure for a.e. \( \lambda \in \left[ \frac{1}{2}, 1 \right) \). Using Solomyak’s theorem, it is enough to prove that Lebesgue measure is either absolutely continuous or singular with respect to \( \nu_\lambda \) for each \( \lambda \). Actually we prove this statement for a more general family of measures. Furthermore, Peres, Solomyak (1996b) [5] proved that if the probabilities of choosing the signs + and − in \( Y_\lambda \) are \((p, 1-p)\) where \( p \in [1/3, 2/3] \), then \( \nu_\lambda \ll m \) holds for a.e. \( \lambda \in [p^2(1-p)^{1-p}, 1] \). Using this, it follows from our result that even in this non-symmetric case the distributions are not only absolutely continuous but equivalent to Lebesgue measure for a.e. \( \lambda \in [p^2(1-p)^{1-p}, 1] \). (For smaller \( \lambda \) the distributions are singular.)

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2. Notation

For an arbitrary \( \lambda \in (\frac{1}{2}, 1) \) we define the ‘projection’ \( \Pi_\lambda : \{-1,1\}^\mathbb{N} \to \left[ -\frac{1}{2\lambda}, \frac{1}{2\lambda} \right] \) by \( \Pi_\lambda(i) = \sum_{k=0}^{\infty} i_k \lambda^k \). Let \( \mu \) be any Borel probability measure on \( \{-1,1\}^\mathbb{N} \) for which

\[ \mu(B) > 0 \implies \mu\{(i, B)\} > 0 \]

holds for all \( B \subset \{-1,1\}^\mathbb{N} \) and \( i \in \{-1,1\} \), where \( (i, B) := \{(i,j) \in \{-1,1\}^\mathbb{N} : j \in B\} \). For example \( \mu \) may be any Bernoulli measure on \( \{-1,1\}^\mathbb{N} \) with probabilities \((p, 1-p)\), \( 0 < p < 1 \). The ‘push down measure’ of \( \mu \) is \( \alpha_{\lambda,\mu}(B) := \mu(\Pi_\lambda^{-1}(B)) \). We denote the interval \([\frac{1}{2\lambda}, \frac{1}{\lambda}]\) by \( I \). Further, we define \( S_i : I \to I \), \( S_i(x) := \lambda x + i \) for \( i = -1,1 \). The iterates of \( S_i \) are

\[ S_{i_1 \ldots i_n}(x) := S_{i_1} \circ \ldots \circ S_{i_n}(x). \]

The image of \( I \) under \( S_{i_1 \ldots i_n} \) is called \( I_{i_1 \ldots i_n} \). The inverse of \( S_{i_1 \ldots i_n} \) is defined only on \( I_{i_1 \ldots i_n} \). So \( S_{i_1 \ldots i_n}^{-1}(A) := S_{i_1 \ldots i_n}^{-1}(A \cap I_{i_1 \ldots i_n}) \). Then \( S_{i_1}^{-1}(x) = \frac{1}{\lambda} x - \frac{i}{\lambda} \) for \( x \in I_i \) \((i = -1,1)\). We denote the Lebesgue measure of a set \( A \) by \( m(A) \).

3. The Theorem and Its Consequences

**Theorem 2.** Either \( m \ll \alpha_{\lambda,\mu} \) or \( m \perp \alpha_{\lambda,\mu} \).

If \( \mu \) is the Bernoulli measure with probabilities \((\frac{1}{2}, \frac{1}{2})\) then \( \nu_\lambda = \alpha_{\lambda,\mu} \). Using Solomyak’s Theorem, we obtain

**Consequence 1.** For almost all \( \lambda \in (\frac{1}{2}, 1) \), \( \nu_\lambda \) is equivalent to Lebesgue measure.

Clearly, any Bernoulli measure \( \mu \) with probabilities \((p, 1-p)\), satisfies (1) (if \( p \neq 0 \)). Thus,
Consequence 2. Let $\eta_\lambda$ be the distribution of the random series $Z_\lambda = \sum \pm \lambda^n$, where the signs are chosen independently with probabilities $(p, 1-p)$ and $0 < \lambda < 1$. Then either $m \ll \eta_\lambda$ or $m \perp \eta_\lambda$.

Let $\eta_\lambda$ be as above. Then $\eta_\lambda$ is singular for all $\lambda < p^p(1-p)^{1-p}$ (see Peres, Solomyak (1996b) [5, Theorem 2 (a)]). Also Peres, Solomyak (1996b) [5, Corollary 1.4] proved that for $p \in [1/3, 2/3]$ and for almost every $\lambda \in [p^p(1-p)^{1-p}, 1]$, $\eta_\lambda \ll \mu$. Thus, using our previous consequence we obtain

Consequence 3. Let $\eta_\lambda$ be the distribution of the random series $Z_\lambda = \sum \pm \lambda^n$, where the signs are chosen independently with probabilities $(p, 1-p)$, Then for each $p \in [1/3, 2/3]$ and for almost every $\lambda \in [p^p(1-p)^{1-p}, 1]$, the distribution $\eta_\lambda$ is equivalent to Lebesgue measure.

4. Lemmas and Proofs

To prove Theorem 2 we need two lemmas.

Lemma 1. Let $A \subset I$. Then $\alpha_{\lambda, \mu}(A) = 0 \implies \alpha_{\lambda, \mu}(S_1^{-1}(A)) = 0$ ($i = -1,1$).

Proof. First observe that

$$\Pi_\lambda^{-1}(A) = \left\{ (-1, \Pi_\lambda^{-1}(S_1^{-1}(A))) \right\} \cup \left\{ (1, \Pi_\lambda^{-1}(S_1^{-1}(A))) \right\}. \tag{2}$$

This is so, since for $i = -1,1$

$$j \in \Pi_\lambda^{-1}(S_1^{-1}(A)) \iff \sum_{k=0}^{\infty} j_k \lambda^k \in S_1^{-1}(A) \iff \sum_{k=0}^{\infty} j_k \lambda^k \in \frac{1}{\lambda} A - \frac{i}{\lambda} \iff i + \sum_{k=0}^{\infty} j_k \lambda^{k+1} \in A \iff (i, j) \in \Pi_\lambda^{-1}(A).$$

To get a contradiction we assume that there exists a set $A$ such that $\alpha_{\lambda, \mu}(A) = 0$ and $\alpha_{\lambda, \mu}(S_1^{-1}(A)) = \mu(\Pi_\lambda^{-1}(S_1^{-1}(A))) > 0$ holds for some $i \in \{-1,1\}$.

Then from (1), it follows that $\mu(\Pi_\lambda^{-1}(S_1^{-1}(A))) > 0$. Using (2), we find that $\mu(\Pi_\lambda^{-1}(A)) = \alpha_{\lambda, \mu}(A) > 0$. This contradiction proves our lemma. \hfill \Box

Let $C \subset I$ be an arbitrary fixed Borel set. Let $C_0 := C$ and

$$C_{-(k+1)} := (S_1^{-1}(C_k) \cup S_1^{-1}(C_{-k})).$$

Then the ‘backward orbit’ of $C$ in $I$ is

$$\Lambda_- := \bigcup_{k \geq 0} C_{-k}. \tag{3}$$

Lemma 2. For any $C \subset I$, the set $\Lambda_-$ defined above is either a set of zero measure or a full measure subset of $I$ with respect to Lebesgue measure.

Proof. Let $\overline{\Lambda}_- := I \setminus \Lambda_-$. Obviously, it is enough to prove the statement of Lemma 2 for the set $\overline{\Lambda}_-$ instead of $\Lambda_-$. Observe that

$$x \in \overline{\Lambda}_- \implies S_i(x) \notin \overline{\Lambda}_- \implies \exists k \geq 0 \text{ such that } S_i(x) \in C_{-k} \cap I_i.$$  

Then $x = S_i^{-1}(S_i(x)) \in C_{-(k+1)} \subset \Lambda_-$. Iterate (4) to obtain

$$S_{i_1 \ldots i_n}(\overline{\Lambda}_-) \subset \overline{\Lambda}_- \tag{5}.$$
for each \( n \in \mathbb{N} \) and \((i_1, \ldots, i_n) \in \{-1, 1\}^n\). Suppose that \( m(\Lambda_-) > 0 \). Then \( d := \frac{m(\Lambda_-)}{|I|} \) is positive. Using (5), we obtain that
\[
\frac{\lambda_n}{\lambda} \geq \frac{m(\Lambda_- \cap I_{i_1} \ldots i_n)}{|I_{i_1} \ldots i_n|} \geq \lambda \cdot d \cdot |I|.
\]
Thus
\[
(6) \quad \frac{\lambda_n}{\lambda} \geq d
\]
holds for each \( i_1 \ldots i_n \).

On the other hand, let \( J \subset I \) be an arbitrary interval. Then we can find \( n \) and \( i_1 \ldots i_n \) such that \( I_{i_1} \ldots i_n \subset J \) and
\[
|I_{i_1} \ldots i_n| \geq \frac{\lambda}{3}.
\]

Now, from (6) and (7) together, it follows that
\[
\frac{\lambda_n}{\lambda} \geq d \cdot \frac{\lambda}{3}.
\]
That is, \( \Lambda_- \) has no density point. Thus \( \Lambda_- \) is a full measure subset of \( I \). This completes the proof of Lemma 2.

Proof of Theorem 2. Suppose that \( m \not\ll \alpha_{\lambda, \mu} \). Then there is a set \( C \subset I \) such that \( m(C) > 0 \) and \( \alpha_{\lambda, \mu}(C) = 0 \). Define \( \Lambda_- \) by (3). Then \( m(\Lambda_-) > 0 \); thus it follows from Lemma 2 that \( \Lambda_- \) is a full measure subset of \( I \) with respect to Lebesgue measure. On the other hand, Lemma 1 implies that \( \alpha_{\lambda, \mu}(\Lambda_-) = 0 \). So \( m \perp \alpha_{\lambda, \mu} \). This completes the proof of Theorem 2.

References


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