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THE EQUIVALENCE OF SOME BERNOULLI CONVOLUTIONS TO LEBESGUE MEASURE

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ABSTRACT. Since the 1930's many authors have studied the distribution ν_λ of the random series $Y_\lambda = \sum \pm\lambda^n$ where the signs are chosen independently with probability $(1/2, 1/2)$ and $0 < \lambda < 1$. Solomyak recently proved that for almost every $\lambda \in [\frac{1}{2}, 1]$, the distribution ν_λ is absolutely continuous with respect to Lebesgue measure. In this paper we prove that ν_λ is even equivalent to Lebesgue measure for almost all $\lambda \in [\frac{1}{2}, 1]$.

1. Introduction

For each $\lambda \in (0, 1)$ we define the random variable

$$Y_\lambda = \sum_{n=0}^{\infty} \theta_n \cdot \lambda^n,$$

where θ_n are independent random variables with $Prob(\theta_n = -1) = Prob(\theta_n = 1) = \frac{1}{2}$. The distribution ν_λ of Y_λ is sometimes called a symmetric infinite Bernoulli convolution. One can easily see that for $0 < \lambda < \frac{1}{2}$ the distribution ν_λ is supported on a Cantor set of zero Lebesgue measure. Since the 1930's a lot of work has been done to characterize ν_λ for $\frac{1}{2} < \lambda$ (for a good survey see e.g. Peres, Solomyak (1996a) [4]). Among these results the most interesting ones are as follows: P. Erdős (1939) [1] proved that ν_λ is singular with respect to Lebesgue measure, if λ is the reciprocal of a PV number. (An algebraic integer is a PV number provided all of its conjugates are less than one in modulus.) On the other hand, Wintner (1935) [7] proved that ν_λ is absolutely continuous for $\lambda = 2^{-\frac{1}{k}}$, for each $k \geq 1$, and Garsia (1962) [3] found some other algebric integers for which ν_λ is absolutely continuous. Moreover, Erdős (1940) [2] also proved that there exists $a < 1$ such that the distribution ν_λ is absolutely continuous with respect to Lebesgue measure for (Lebesgue) a.e. $\lambda \in (a, 1)$. Then P. Erdős asked:

$$\text{Is this statement true with } a = \frac{1}{2}?$$

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This exciting problem remained open for more than fifty years. Then Solomyak (1995) [6] gave a positive answer (see also Peres, Solomyak (1996a) [4] for a shorter proof). Namely,

Theorem 1 (Solomyak)**.**

$$\nu_\lambda \ll m \text{ for Lebesgue a.e. } \lambda \in (\frac{1}{2}, 1),$$

where m is Lebesgue measure.

Answering a problem of the first author, posed to the Conference on Fractals and Stochastics (1994, Finsterbergen), we prove that that ν_λ is even equivalent to Lebesgue measure for a.e. $\lambda \in [\frac{1}{2}, 1]$. Using Solomyak's theorem, it is enough to prove that Lebesgue measure is either absolutely continuous or singular with respect to ν_λ for each λ. Actually we prove this statement for a more general family of measures. Furthermore, Peres, Solomyak (1996b) [5] proved that if the probabilities of choosing the signs $+$ and $-$ in Y_λ are $(p, 1-p)$ where $p \in [1/3, 2/3]$, then $\nu_\lambda \ll m$ holds for a.e. $\lambda \in [p^p(1-p)^{1-p}, 1]$. Using this, it follows from our result that even in this non-symmetric case the distributions are not only absolutely continuous but equivalent to Lebesgue measure for a.e. $\lambda \in [p^p(1-p)^{1-p}, 1]$. (For smaller λ the distributions are singular.)

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2. Notation

For an arbitrary $\lambda \in (\frac{1}{2}, 1)$ we define the 'projection' $\Pi_\lambda : \{-1,1\}^{\mathbf{N}} \to [\frac{-1}{1-\lambda}, \frac{1}{1-\lambda}]$ by $\Pi_\lambda(\mathbf{i}) = \sum_{k=0}^{\infty} i_k \lambda^k$. Let μ be any Borel probability measure on $\{-1,1\}^{\mathbf{N}}$ for which

$$\mu(B) > 0 \Longrightarrow \mu\{(i, B)\} > 0 \tag{1}$$

holds for all $B \subset \{-1,1\}^{\mathbf{N}}$ and $i \in \{-1,1\}$, where $(i, B) := \{(i, \mathbf{j}) \in \{-1,1\}^{\mathbf{N}} : \mathbf{j} \in B\}$. For example μ may be any Bernoulli measure on $\{-1,1\}^{\mathbf{N}}$ with probabilities $(p, 1-p)$, $0 < p < 1$. The 'push down measure' of μ is $\alpha_{\lambda,\mu}(B) := \mu\left(\Pi_\lambda^{-1}(B)\right)$. We denote the interval $[\frac{-1}{1-\lambda}, \frac{1}{1-\lambda}]$ by I. Further, we define $S_i : I \to I$, $S_i(x) := \lambda x + i$ for $(i = -1, 1)$. The iterates of S_i are

$$S_{i_1 \dots i_n}(x) := S_{i_1} \circ \dots \circ S_{i_n}(x).$$

The image of I under $S_{i_1 \dots i_n}$ is called $I_{i_1 \dots i_n}$. The inverse of $S_{i_1 \dots i_n}$ is defined **only** on $I_{i_1 \dots i_n}$. So $S^{-1}_{i_1 \dots i_n}(A) := S^{-1}_{i_1 \dots i_n}(A \cap I_{i_1 \dots i_n})$. Then $S_i^{-1}(x) = \frac{1}{\lambda}x - \frac{i}{\lambda}$ for $x \in I_i$ $(i = -1, 1)$. We denote the Lebesgue measure of a set A by $m(A)$.

3. The theorem and its consequences

Theorem 2. *Either $m \ll \alpha_{\lambda,\mu}$ or $m \perp \alpha_{\lambda,\mu}$.*

If μ is the Bernoulli measure with probabilities $(\frac{1}{2}, \frac{1}{2})$ then $\nu_\lambda = \alpha_{\lambda,\mu}$. Using Solomyak's Theorem, we obtain

Consequence 1. *For almost all $\lambda \in (\frac{1}{2}, 1)$, ν_λ is equivalent to Lebesgue measure.*

Clearly, any Bernoulli measure μ with probabilities $(p, 1-p)$, satisfies (1) (if $p \neq 0$). Thus,

Consequence 2. *Let η_λ be the distribution of the random series $Z_\lambda = \sum \pm\lambda^n$, where the signs are are chosen independently with probabilities $(p, 1-p)$ and $0 < \lambda < 1$. Then either $m \ll \eta_\lambda$ or $m \perp \eta_\lambda$.*

Let η_λ be as above. Then η_λ is singular for all $\lambda < p^p(1-p)^{1-p}$ (see Peres, Solomyak (1996b) [5, Theorem 2 (a)]). Also Peres, Solomyak (1996b) [5, Corollary 1.4] proved that for $p \in [1/3, 2/3]$ and for a.e. $\lambda \in [p^p(1-p)^{1-p}, 1]$, $\eta_\lambda \ll m$. Thus, using our previous consequence we obtain

Consequence 3. *Let η_λ be the distribution of the random series $Z_\lambda = \sum \pm\lambda^n$, where the signs are are chosen independently with probabilities $(p, 1-p)$. Then for each $p \in [1/3, 2/3]$ and for almost every $\lambda \in [p^p(1-p)^{1-p}, 1]$, the distribution η_λ is equivalent to Lebesgue measure.*

4. Lemmas and proofs

To prove Theorem 2 we need two lemmas.

Lemma 1. *Let $A \subset I$. Then $\alpha_{\lambda,\mu}(A) = 0 \Longrightarrow \alpha_{\lambda,\mu}(S_i^{-1}(A)) = 0$ $(i = -1, 1)$.*

Proof. First observe that

$$\Pi_\lambda^{-1}(A) = \left\{\left(-1, \Pi_\lambda^{-1}(S_{-1}^{-1}(A))\right)\right\} \bigcup \left\{\left(1, \Pi_\lambda^{-1}(S_1^{-1}(A))\right)\right\}. \tag{2}$$

This is so, since for $i = -1, 1$

$$\mathbf{j} \in \Pi_\lambda^{-1}(S_i^{-1}(A)) \Longleftrightarrow \sum_{k=0}^{\infty} j_k\lambda^k \in S_i^{-1}(A) \Longleftrightarrow \sum_{k=0}^{\infty} j_k\lambda^k \in \frac{1}{\lambda}A - \frac{i}{\lambda}$$
$$\Longleftrightarrow i + \sum_{k=0}^{\infty} j_k\lambda^{k+1} \in A \Longleftrightarrow (i, \mathbf{j}) \in \Pi_\lambda^{-1}(A).$$

To get a contradiction we assume that there exists a set A such that $\alpha_{\lambda,\mu}(A) = 0$ and $\alpha_{\lambda,\mu}(S_i^{-1}(A)) = \mu\left(\Pi_\lambda^{-1}(S_i^{-1}(A))\right) > 0$ holds for some $i \in \{-1, 1\}$.

Then from (1), it follows that $\mu\left(\left(i, \Pi_\lambda^{-1}(S_i^{-1}(A))\right)\right) > 0$. Using (2), we find that $\mu(\Pi_\lambda^{-1}(A)) = \alpha_{\lambda,\mu}(A) > 0$. This contradiction proves our lemma. ☐

Let $C \subset I$ be an arbitrary fixed Borel set. Let $C_0 := C$ and

$$C_{-(k+1)} := \left(S_{-1}^{-1}(C_{-k}) \cup S_1^{-1}(C_{-k})\right).$$

Then the 'backward orbit' of C in I is

$$\Lambda_- := \bigcup_{k \geq 0} C_{-k}. \tag{3}$$

Lemma 2. *For any $C \subset I$, the set Λ_- defined above is either a set of zero measure or a full measure subset of I with respect to Lebesgue measure.*

Proof. Let $\overline{\Lambda}_- := I \setminus \Lambda_-$. Obviously, it is enough to prove the statement of Lemma 2 for the set $\overline{\Lambda}_-$ instead of Λ_-. Observe that

$$x \in \overline{\Lambda}_- \Longrightarrow S_i(x) \in \overline{\Lambda}_- \tag{4}$$

holds, since $S_i(x) \notin \overline{\Lambda}_- \Longrightarrow \exists k \geq 0$ such that $S_i(x) \in C_{-k} \cap I_i$. Then $x = S_i^{-1}(S_i(x)) \in C_{-(k+1)} \subset \Lambda_-$. Iterate (4) to obtain

$$S_{i_1 \ldots i_n}\left(\overline{\Lambda}_-\right) \subset \overline{\Lambda}_-, \tag{5}$$

for each $n \in \mathbf{N}$ and $(i_1, \dots, i_n) \in \{-1, 1\}^n$. Suppose that $m(\overline{\Lambda}_-) > 0$. Then $d := \frac{m(\overline{\Lambda}_-)}{|I|}$ is positive. Using (5), we obtain that $m\left(\overline{\Lambda}_- \cap I_{i_1 \dots i_n}\right) \geq m(S_{i_1 \dots i_n}(\overline{\Lambda}_-)) = \lambda^n \cdot d \cdot |I|$. Thus

$$\frac{m\left(\overline{\Lambda}_- \cap I_{i_1 \dots i_n}\right)}{|I_{i_1 \dots i_n}|} \geq d \tag{6}$$

holds for each $i_1 \dots i_n$.

On the other hand, let $J \subset I$ be an arbitrary interval. Then we can find n and $i_1 \dots i_n$ such that $I_{i_1 \dots i_n} \subset J$ and

$$\frac{|I_{i_1 \dots i_n}|}{|J|} \geq \frac{\lambda}{3}. \tag{7}$$

Now, from (6) and (7) together, it follows that

$$\frac{m\left(\overline{\Lambda}_- \cap J\right)}{|J|} \geq d \cdot \frac{\lambda}{3}.$$

That is, Λ_- has no density point. Thus $\overline{\Lambda}_-$ is a full measure subset of I. This completes the proof of Lemma 2. □

Proof of Theorem 2. Suppose that $m \not\ll \alpha_{\lambda,\mu}$. Then there is a set $C \subset I$ such that $m(C) > 0$ and $\alpha_{\lambda,\mu}(C) = 0$. Define Λ_- by (3). Then $m(\Lambda_-) > 0$; thus it follows from Lemma 2 that Λ_- is a full measure subset of I with respect to Lebesgue measure. On the other hand, Lemma 1 implies that $\alpha_{\lambda,\mu}(\Lambda_-) = 0$. So $m \perp \alpha_{\lambda,\mu}$. This completes the proof of Theorem 2. □

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