M-IDEALS OF COMPACT OPERATORS
ARE SEPARABLY DETERMINED

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(Communicated by Palle E. T. Jorgensen)

Abstract. We prove that the space $K(X)$ of compact operators on a Banach space $X$ is an $M$-ideal in the space $L(X)$ of bounded operators if and only if $X$ has the metric compact approximation property (MCAP), and $K(Y)$ is an $M$-ideal in $L(Y)$ for all separable subspaces $Y$ of $X$ having the MCAP. It follows that the Kalton-Werner theorem characterizing $M$-ideals of compact operators on separable Banach spaces is also valid for non-separable spaces: for a Banach space $X$, $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ has the MCAP, contains no subspace isomorphic to $\ell_1$, and has property (M). It also follows that $K(Z, X)$ is an $M$-ideal in $L(Z, X)$ for all Banach spaces $Z$ if and only if $X$ has the MCAP, and $K(\ell_1, X)$ is an $M$-ideal in $L(\ell_1, X)$.

Introduction

A (closed) subspace $Y$ of a Banach space $X$ is called an $M$-ideal if there exists a projection $P$ on the dual space $X^*$ such that $\ker P = Y^\perp$, and $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$ for all $x^* \in X^*$.

Already more than twenty years, many authors have studied conditions for the space $K(X)$ of compact operators on $X$ to be an $M$-ideal in the space $L(X)$ of bounded operators (see [8, pp. 333-336] for a brief history and references) with the main aim to characterize those Banach spaces $X$ for which $K(X)$ is an $M$-ideal in $L(X)$. Some years ago, N. J. Kalton and D. Werner [10] showed that, for a separable Banach space $X$, $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ has the metric compact approximation property (MCAP), contains no subspace isomorphic to $\ell_1$, and has property (M) (property (M), which is an internal geometric property of $X$, will be defined in Section 2 below). Since the method of proof of this characterization implies the separability of $X$, the question whether the Kalton-Werner theorem is also valid for non-separable $X$ remained open.

In Section 1 of the present note, we prove that $M$-ideals of compact operators are separably determined: $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ has the MCAP, and $K(Y)$ is an $M$-ideal in $L(Y)$ for all separable subspaces $Y$ of $X$ having the MCAP. In Section 2, this enables us to extend the Kalton-Werner theorem to non-separable spaces $X$. This also enables us to show that $K(Z, X)$ is an $M$-ideal in $L(Z, X)$ for all Banach spaces $Z$ if and only if $X$ has the MCAP, and $K(\ell_1, X)$ is an $M$-ideal in $L(\ell_1, X)$ (cf. Section 3).

Received by the editors February 14, 1997.
1991 Mathematics Subject Classification. Primary 46B28, 47D15, 46B20.
The author was partially supported by the Estonian Science Foundation Grant 3055.

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2747
Let us fix some more notation. In a Banach space $X$, we denote the closed unit ball by $B_X$. For a set $A \subseteq X$, its norm closure is denoted by $\overline{A}$, its linear span by $\text{span} \ A$, and its convex hull by $\text{conv} \ A$. The set of all weak* strongly exposed points of $B_X$ is denoted by $w^\ast$-sexp $B_{X^\ast}$, and the identity operator of $X$ is denoted by $I_X$ or simply by $I$. Recall that $X$ is said to have the MCAP if there is a net in $B_{K(X)}$ converging strongly to $I$. (This means that $K(X)$ contains a left 1-approximate unit (cf. e.g. [8, p. 294]).)

1. $M$-ideals of compact operators

The following characterization of $M$-ideals of compact operators will be needed below to prove that $M$-ideals of compact operators are separably determined.

Theorem 1. Let $X$ be a Banach space. Then $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ an $M$-ideal in $X^\ast\ast$, and for all $S \in B_{K(X)}$ there is a net $(K_\alpha)$ in $B_{K(X)}$ such that $K_\alpha \to I$ strongly and

$$\limsup \|S + I - K_\alpha\| \leq 1.$$

Proof. The necessity is well known (cf. [11] or [8, p. 291] together with [19] or [8, p. 299]). Sufficiency. Recall that $K(X)$ is an $M$-ideal in $L(X)$ if and only if $K(X)$ is an $M$-ideal in $\mathcal{L} = \text{span} (K(X) \cup \{I\})$ (cf. [14] or [8, p. 299], or [9] for separable $X$). Recall also that the MCAP of $X$ implies the existence of a linear norm preserving extension operator $\Phi : K(X)^* \to \mathcal{L}^\ast$ (cf. e.g. [12]). This makes it possible to consider the topology $\sigma = \sigma (\mathcal{L}, \Phi (K(X)^*))$. In [18], it is essentially proved (for a simpler proof cf. [6]) that $K(X)$ is an $M$-ideal in $\mathcal{L}$ if and only if for all $S \in B_{K(X)}$ and $T \in B_\mathcal{L}$ there is a net $(L_\alpha)$ in $B_{K(X)}$ such that $(L_\alpha)$ is $\sigma$-convergent to $T$ and

$$\limsup \|S + T - L_\alpha\| \leq 1.$$

Consider $T = K + \lambda I \in B_\mathcal{L}$ (with $K \in K(X)$). Note that $|\lambda| \leq 1$ because otherwise $K$ would be invertible. Therefore $\lambda = re^{i\varphi}$ with $r \in [0,1]$. For $e^{-i\varphi}S$, choose the net $(K_\alpha)$. Since $X$ has the unique extension property (following from the fact that $X$ is an $M$-ideal in $X^\ast\ast$), we have $x^* (K_\alpha x^\ast) \to x^* (x^*)$ for all $x^* \in X^\ast$, $x^\ast \in X^\ast\ast$ (cf. [5] or [8, p. 118]). Set $L_\alpha = K_\alpha T = K_\alpha K + \lambda K_\alpha$. Then $L_\alpha \in B_{K(X)}$,

$$x^* (L_\alpha^\ast x^\ast) = (K_\alpha^\ast x^\ast) (K_\alpha^\ast x^\ast) + \lambda x^\ast (K_\alpha^\ast x^\ast) \to x^* (T^\ast x^\ast),$$

i.e. $(x^\ast \otimes x^*) (L_\alpha - T) \to 0$ for all $x^\ast \otimes x^\ast \in \mathcal{L}^\ast$, and (since $\|K_\alpha K - K\| \to 0$)

$$\limsup \|S + T - L_\alpha\| = \limsup \|S + re^{i\varphi} I - re^{i\varphi} K_\alpha\|$$

$$= \limsup \|e^{-i\varphi} S + r I - r K_\alpha\|$$

$$\leq r \limsup \|e^{-i\varphi} S + I - K_\alpha\| + 1 - r$$

$$\leq 1.$$
Remark 1. It is known (cf. [14] or [8, p. 299], or [9] for separable $X$) that $K(X)$ is an $M$-ideal in $L(X)$ if and only if there is a net $(K_\alpha)$ in $B_{K(X)}$ such that both $K_\alpha \to I_X$ and $K_\alpha^* \to I_{X^*}$ strongly, and $\limsup \|S+I-K_\alpha\| \leq 1$ for all $S \in B_{K(X)}$.

Remark 2. The following is clear from the proof of Theorem 1: if $X$ is an $M$-ideal in $X^{**}$ (in particular, if $K(X)$ is an $M$-ideal in $L(X)$), and $K_\alpha \to I$ strongly for some net $K_\alpha \in B_{K(X)}$, then $K_\alpha \to I$ in the $\sigma(L(X),\Phi(K(X)^*))$-topology for any linear norm preserving extension operator $\Phi : K(X)^* \to L(X)^*$.

The next theorem is the main result of the present note.

Theorem 2. Let $X$ be a Banach space. Then $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ has the MCAP, and $K(Y)$ is an $M$-ideal in $L(Y)$ for all separable subspaces $Y$ of $X$ having the MCAP.

Proof. The necessity is well known (cf. [14] or [8, p. 301], or [9] for separable $X$). Sufficiency. We shall apply Theorem 1. To prove that $X$ is an $M$-ideal in $X^{**}$, we need to show that every separable subspace of $X$ is an $M$-ideal in its bidual (cf. [13] or [8, p. 115]). Consider a separable subspace $Y$ of $X$. Since $X$ has the MCAP, $Y$ is contained in a separable subspace $Z$ of $X$ having the MCAP (the proof of this fact is the same as of the similar fact for the metric approximation property (cf. e.g. [17, p. 606])). Thus, $K(Z)$ is an $M$-ideal in $L(Z)$, which implies that $Z$ is an $M$-ideal in $Z^{**}$. But then also its subspace $Y$ is an $M$-ideal in $Y^{**}$.

Let us now make the following observation. If $K(Y)$ is an $M$-ideal in $L(Y)$ for a separable Banach space $Y$, and $K_\alpha \to I_Y$ strongly for some sequence $(K_\alpha) \subset B_{K(Y)}$, then, for all $S \in B_{K(Y)}$ and $\varepsilon > 0$, there is some $K \in \text{conv} \{K_1, K_2, \ldots\}$ such that $\|S-K+I_Y\| \leq 1 + \varepsilon/2$. [Due to Remark 2, the proof of this fact is the same as of the similar assertion about Banach spaces being $M$-ideals in their biduals in [13, Proposition 2.8, (i) $\implies$ (ii)] (cf. also [8], p. 113), only using instead of the weak* topology the $\sigma(L(Y),\Phi(K(Y)^*))$-topology (where $\Phi : K(Y)^* \to L(Y)^*$ is the (unique linear) norm preserving extension operator)].

We denote by $s_{op}$ the strong operator topology on $L(X)$, and suppose that the condition of Theorem 1 is not satisfied: for some $S \in B_{K(X)}$, there is no such net. Then there are $\varepsilon > 0$ and a convex $s_{op}$- neighbourhood $U_0$ of $I$ such that

$$\|S-K+I\| > 1 + \varepsilon \quad \forall K \in B_{K(X)} \cap U_0. \tag{1}$$

For all $n \in \mathbb{N}$, denote by $\Lambda_n$ a finite $\varepsilon/4$-net in the subset $\{K_{\alpha} : \lambda_k \geq 0, \lambda_1 + \cdots + \lambda_n = 1\}$ of $l_1^n$. Let $(K_\alpha)_{\alpha \in \mathcal{A}}$ be a net in $B_{K(X)}$ converging to $I$ in the $s_{op}$. We shall follow some ideas from the proofs of Proposition 2.8, (iii) $\implies$ (iv), in [13] (cf. [8, p. 114]) and Theorem 18.2 in [17, p. 606] to pick a sequence $\alpha_1, \alpha_2, \ldots$ in $\mathcal{A}$ and to define a separable subspace $Y \subset X$ so that $S(X) \subset Y$, $K_n(X) \subset Y$ for all $K_n = K_{\alpha_n}$, $K_n y \to y$ for all $y \in Y$, and $\|S-K+I\|_Y > 1 + \varepsilon/2$ for all $K \in \text{conv} \{K_1, K_2, \ldots\}$. This will contradict the observation above, and complete the proof.

To begin, choose $K_1 = K_{\alpha_1} \in U_0$ such that $\|K_1 x - x\| < 1$ for all $x \in S(B_X)$. Assume that a convex $s_{op}$- neighbourhood $U_{n-1} \subset U_{n-2}$ (where $U_{-1} = U_0$) and $K_n = K_{\alpha_n} \in U_{n-1}$ have been chosen. Consider $S_{\lambda} \in L(X), \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_n$, defined by $S_{\lambda} = S - (\lambda_1 K_1 + \cdots + \lambda_n K_n) + I$. For all $\lambda \in \Lambda_n$, select $x_\lambda \in B_X$ such that $\|S_{\lambda} x_\lambda\| > \|S_{\lambda}\| - \varepsilon/4$, and denote $C_\lambda = \{x_\lambda : \lambda \in \Lambda_n\}$. Put

$$F_n = (1 + \varepsilon)B_{L(X)} + \text{conv}\{K_1, \ldots, K_n\} - S.$$
Proof. (a) = separable case). Only if $X$ is not isomorphic to $\mathcal{L}$ or e.g. [8, p. 142]). Show that if a Banach space $X$ has the MCAP, and has property $(M)$, then $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ has the MCAP, contains no subspace isomorphic to $\ell_1$, and has property $(M)$. By [9], since $Z$ is separable, $Z$ has property $(M)$, and is an $M$-ideal in $Z^{**}$, in particular (cf. [11] or e.g. [8, p. 126]), $Z$ is an Asplund space.

2. Kalton-Werner theorem

Recall (cf. [9]) that a Banach space $X$ is said to have property $(M)$ if

$$\limsup ||x + x_n|| = \limsup ||y + x_n||$$

whenever $||x|| = ||y||$, and $(x_n)$ is a weakly null sequence in $X$; if

$$\limsup ||x^* + x^*_n|| = \limsup ||y^* + x^*_n||$$

whenever $||x^*|| = ||y^*||$, and $(x_n^*)$ is a weak*-null sequence in $X^*$, then $X$ is said to have property $(M^*)$. We also need the strong version of property $(M^*)$, which we call property $(sM^*)$, defined by bounded weak*-null nets $(x_n^*)$ instead of weak*-null sequences $(x_n^*)$. It is shown in [14], that if $X$ is separable, then properties $(M^*)$ and $(sM^*)$ are equivalent. The Kalton–Werner theorem mentioned above asserts that if a Banach space $X$ is separable, then $K(X)$ is an $M$-ideal in $L(X)$ if and only if $X$ has the MCAP, contains no subspace isomorphic to $\ell_1$, and has property $(M)$.

Theorem 3. For a Banach space $X$ the following assertions are equivalent.

(a) $K(X)$ is an $M$-ideal in $L(X)$.
(b) $X$ has the MCAP, and has property $(sM^*)$.
(c) $X$ has the MCAP, is weakly compactly generated, and has property $(M^*)$.
(d) $X$ has the MCAP, contains no subspace isomorphic to $\ell_1$, and has property $(M)$.

Proof. (a) $\implies$ (b) is well known (cf. [7] and [14] or e.g. [8, p. 299]; cf. [9] for separable case). (b) $\implies$ (c). Property $(sM^*)$ implies that $X$ is an $M$-ideal in $X^{**}$ (cf. [14] or [8, p. 297]; cf. [9] for separable case). But then $X$ is weakly compactly generated (cf. [3] or e.g. [8, p. 142]). (c) $\implies$ (d). Let $Y$ be an arbitrary separable subspace of $X$. We have to show that $Y$ is not isomorphic to $\ell_1$, and has property $(M)$. Since $X$ is weakly compactly generated, there exists a separable subspace $Z$ containing $Y$, and a norm-one projection $P$ of $X$ onto $Z$ (cf. e.g. [2, p. 149]). But then $Z^*$ isometrically embeds into $X^*$ by means of the formula $z^* \in Z^* \mapsto z^*P \in X^*$. This implies that $Z$ also has property $(M^*)$. By [9], since $Z$ is separable, $Z$ has property $(M)$, and is an $M$-ideal in $Z^{**}$, in particular (cf. [11] or e.g. [8, p. 126]), $Z$ is an Asplund space.
(i.e. every separable subspace of $Z$ has a separable dual). Hence, $Y$ has property $(M)$, and is not isomorphic to $\ell_1$.

(d) $\implies$ (a) follows from Theorem 2 and the Kalton-Werner theorem. 

Remark. In (c) of Theorem 3, the condition that $X$ is weakly compactly generated clearly may be replaced by the separable complementation property (i.e. every separable subspace of $X$ is contained in a separable subspace which is a range of a norm-one projection on $X$).

A Banach space $X$ is said to have the compact approximation property (CAP) if there is a net in $K(X)$ converging strongly to the identity. Since a reflexive space with the CAP has the MCAP (cf. [1] or [5]), we can refine Theorem 3 for reflexive $X$ as follows.

Corollary 4. For a reflexive Banach space $X$, the following assertions are equivalent.

(a) $K(X)$ is an $M$-ideal in $L(X)$.
(b) $X$ has the CAP, and has property $(sM^*)$.
(c) $X$ has the CAP, and has property $(M^*)$.
(d) $X$ has the CAP, and has property $(M)$.

Remark. For separable reflexive spaces $X$, Corollary 4 was obtained in [10]. The equivalence (a) $\iff$ (b) of Corollary 4 was established in [12] (using an entirely different proof).

3. $(M_p)$-spaces

Let $1 \leq p \leq \infty$. Following [15] (cf. also [8, p. 306]), we say that a Banach space $X$ is an $(M_p)$-space if $K(X \oplus_p X)$ is an $M$-ideal in $L(X \oplus_p X)$. Note that $(M_1)$-spaces are finite dimensional [16], [8, p. 306], and therefore not of interest in the present context. Since every separable subspace of $X \oplus_p X$ is contained in $Y \oplus_p Y$ for some separable subspace $Y$ of $X$ with the MCAP whenever $X$ has the MCAP (cf. the proof of Theorem 2), the next result follows immediately from Theorem 2.

Corollary 5. Let $1 < p \leq \infty$. A Banach space $X$ is an $(M_p)$-space if and only if $X$ has the MCAP, and all separable subspaces of $X$ with the MCAP are $(M_p)$-spaces.

In [10], N. J. Kalton and D. Werner characterized separable $(M_p)$-spaces using the following stronger version of property $(M)$. A Banach space $X$ is said to have property $(m_p)$ if

$$\limsup \|x + x_n\| = \|(|x|, \limsup \|x_n\|)\|_p$$

(where $\| \cdot \|_p$ denotes the $\ell^2_p$-norm) whenever $(x_n)$ is a weakly null sequence in $X$. For separable Banach spaces $X$, the following result was obtained in [10].

Corollary 6. Let $1 < p \leq \infty$. For a Banach space $X$, the following assertions are equivalent.

(a) $X$ is an $(M_p)$-space.
(b) $K(X)$ is an $M$-ideal in $L(X)$, and $X$ has property $(m_p)$.
(c) $X$ has the MCAP, contains no subspace isomorphic to $\ell_1$, and has property $(m_p)$. 

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(d) $X$ has the MCAP, and every separable subspace of $X$ is almost isometric (in the sense of Banach-Mazur distance) to a subspace of an $\ell_p$-sum of finite-dimensional spaces when $p < \infty$, respectively, to a subspace of $c_0$ when $p = \infty$.

Proof. The equivalence (b) $\iff$ (c) is clear from Theorem 3 since $(m_p)$ implies (M); (c) $\iff$ (d) is clear from Theorems 3.3 and 3.5 in [10] since considered $\ell_p$-sums are reflexive, and subspaces of $c_0$ fail to contain a copy of $\ell_1$. The equivalence (a) $\iff$ (c) is proved in [10] for separable $X$; it extends to the general case by Corollary 5 using the fact that if $X$ has the MCAP, then every separable subspace of $X$ is contained in a separable subspace having the MCAP. \hfill \Box

Finally, we come to the most important application of this paper – a characterization of $(M_\infty)$-spaces. The class of $(M_\infty)$-spaces was introduced and studied by R. Payá and W. Werner in [16], where it is proved that $X$ is an $(M_\infty)$-space if and only if $K(Z,X)$ is an $M$-ideal in $L(Z,X)$ for every Banach space $X$. One of the main results of [13] states that a separable Banach space $Z$ with the MCAP is an $(M_\infty)$-space if and only if $K(\ell_1,Y)$ is an $M$-ideal in $L(\ell_1,Y)$. As we now see, this is also true for non-separable spaces.

Corollary 7. A Banach space $X$ is an $(M_\infty)$-space if and only if $X$ has the MCAP, and $K(\ell_1,X)$ is an $M$-ideal in $L(\ell_1,X)$.

Proof. The necessity is clear from the above. The sufficiency immediately follows from Corollary 5 and the result of [13] stated just before Corollary 7, because the $M$-ideal property of $K(\ell_1,X)$ in $L(\ell_1,X)$ implies that $K(\ell_1,Y)$ is an $M$-ideal in $L(\ell_1,Y)$ for all subspaces $Y$ of $X$ [13]. \hfill \Box

ACKNOWLEDGMENT

The author wishes to thank M. Pőldvere for interesting discussions on the topic of this paper.

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