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REGULARIZATION OF SEMIGROUPS THAT ARE STRONGLY CONTINUOUS FOR $t > 0$

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ABSTRACT. Let E be a Banach space and $T :]0,\infty[\to L(E)$ a strongly continuous semigroup with $\bigcap_{t>0} \operatorname{Kern} T_t = \{0\}$. We show that the generator A of (T_t) generates a regularized semigroup. Our construction of a regularizing operator uses an existence result of J. Esterle.

1. INTRODUCTION AND MAIN RESULT

Regularized semigroups are an active field of research. They were independently introduced in [1] and [2]. Some other references on this subject are [7], [9] and especially [3]. However, the problem of their relation to semigroups that are strongly continuous for $t > 0$ seems not to have been fully cleared, and the quite natural question posed in [6], of whether *any* semigroup that is strongly continuous for $t > 0$ can be regularized, was still unanswered.

It has been known that strongly continuous semigroups of growth order $\alpha > 0$ (see [1], [7]) and semigroups of class $(C_{(k)})$ (see [8], [7]) can be regularized. In [6], the first result was extended to semigroups whose norms grow not faster than $\exp(\omega(t^{-1}))$ as $t \to 0$, where ω is an increasing function satisfying $\int_1^\infty t^{-2} \log \omega(t)\, dt < \infty$.

The purpose of this note is to show that no growth assumptions are needed, thus giving a positive answer to the question mentioned above.

Theorem 1.1 (Main Result)**.** *Let $(T_t)_{t>0}$ be a semigroup that is strongly continuous for $t > 0$ in a Banach space E satisfying $\bigcap_{t>0} \operatorname{Kern} T_t = \{0\}$. Then the generator A of (T_t) generates a regularized semigroup.*

Recall that in [6] we defined the *generator* of (T_t) to be the operator

$$A := \{(x, y) \in E \times E : \forall s > 0 : \lim_{t\to 0} \frac{1}{t}(T_{t+s}x - T_s x) = T_s y\},$$

which is single-valued, linear and closed and satisfies

$$A = \{(x, y) : \forall s > 0 : (T_s x, T_s y) \in A_0\},$$

where $A_0 = \{(x, y) : \lim_{t\to 0}(T_t x - x)/t = y\}$ is the *infinitesimal generator* of (T_t) introduced in [5].

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2. Proof of the Main Result

Let E be a Banach space, let $T :]0,\infty[\to L(E)$ be a strongly continuous semigroup satisfying $\bigcap_{t>0} \operatorname{Kern} T_t = \{0\}$, and let A denote the generator of (T_t). Moreover we assume without loss of generality that the type of (T_t) is negative. Concerning the generator A of (T_t), we remark that

$$(x,y) \in A \Longleftrightarrow \forall t > s > 0 : T_t x - T_s x = \int_s^t T_r y \, dr. \tag{1}$$

For the definitions and properties of regularized semigroups we refer to [3]. Recall that a *regularized semigroup* is a strongly continuous mapping $S : [0,\infty[\to L(E)$ satisfying $S_s S_t = S_{t+s} S_0$ for all $s, t \geq 0$ and $\operatorname{Kern} S_0 = \{0\}$, and that its generator B is the closed linear operator defined by

$$B := \{(x,y) : \lim_{t\to 0} (S_t x - S_0 x)/t = S_0 y\}.$$

If the operator S_0 is to be specified then (S_t) is called an *S_0-regularized semigroup.* Recall also that

$$(x,y) \in B \Longleftrightarrow \forall t > 0 : S_t x - S_0 x = \int_0^t S_r y \, dr. \tag{2}$$

We start with the following lemma.

Lemma 2.1. *The generator A of (T_t) generates a regularized semigroup if and only if there is an injective $C \in L(E)$ which commutes with all T_t, $t > 0$, and satisfies*

$$\operatorname{im} C \subset \Sigma := \{x \in E : \lim_{t\to 0} T_t x = x\}$$

(in which case A generates a C-regularized semigroup).

Proof. If A generates a regularized semigroup (S_t), then $C := S_0$ has the desired properties since in this case $T_t = S_0^{-1} S_t$ for all $t > 0$ (see e.g. [3]).

On the other hand, if C is as in the assertion, then $S_0 := C$, $S_t := T_t C$ for $t > 0$, defines a regularized semigroup. Let B denote the generator of (S_t). If $(x,y) \in B$, then we get from (2) that $C(T_t x - T_s x) = S_t x - S_s x = \int_s^t S_r y \, dr = C \int_s^t T_r y \, dr$. By the injectivity of C and (1) this implies $(x,y) \in A$. On the other hand, if $(x,y) \in A$, then by applying C to both sides of the right-hand side of (1) and using the definition and the strong continuity of (S_t) we get $(x,y) \in B$ by (2). Hence $A = B$. □

We will now construct a suitable regularizing operator C for (T_t). Our construction will use a result of J. Esterle on the existence of functions which generate dense principal ideals in certain convolution algebras.

We first choose a decreasing continuous function $w :]0,\infty[\to [1,\infty[$ such that $w(t) \geq \|T_t\|$ for all $t > 0$. This is possible since the type of (T_t) is negative.

Let $L^1(w)$ denote the Banach space of all measurable complex valued functions f on $]0,\infty[$ such that $\|f\|_w := \int_0^\infty |f(t)| w(t) \, dt$ is finite. Since $w \geq 1$ is decreasing, $L^1(w)$ is a Banach algebra with respect to convolution.

Lemma 2.2. *(a) The mapping $f \mapsto \int_0^\infty f(t) T_t \, dt$, where the integral is taken in the strong sense, defines a continuous algebra homomorphism $\tilde{T} : L^1(w) \to L(E)$.*

(b) For every $f \in L^1(w)$ the operator $\tilde{T}(f)$ commutes with all T_t, $t > 0$, and satisfies $\operatorname{im} \tilde{T}(f) \subset \Sigma$.

(c) If $f \in L^1(w)$ is such that the set of right translates $\{\tau_t f : t > 0\}$ is total in $L^1(w)$, then $\tilde{T}(f)$ is injective.

Proof. (a) is clear from the choice of w and the semigroup property. For (b) and (c), observe first that $T_t\tilde{T}(f) = \tilde{T}(\tau_t f)$ for all $t > 0$. Since w is decreasing, the right translation (τ_t) is by dominated convergence strongly continuous on $L^1(w)$. This and (a) prove (b). To prove (c), assume that $\tilde{T}(f)x = 0$ and let $x^* \in E^*$. Then

$$\int \tau_t f(s)\langle x^*, T_s x\rangle \, ds = \langle x^*, \tilde{T}(\tau_t f)x\rangle = \langle x^*, T_t\tilde{T}(f)\rangle = 0$$

for every $t > 0$. Since $t \mapsto \langle x^*, T_t x\rangle$ is a continuous linear functional on $L^1(w)$, the assumption on f implies $\langle x^*, T_\cdot x\rangle = 0$. By Hahn-Banach this means $T_\cdot x = 0$, and the assumption on (T_t) implies $x = 0$. □

We give another formulation of the condition in (c).

Remark 2.3. Let $f \in L^1(w)$. Then $\{\tau_t f : t > 0\}$ is total in $L^1(w)$ if and only if $f * L^1(w)$ is dense in $L^1(w)$.

Proof. If $t > 0$ then $\tau_t f = \lim_{n\to\infty} f * \tau_t\varphi_n$, where $\varphi_n := n\varphi(n\cdot)$ and $\varphi \geq 0$ with $\operatorname{supp}\varphi \subset]0,\infty[$ and $\int \varphi = 1$. This clearly holds in L^1, and also in $L^1(w)$ since on the space of functions with support in $[t,\infty[$ both norms are equivalent (recall that w is decreasing and ≥ 1).

On the other hand, let g be continuous with compact support in $]0,\infty[$. Then $g \in L^1(w)$, and the function $h :]0,\infty[\to L^1(w), t \mapsto (\tau_t f)g(t)$ is continuous with compact support, hence integrable. Thus we have $f * g = \int_0^\infty h(t)\, dt$, which shows that $f * g$ belongs to the closure of the linear span of $\{\tau_t f : t > 0\}$. Since right translation is continuous in $L^1(w)$, the set of continuous functions with compact support in $]0,\infty[$ is dense in $L^1(w)$. The continuity of $*$ in $L^1(w)$ implies that we can replace g above by any $g \in L^1(w)$. □

Now we use the following result ([4], Theorem 6.7) to finish the proof of our theorem.

Theorem 2.4. *There is a function $f \in L^1(w)$ such that $f * L^1(w)$ is dense in $L^1(w)$.*

Actually, Theorem 6.7 in [4] stated even more, but this version suffices for our purpose. Take $f \in L^1(w)$ as in Theorem 2.4; then by Lemma 2.2 $C := \tilde{T}(f)$ satisfies all the conditions in Lemma 2.1, and thus A generates a C-regularized semigroup.

Remark 2.5. (a) The assumption $\bigcap \operatorname{Kern} T_t = \{0\}$ in our Main Result is also necessary for the existence of an operator $C \in L(E)$ such that $CT_t = T_tC$ defines a regularized semigroup.

(b) The regularized semigroup we constructed is of the form $(\tilde{T}(\tau_t f))$ for a certain $f \in L^1(\omega)$. Since (τ_t) is a C_0-semigroup in $L^1(\omega)$ and $\tilde{T}$ is continuous, we even get a norm-continuous regularized semigroup.

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