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NONSYMMETRIC OSSERMAN PSEUDO–RIEMANNIAN MANIFOLDS

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Abstract. Examples of Osserman pseudo–Riemannian manifolds with metric of any signature (p, q), p, $q > 1$ which are not locally symmetric are exhibited.

1. Introduction

A Riemannian manifold (M, g) is said to be an *Osserman space* if the eigenvalues of the Jacobi operator are constant on the unit sphere bundle. Osserman conjectured that such spaces must be flat or locally isometric to a rank–one symmetric space [11]. This was proved by Chi in many cases, and specially, for any n–dimensional Riemannian manifold with $n \neq 4k$, $k > 1$ [3]. (See [4], [5], [9] for related work and further references.)

The study of the Osserman problem for semi–Riemannian metrics was initiated in [8] (see also [7]). Due to the indefiniteness of the metric, the Jacobi operator is not necessarily diagonalizable, and a Lorentzian manifold is said to be *timelike* (resp., *spacelike*) *Osserman* if and only if the coefficients of the characteristic polynomial of the Jacobi operators R_X are independent of the timelike (resp., spacelike) unit vector X. This is equivalent to the constancy of the (possibly complex) eigenvalues of the corresponding Jacobi operators. It was proved in [8] that any timelike Osserman Lorentzian manifold is of constant curvature, and that the same result holds for 4–dimensional spacelike Osserman Lorentzian manifolds. Later on, Blažić, Bokan and Gilkey [2] proved that any spacelike Osserman Lorentzian manifold is of constant curvature, which shows that a Lorentzian manifold is Osserman if and only if it has constant sectional curvature.

The situation is different as concerns indefinite metrics of non–Lorentzian signature. Spaces of constant curvature are the simplest examples of spacelike and timelike Osserman manifolds of any signature. Indefinite Kähler manifolds of constant holomorphic sectional curvature and para–Kähler manifolds of constant paraholomorphic sectional curvature are Osserman spaces of signature $(2p, 2q)$ and (p, p) respectively (see [1], [6]). Note that there is no Riemannian counterpart of paracomplex space forms, and that all the above-mentioned examples are flat or rank–one symmetric. Moreover, it has been reported recently in [12] that there exist Osserman manifolds with metric of signature $(2, 2)$ which are symmetric of rank–two.

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The purpose of this note is to point out a further significant difference with respect to both the Riemannian and Lorentzian cases: *the existence of spacelike and timelike Osserman pseudo–Riemannian manifolds with metric of any signature* (p,q)*,* $p,\ q>1$*, which are not locally symmetric.*

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2. Four–dimensional examples

In this section we will construct a family of four–dimensional Osserman spaces with metric of signature $(+,+,-,-)$. In all the cases described below the characteristic polynomial of the Jacobi operator is $p_\lambda(R_X)=\lambda^4$. However the behaviour of the minimal polynomial differs, varying from $m_\lambda(R_X)=\lambda$ to $m_\lambda(R_X)=\lambda^3$.

Let $M=\mathbb{R}^4$ be the 4–dimensional Euclidean space with usual coordinates (x_1,x_2,x_3,x_4). Then

$$\begin{aligned} g_{(f_1,f_2)} &= x_3 f_1(x_1,x_2)dx_1\otimes dx_1 + x_4 f_2(x_1,x_2)dx_2\otimes dx_2 \\ &\quad + a[dx_1\otimes dx_2 + dx_2\otimes dx_1] \\ &\quad + b[dx_1\otimes dx_3 + dx_3\otimes dx_1 + dx_2\otimes dx_4 + dx_4\otimes dx_2] \end{aligned} \tag{1}$$

defines a pseudo–Riemannian metric on M of signature $(+,+,-,-)$ for any real constants a, b, with $b\neq 0$, and for any smooth real valued functions f_1, f_2.

Further, assume that f_1 and f_2 satisfy

$$\frac{\partial f_1}{\partial x_2}+\frac{\partial f_2}{\partial x_1}=0. \tag{2}$$

Then the Christoffel symbols of $g_{(f_1,f_2)}$,

$$\Gamma^k_{ij}=\frac{1}{2}\sum g^{kl}_{(f_1,f_2)}\left\{\frac{\partial g_{(f_1,f_2)_{il}}}{\partial x_j}+\frac{\partial g_{(f_1,f_2)_{jl}}}{\partial x_i}-\frac{\partial g_{(f_1,f_2)_{ij}}}{\partial x_l}\right\}$$

are given by

$$\begin{aligned} &\Gamma^1_{11}=-\frac{1}{2b}f_1, && \Gamma^3_{11}=\frac{1}{2b}x_3\frac{\partial f_1}{\partial x_1}+\frac{1}{2b^2}x_3 f_1^2,\\ &\Gamma^3_{12}=\frac{1}{2b}x_3\frac{\partial f_1}{\partial x_2}, && \Gamma^4_{11}=-\frac{1}{2b}x_3\frac{\partial f_1}{\partial x_2}+\frac{a}{2b^2}f_1,\\ &\Gamma^4_{12}=-\frac{1}{2b}x_4\frac{\partial f_1}{\partial x_2}, && \Gamma^3_{13}=\frac{1}{2b}f_1,\\ &\Gamma^2_{22}=-\frac{1}{2b}f_2, && \Gamma^3_{22}=\frac{1}{2b}x_4\frac{\partial f_1}{\partial x_2}+\frac{a}{2b^2}f_2,\\ &\Gamma^4_{24}=\frac{1}{2b}f_2, && \Gamma^4_{22}=\frac{1}{2b}x_4\frac{\partial f_2}{\partial x_2}+\frac{1}{2b^2}x_4 f_2^2, \end{aligned} \tag{3}$$

the others being zero.

Next, if we put $E_i = \frac{\partial}{\partial x_i}$ $(i = 1, 2, 3, 4)$, the coordinate vector fields, the only nonvanishing covariant derivatives $(\nabla_{E_i} E_j = \Gamma_{ij}^k E_k)$ are given by

$$
\begin{aligned}
\nabla_{E_1} E_1 &= -\frac{1}{2b} f_1 E_1 + \left[\frac{1}{2b} x_3 \frac{\partial f_1}{\partial x_1} + \frac{1}{2b^2} x_3 f_1^2\right] E_3 \\
&\quad + \left[-\frac{1}{2b} x_3 \frac{\partial f_1}{\partial x_2} + \frac{a}{2b^2} f_1\right] E_4, \\
\nabla_{E_1} E_2 &= \frac{1}{2b} x_3 \frac{\partial f_1}{\partial x_2} E_3 - \frac{1}{2b} x_4 \frac{\partial f_1}{\partial x_2} E_4, \\
\nabla_{E_1} E_3 &= \frac{1}{2b} f_1 E_3, \\
\nabla_{E_2} E_2 &= -\frac{1}{2b} f_2 E_2 + \left[\frac{1}{2b} x_4 \frac{\partial f_1}{\partial x_2} + \frac{a}{2b^2} f_2\right] E_3 \\
&\quad + \left[\frac{1}{2b} x_4 \frac{\partial f_2}{\partial x_2} + \frac{1}{2b^2} x_4 f_2^2\right] E_4, \\
\nabla_{E_2} E_4 &= \frac{1}{2b} f_2 E_4.
\end{aligned} \tag{4}
$$

From (4) it follows that the only nonvanishing components of the curvature tensor $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ are those determined by

(5)

$$
\begin{aligned}
R(E_1, E_2)E_3 &= -\frac{1}{2b} \frac{\partial f_1}{\partial x_2} E_3, \\
R(E_1, E_2)E_4 &= -\frac{1}{2b} \frac{\partial f_1}{\partial x_2} E_4, \\
R(E_1, E_3)E_1 &= \frac{1}{2b} \frac{\partial f_1}{\partial x_2} E_4, \\
R(E_1, E_3)E_2 &= -\frac{1}{2b} \frac{\partial f_1}{\partial x_2} E_3, \\
R(E_2, E_4)E_1 &= \frac{1}{2b} \frac{\partial f_1}{\partial x_2} E_4, \\
R(E_2, E_4)E_2 &= -\frac{1}{2b} \frac{\partial f_1}{\partial x_2} E_3, \\
R(E_1, E_2)E_1 &= \frac{1}{2b} \frac{\partial f_1}{\partial x_2} E_1 - \frac{1}{2b^2} x_3 f_1 \frac{\partial f_1}{\partial x_2} E_3 \\
&\quad + \frac{1}{4b^3} \left[2b^2 x_3 \frac{\partial^2 f_1}{\partial x_2^2} - 2b^2 x_4 \frac{\partial^2 f_1}{\partial x_1 \partial x_2} + b(x_3 f_2 - x_4 f_1 - 2a) \frac{\partial f_1}{\partial x_2} - a f_1 f_2\right] E_4, \\
R(E_1, E_2)E_2 &= \frac{1}{2b} \frac{\partial f_1}{\partial x_2} E_2 - \frac{1}{2b^2} x_4 f_2 \frac{\partial f_1}{\partial x_2} E_4 \\
&\quad - \frac{1}{4b^3} \left[2b^2 x_3 \frac{\partial^2 f_1}{\partial x_2^2} - 2b^2 x_4 \frac{\partial^2 f_1}{\partial x_1 \partial x_2} + b(x_3 f_2 - x_4 f_1 + 2a) \frac{\partial f_1}{\partial x_2} - a f_1 f_2\right] E_3.
\end{aligned}
$$

Next, if $X = \sum_{i=1}^4 \alpha_i E_i$ is a vector on M, the associated Jacobi operator $R_X = R(\cdot, X)X$ defines a self–adjoint endomorphism of the tangent space at each point of M. Moreover, the matrix associated to R_X with respect to the basis $\{E_i; i =$

$1,2,3,4\}$ is given by

$$R_X = \begin{pmatrix} A & 0 \\ B & {}^tA \end{pmatrix}, \tag{6}$$

where A is the 2×2 matrix

$$A = \frac{1}{2b}\left(\frac{\partial f_1}{\partial x_2}\right)\begin{pmatrix} \alpha_1\alpha_2 & -\alpha_1^2 \\ \alpha_2^2 & -\alpha_1\alpha_2 \end{pmatrix}$$

and B is given by the coefficients

$$\begin{aligned}
b_{11} &= \frac{1}{4b^3}\Bigg[-2b^2\alpha_2^2x_3\frac{\partial^2 f_1}{\partial x_2^2} + 2b^2\alpha_2^2x_4\frac{\partial^2 f_1}{\partial x_1\partial x_2} \\
&\quad -b(2\alpha_1\alpha_2x_3f_1 + \alpha_2^2x_3f_2 - \alpha_2^2x_4f_1 + 4b\alpha_2\alpha_3 + 2a\alpha_2^2)\frac{\partial f_1}{\partial x_2} \\
&\qquad\qquad + a\alpha_2^2f_1f_2\Bigg], \\
b_{12} &= \frac{1}{4b^3}\Bigg[2b^2\alpha_1\alpha_2x_3\frac{\partial^2 f_1}{\partial x_2^2} - 2b^2\alpha_1\alpha_2x_4\frac{\partial^2 f_1}{\partial x_1\partial x_2} \\
&\quad +b(2\alpha_1^2x_3f_1 + \alpha_1\alpha_2x_3f_2 - \alpha_1\alpha_2x_4f_1 + 2b\alpha_1\alpha_3 + 2a\alpha_1\alpha_2 \\
&\qquad\qquad -2b\alpha_2\alpha_4)\frac{\partial f_1}{\partial x_2} - a\alpha_1\alpha_2f_1f_2\Bigg], \\
b_{21} &= \frac{1}{4b^3}\Bigg[2b^2\alpha_1\alpha_2x_3\frac{\partial^2 f_1}{\partial x_2^2} - 2b^2\alpha_1\alpha_2x_4\frac{\partial^2 f_1}{\partial x_1\partial x_2} \\
&\quad -b(\alpha_1\alpha_2x_4f_1 + 2\alpha_2^2x_4f_2 - \alpha_1\alpha_2x_3f_2 - 2b\alpha_1\alpha_3 + 2a\alpha_1\alpha_2 \\
&\qquad\qquad +2b\alpha_2\alpha_4)\frac{\partial f_1}{\partial x_2} - a\alpha_1\alpha_2f_1f_2\Bigg], \\
b_{22} &= \frac{1}{4b^3}\Bigg[-2b^2\alpha_1^2x_3\frac{\partial^2 f_1}{\partial x_2^2} + 2b^2\alpha_1^2x_4\frac{\partial^2 f_1}{\partial x_1\partial x_2} \\
&\quad +b(2\alpha_1\alpha_2x_4f_2 + \alpha_1^2x_4f_1 - \alpha_1^2x_3f_2 + 4b\alpha_1\alpha_4 + 2a\alpha_1^2)\frac{\partial f_1}{\partial x_2} \\
&\qquad\qquad + a\alpha_1^2f_1f_2\Bigg].
\end{aligned} \tag{7}$$

Then we have the following

Theorem 1. *$(\mathbb{R}^4, g_{(f_1,f_2)})$ is a spacelike and timelike Osserman pseudo–Riemannian manifold with metric of signature $(+,+,-,-)$. Moreover, the characteristic polynomial of the Jacobi operators is $p_\lambda(R_X) = \lambda^4$ and the minimal polynomial $m_\lambda(R_X)$ of the Jacobi operators satisfies the following conditions*:

(i) *$m_\lambda(R_X) = \lambda^3$ at any point where $\dfrac{\partial f_1}{\partial x_2} \neq 0$.*

(ii) *At any point where $\dfrac{\partial f_1}{\partial x_2} = 0$ the function*

$$F(x_1,x_2,x_3,x_4) = 2b^2\frac{\partial}{\partial x_2}\left(x_3\frac{\partial f_1}{\partial x_2} - x_4\frac{\partial f_1}{\partial x_1}\right) - af_1f_2 \tag{8}$$

determines the minimal polynomial as follows:

(ii.a) $(\mathbb{R}^4, g_{(f_1,f_2)})$ *is of constant zero curvature* ($m_\lambda(R_X) = \lambda$) *at any point where* $\dfrac{\partial f_1}{\partial x_2}$ *and* F *vanish,*

(ii.b) *the minimal polynomial* $m_\lambda(R_X) = \lambda^2$ *at those points where* $\dfrac{\partial f_1}{\partial x_2} = 0$ *and* F *is different from zero.*

Proof. It follows from the expression of the Jacobi operator of any vector X that the corresponding characteristic polynomial satisfies $p_\lambda(R_X) = \det(R_X - \lambda I_4) = \lambda^4$, and thus all the eigenvalues vanish. This proves that $(\mathbb{R}^4, g_{(f_1,f_2)})$ is spacelike and timelike Osserman.

Moreover, after some calculations, from (6) one has

$$R_X^2 = \begin{pmatrix} 0 & 0 \\ BA + {}^tAB & 0 \end{pmatrix} = \frac{1}{4b^3} g(X,X)(\frac{\partial f_1}{\partial x_2})^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha_2^2 & \alpha_1\alpha_2 & 0 & 0 \\ \alpha_1\alpha_2 & -\alpha_1^2 & 0 & 0 \end{pmatrix}$$

and

$$R_X^3 = \begin{pmatrix} 0 & 0 \\ (BA + {}^tAB)A & 0 \end{pmatrix} = 0$$

for all vector fields X. Therefore the minimal polynomial is $m_\lambda(R_X) = \lambda^3$ whenever $\frac{\partial f_1}{\partial x_2} \neq 0$. At those points where $\frac{\partial f_1}{\partial x_2} = 0$ the minimal polynomial is $m_\lambda(R_X) = \lambda^2$ or $m_\lambda(R_X) = \lambda$ depending on whether the function F above vanishes, since in this case (6) becomes

$$R_X = \frac{1}{4b^3} F \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha_2^2 & \alpha_1\alpha_2 & 0 & 0 \\ \alpha_1\alpha_2 & -\alpha_1^2 & 0 & 0 \end{pmatrix}.$$

□

Remark 2. Condition (2) imposed on the functions f_1 and f_2 allowed us to simplify the calculations above. Proceeding in the same way as before, it can be shown that such a condition is indeed equivalent to $(M, g_{(f_1,f_2)})$ being Osserman.

Theorem 3. $(\mathbb{R}^4, g_{(f_1,f_2)})$ *is a locally symmetric space if and only if, in addition to (2), the functions* f_1 *and* f_2 *are solutions of the following*:

(i) $\dfrac{\partial^2 f_1}{\partial x_1 \partial x_2} + \dfrac{1}{2b} f_1 \dfrac{\partial f_1}{\partial x_2} = 0,$

(ii) $\dfrac{\partial^2 f_1}{\partial x_2^2} + \dfrac{1}{2b} f_2 \dfrac{\partial f_1}{\partial x_2} = 0,$

(iii) $\dfrac{1}{4b} a \left[3 f_1 \dfrac{\partial f_1}{\partial x_2} - f_2 \dfrac{\partial f_1}{\partial x_1} - \dfrac{1}{b} f_1^2 f_2 \right] - x_3 (\dfrac{\partial f_1}{\partial x_2})^2 = 0,$

(iv) $\dfrac{1}{4b} a \left[3 f_2 \dfrac{\partial f_1}{\partial x_2} + f_1 \dfrac{\partial f_2}{\partial x_2} + \dfrac{1}{b} f_1 f_2^2 \right] + x_4 (\dfrac{\partial f_1}{\partial x_2})^2 = 0.$

Proof. It follows from (4) and (5) after a long but straightforward calculation. □

Remark 4. Using the metrics (1) described above, one can construct examples of nonsymmetric Osserman pseudo–Riemannian manifolds of signature (+,+,−,−)

where the behaviour of the minimal and the characteristic polynomials of the Jacobi operators are as follows.

1. The minimal polynomial is $m_\lambda(R_X) = \lambda^3$. For the special choice of $f_1(x_1, x_2) = x_2$, $f_2(x_1, x_2) = -x_1$ and any value of a with $b \neq 0$.
2. The minimal polynomial is $m_\lambda(R_X) = \lambda^2$. For the special choice of $f_1(x_1, x_2) = k_1$, $f_2(x_1, x_2) = k_2$ and any value of a and b, all the constants k_1, k_2, a, b being different from zero.
3. If we take $f_1(x_1, x_2) = x_1$, $f_2(x_1, x_2) = k$, and a, b and k are different from zero, the resulting manifold has minimal polynomial $m_\lambda(R_X) = \lambda^2$ at any point with $x_1 \neq 0$ and $m_\lambda(R_X) = \lambda$ at those points with $x_1 = 0$.
4. For the special choice of $f_1(x_1, x_2) = x_1x_2$, $f_2(x_1, x_2) = -\frac{1}{2}x_1^2$ and any constants a, b with $b \neq 0$, the minimal polynomial corresponding to the resulting manifold is $m_\lambda(R_X) = \lambda^3$ at any point with $x_1 \neq 0$, $m_\lambda(R_X) = \lambda^2$ at those points with $x_1 = 0$, $x_4 \neq 0$ and $m_\lambda(R_X) = \lambda$ at points with $x_1 = x_4 = 0$.

   Note that on the open subset determined by $x_4 \neq 0$ the minimal polynomial varies from $m_\lambda(R_X) = \lambda^3$ to $m_\lambda(R_X) = \lambda^2$ according to whether x_1 is different from or equal to zero.
5. Define $f_1(x_1, x_2) = x_1^3x_2^3$ and $f_2(x_1, x_2) = -\frac{3}{4}x_1^4x_2^2$. Then, for any value of a and $b \neq 0$, $(\mathbb{R}^4, g_{(f_1,f_2)})$ has minimal polynomial $m_\lambda(R_X) = \lambda^3$ at those points with $x_1x_2 \neq 0$, and $m_\lambda(R_X) = \lambda$ when $x_1x_2 = 0$.

Remark 5. The previous examples show that in general the roots of the minimal polynomial and their multiplicities may change from point to point. However they are necessarily constant at each point, since the conditions discussed in Theorem 1 do not depend on any particular direction.

Remark 6. Let M be the open subset of $\mathbb{R}^4$ determined by $x_1 \neq 0$ and $x_2 \neq 0$ equipped with the metric (1) determined by the functions $f_1(x_1, x_2) = \frac{b}{x_1}$ and $f_2(x_1, x_2) = \frac{b}{x_2}$, $a \neq 0$. Then $(M, g_{(f_1,f_2)})$ is a four–dimensional Osserman pseudo–Riemannian manifold with $p_\lambda(R_X) = \lambda^4$ and minimal polynomial $m_\lambda(R_X) = \lambda^2$; this is a direct application of Theorem 1. Moreover, it follows from Theorem 3 that $(M, g_{(f_1,f_2)})$ is locally symmetric.

3. Higher–dimensional examples

Although the Osserman problem is not completely solved in Riemannian geometry, it was proved by Chi that any odd–dimensional Osserman Riemannian manifold is of constant curvature and hence locally symmetric (see [3] and [9] for an extension of that result under the weaker assumption of M being pointwise Osserman). This is no longer true for pseudo–Riemannian metrics of signature (p, q), p, $q > 1$. Next we will show the *existence of nonsymmetric Osserman pseudo–Riemannian manifolds of any signature* (p, q), p, $q > 1$.

Let (M_1, g_1) and (M_2, g_2) be pseudo–Riemannian manifolds of dimension n_1 and n_2 respectively. The product $M = M_1 \times M_2$ endowed with the product metric $g = g_1 \oplus g_2$ is a pseudo–Riemannian manifold. Moreover, for each vector $X = (X_1, X_2)$ on M the characteristic polynomial of the associated Jacobi operator satisfies

$$p_\lambda(R_X) = \det(R_X - \lambda I_{n_1+n_2}) = \det(R^{(1)}_{X_1} - \lambda I_{n_1})\det(R^{(2)}_{X_2} - \lambda I_{n_2}),$$

where $R^{(1)}$ and $R^{(2)}$ are the curvature tensors on M_1 and M_2 respectively.

Now, it follows that the product manifold M is spacelike and timelike Osserman if both factors M_1 and M_2 are spacelike and timelike Osserman with all eigenvalues of the Jacobi operator vanishing. (Note that in the Riemannian case a pointwise Osserman manifold is flat if it is locally reducible [9, Lemma 2.2].)

Therefore, if $(N, g_{(f_1,f_2)})$ denotes any one of the examples constructed in the previous section, the product manifold $\mathbb{R}^n_{(p-2,q-2)} \times N$ endowed with the product metric is a pseudo–Riemannian manifold of signature (p,q) which is Osserman but not locally symmetric. ($\mathbb{R}^n_{(p-2,q-2)}$ denotes here the Euclidean space with the usual indefinite metric of signature $(p-2, q-2)$, $p+q = n+4$.)

Remark 7. Note that, at each point of $\mathbb{R}^n_{(p-2,q-2)} \times N$ there exist unit vectors whose Jacobi operators have different minimal polynomial. Indeed, if $X = (X_1, 0)$, the associated Jacobi operator vanishes identically, and thus the minimal polynomial is $m_\lambda(R_X) = \lambda$. However, since R_X is nonzero for $X = (0, X_2)$, where N is chosen as in the previous section, its minimal polynomial is $m_\lambda(R_X) = \lambda^s$, $s = 1, 2, 3$, depending on the point and the metric $g_{(f_1,f_2)}$ considered on N. This shows that, at the same point, the Jacobi operator may be diagonalizable for some directions but nondiagonalizable for other directions.

Remark 8. For any locally symmetric Osserman space with $p_\lambda(R_X) = \lambda^4$ and $m_\lambda(R_X) = \lambda^2$ (cf. Remark 6 and [12]) the product manifolds constructed above are also locally symmetric. Note that, even in this case, at each point the minimal polynomial of the Jacobi operator has nonconstant roots.

Remark 9. Note that the characteristic and the minimal polynomials of the Jacobi operators play a different role, and the latter may have nonconstant roots even when the characteristic polynomial has constant roots (see examples 3, 4 and 5 in Remark 4). Moreover, note also that even in the case that both the characteristic and the minimal polynomial have constant roots, the manifold is not necessarily symmetric (examples 1 and 2 in Remark 4).

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