PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 9, September 1998, Pages 2737–2745
S 0002-9939(98)04703-0

VECTORS OF MINIMAL NORM

ANGELA SPALSBURY

(Communicated by Theodore W. Gamelin)

Abstract. We characterize the minimal vectors (or extremal vectors) for the operator of multiplication by $(z-1)$ on H^2. We then give an application to infinite systems of equations.

1. Main result

In this paper we study the extremal vectors of a concrete operator. Extremal vectors were recently introduced by Ansari and Enflo [1], and were used to show the existence of invariant subspaces for certain classes of operators.

Our setting is a Hilbert space H, and we consider a bounded, linear operator T on H that has dense range. Fix a vector $x_0 \in H$, fix $\epsilon > 0$, with $\epsilon < ||x_0||$, and fix $n \geq 1$. Since we are working in a Hilbert space, we can find a unique vector y^{ϵ}_{n,x_0} of minimal norm such that

$$||T^n y^{\epsilon}_{n,x_0} - x_0|| \leq \epsilon,$$

i.e.,

$$||y^{\epsilon}_{n,x_0}|| = \inf\{||y|| : ||T^n y - x_0|| \leq \epsilon\}.$$

We call the vector y^{ϵ}_{n,x_0} (or when the context is clear, y_n) the minimal vector for T^n.

Ansari and Enflo used these minimal vectors to construct invariant subspaces for a large class of operators containing both the compact operators and the normal operators (but larger than their union). They showed that if T is compact then there is a subsequence of $(T^n y_n)$ that converges weakly to a vector that is noncyclic, and if T is normal then $(T^n y_n)$ converges in norm to a vector that is noncyclic. (In fact in both cases the vector is noncyclic for all operators commuting with T.) Thus the orbit of this vector under the operator T is a (hyper) invariant subspace for T.

We study an operator that is neither compact nor normal. Let T be the operator of "multiplication by $(z-1)$" on H^2, i.e.,

$$(Tf)(z) = (z-1)f(z)$$

for any $f \in H^2$. Note that T has dense range but is not invertible. We begin by characterizing the minimal vectors of T^n.

Received by the editors February 11, 1997.
1991 *Mathematics Subject Classification.* Primary 47A15, 47B38.
Key words and phrases. Invariant subspaces, extremal vectors.

Theorem 1. *Let $T : H^2 \to H^2$ be the operator of multiplication by $(z-1)$. Fix $\epsilon > 0$, $n \geq 1$, and set $x_0 = 1 \in H^2$. Let y_n be the unique function in H^2 of minimal norm such that*

$$||T^n y_n(z) - 1|| = ||(z-1)^n y_n(z) - 1|| \leq \epsilon.$$

Then for each $i = 1, \dots, n$, there exist numbers a_i, $|a_i| > 1$, and there exists a constant A such that

$$y_n(z) = \frac{A}{(z-a_1)\cdots(z-a_n)}.$$

In proving this theorem we need a geometric result from Ansari and Enflo [1, Theorem 1].

Theorem A (Ansari, Enflo)**.** *Let $T \in L(H)$, with dense range. Fix $x_0 \in H$, fix $\epsilon > 0$, with $\epsilon < ||x_0||$, and fix $n \geq 1$. Let $y_n = y^{\epsilon}_{n,x_0}$ be the minimal vector for T^n. Then there exists a $\delta < 0$ such that*

$$y_n = (\delta(T^*)^n T^n - I)^{-1} \delta (T^*)^n (x_0).$$

Moreover, the following orthogonality relation holds:

if $z \in H$ such that $z \perp y_n$, then $T^n z \perp T^n y_n - x_0$.

We now give a proof of Theorem 1.

Proof. By Theorem A there exists a $\delta < 0$ such that

$$y_n = (\delta(T^*)^n T^n - I)^{-1} \delta (T^*)^n (1). \tag{1.1}$$

We need to introduce some notation. Let $\mathcal{P}_+$ denote the operator of taking the "analytical part" of a power series, i.e.,

$$\mathcal{P}_+ \left[\sum_{n=-\infty}^{\infty} a_n z^n \right] = \sum_{n=0}^{\infty} a_n z^n.$$

Since T is multiplication by $(z-1)$, T^* is

$$\mathcal{P}_+ \left[\text{multiplication by} \left(\frac{1}{z} - 1 \right) \right],$$

i.e., for any $g \in H^2$, and any $z \in \mathbf{C}$,

$$(T^* g)(z) = \mathcal{P}_+ \left[\left(\frac{1}{z} - 1 \right) g(z) \right].$$

Without loss of generality, we may assume that

$$(T^*)^n (1) = 1.$$

Therefore by (1.1)

$$\frac{1}{\delta} y_n = (\delta(T^*)^n T^n - I)^{-1} (1),$$

so

$$(\delta(T^*)^n T^n - I) \left(\frac{1}{\delta} y_n \right) = 1.$$

In other words, we have that for any $z \in \mathbf{C}$,

$$(T^*)^n T^n y_n(z) - \frac{1}{\delta} y_n(z) = 1. \tag{1.2}$$

Using our notation it is easy to check that for any $g \in H^2$, and $z \in \mathbf{C}$,

$$((T^*)^n T^n g)(z) = \mathcal{P}_+ \left[\left(\left(\frac{1}{z} - 1 \right) (z-1) \right)^n g(z) \right] = \mathcal{P}_+ \left[\left(2 - \frac{1}{z} - z \right)^n g(z) \right].$$

Now fix $z \in \mathbf{C}$. Then by (1.2)

$$\mathcal{P}_+ \left[\left(2 - \frac{1}{z} - z \right)^n y_n(z) \right] - \frac{1}{\delta} y_n(z) = 1.$$

Hence there exists a polynomial Q_n of degree less than or equal to n such that

$$\left(2 - \frac{1}{z} - z \right)^n y_n(z) + Q_n \left(\frac{1}{z} \right) - \frac{1}{\delta} y_n(z) = 1.$$

Therefore

$$y_n(z) \left(\left(2 - \frac{1}{z} - z \right)^n - \frac{1}{\delta} \right) = 1 - Q_n \left(\frac{1}{z} \right),$$

or

$$\frac{1}{\delta} y_n(z) \left(\delta \left(2 - \frac{1}{z} - z \right)^n - 1 \right) = 1 - Q_n \left(\frac{1}{z} \right).$$

We have then that

$$\frac{1}{\delta} y_n(z) = \frac{1 - Q_n \left(\frac{1}{z} \right)}{\delta \left(2 - \frac{1}{z} - z \right)^n - 1}.$$

Write $Q_n(z) = b_0 + b_1 z + \cdots + b_n z^n$. Then

$$y_n(z) = \frac{\delta \left(1 - b_0 - \frac{b_1}{z} - \cdots - \frac{b_n}{z^n} \right)}{\delta (2 - \frac{1}{z} - z)^n - 1} = \frac{\delta \left(z^n (1 - b_0) - b_1 z^{n-1} - \cdots - b_n \right)}{z^n \left(\delta \left(2 - \frac{1}{z} - z \right)^n - 1 \right)}. \tag{1.3}$$

Set $p_n(z) = \delta(z^n(1-b_0) - b_1 z^{n-1} - \cdots - b_n)$, and $s_n(z) = \delta(2 - \frac{1}{z} - z)^n - 1$. Then

$$y_n(z) = \frac{p_n(z)}{z^n s_n(z)}.$$

Now $y_n \in H^2(U)$, so y_n can have no poles inside the unit disk (or even *on* the unit circle). Clearly z_0 is a pole of $y_n(z)$ only if

$$z_0 = 0 \ \text{ or } \ s_n(z_0) = 0.$$

It is easy to see that $s_n(z)$ has exactly $2n$ zeroes. Also for any z

$$s_n \left(\frac{1}{z} \right) = s_n(z).$$

Thus if z_0 is a zero of s_n then so is $\frac{1}{z_0}$. Therefore the same number of zeroes of $s_n(z)$ lie inside the disk as lie outside the unit disk.

Since $s_n(z)$ has at most n zeroes outside the unit disk, it follows that $y_n(z)$ can have at most n poles.

On the other hand, y_n has at least n poles. To see this just note that (1.3) tells us that

$$\lim_{z\to\infty} y_n(z) = \lim_{z\to\infty} \frac{\text{constant}}{z^n}.$$

This also shows that y_n must be a rational function.

Therefore $y_n(z)$ is a rational function with exactly n poles. Hence there exist numbers a_i, $|a_i| > 1$ (for each $i = 1, \dots, n$), and there exists a constant A such that

$$y_n(z) = \frac{A}{(z-a_1)(z-a_2)\cdots(z-a_n)}. \quad \square$$

We would like to thank A. Volberg who introduced the idea of using "$\mathcal{P}_+$" and who showed that y_n has at least n poles.

Now that we have characterized the minimal vectors y_n for the operator T of multiplication by $(z-1)$ on H^2 we consider the question of convergence of $(T^n y_n)$. From the proof of Theorem 1 we see that y_n is of the form

$$y_n(z) = \frac{A}{(z-a_1)(z-a_2)\cdots(z-a_n)},$$

where $a_1, a_2, \dots, a_n$ are the n zeros of the function

$$s_n(z) = \delta\left(2 - \frac{1}{z} - z\right)^n - 1$$

found outside the unit disk, i.e., $a_1, a_2, \dots, a_n$ are the n solutions to the quadratic equation

$$z^2 + \left(\left(\frac{1}{\delta}\right)^n - 2\right) z + 1 = 0$$

found outside the unit disk. We are currently working to show that $(T^n y_n)$ is norm convergent. By arguments similar to those used in [1] the limit would be a noncylic vector for T. By Beurling's theory of the invariant subspaces for the shift operator we would then have that $(T^n y_n)$ converges to a nonsingular inner function.

2. An application to infinite systems of equations

In this section we look at the relationship between the Taylor coefficients of the minimal vectors of our operator T (multiplication by $(z-1)$ on H^2). It is interesting to note that the Taylor coefficients satisfy certain recurrence relations, or equivalently they are solutions to certain infinite systems of equations. In Theorem 4 we characterize the solutions to such systems of equations using Theorem 1.

Proposition 2. *Let $T : H^2 \to H^2$ be the operator of multiplication by $(z-1)$. Fix $\epsilon > 0$, and let y $(= y_1)$ be the unique function in H^2 of minimal norm such that*

$$||Ty - 1|| = ||(z-1)y(z) - 1|| \le \epsilon.$$

If we express y as a power series

$$\sum_{n=0}^{\infty} a_n z^n,$$

then the Taylor coefficients must satisfy then following recursive relation:

$$\begin{aligned} a_0 &= \delta(2a_0 + 1 - a_1), \\ a_1 &= \delta(-a_0 + 2a_1 - a_2), \\ a_2 &= \delta(-a_1 + 2a_2 - a_3), \\ &\vdots \\ a_k &= (-a_{k-1} + 2a_k - a_{k+1}), \end{aligned}$$

or equivalently,

$$\begin{aligned} a_1 &= \left(\frac{2\delta - 1}{\delta}\right) a_0 + 1, \\ a_2 &= \left(\frac{2\delta - 1}{\delta}\right) a_1 - a_0, \\ &\vdots \\ a_k &= \left(\frac{2\delta - 1}{\delta}\right) a_k - a_{k-1}, \end{aligned}$$

for all $k \geq 2$. *(Here* δ *is as in Theorem A.)*

Proof. If we write

$$y(z) = \sum_{i=0}^{\infty} a_i z^i,$$

then we see that

$$\begin{aligned} (z-1)y(z) &= z\sum_{i=0}^{\infty} a_i z^i - \sum_{i=0}^{\infty} a_i z^i \\ &= \sum_{i=0}^{\infty} a_i z^{i+1} - \sum_{i=0}^{\infty} a_i z^i \\ &= \sum_{j=1}^{\infty} a_{j-1} z^j - \sum_{i=0}^{\infty} a_i z^i \\ &= -a_0 + \sum_{j=1}^{\infty} (a_{j-1} - a_j) z^j, \end{aligned}$$

and therefore

$$(z-1)y(z) - 1 = (-a_0 - 1) + \sum_{j=1}^{\infty} (a_{j-1} - a_j) z^j.$$

Now suppose that $g \in H^2$ with $g \perp y$. By Theorem A we know that

$$Tg \perp Ty - 1.$$

Therefore if we write

$$g(z) = \sum_{i=0}^{\infty} b_i z^i,$$

then

$$(z-1)g(z) \perp (z-1)y(z) - 1,$$

and so

$$-b_0(\overline{-a_0 - 1}) + \sum_{j=1}^{\infty}(b_{j-1} - b_j)(\overline{a_{j-1} - a_j}) = 0.$$

Therefore

$$\begin{aligned}
0 &= -b_0(\overline{-a_0 - 1}) + \sum_{j=1}^{\infty} b_{j-1}\overline{a_{j-1}} + \sum_{j=1}^{\infty} b_j \overline{a_j} - \sum_{j=1}^{\infty} b_j \overline{a_{j-1}} - \sum_{j=1}^{\infty} b_{j-1}\overline{a_j} \\
&= -b_0(\overline{-a_0 - 1}) + \sum_{n=0}^{\infty} b_n\overline{a_n} + \sum_{n=1}^{\infty} b_n\overline{a_n} - \sum_{n=1}^{\infty} b_n\overline{a_{n-1}} - \sum_{n=0}^{\infty} b_n\overline{a_{n+1}} \\
&= b_0(\overline{a_0 + 1}) + b_0\overline{a_0} + \sum_{n=1}^{\infty} b_n\overline{a_n} + \sum_{n=1}^{\infty} b_n\overline{a_n} - \sum_{n=1}^{\infty} b_n\overline{a_{n-1}} - \sum_{n=1}^{\infty} b_n\overline{a_{n+1}} - b_0\overline{a_1} \\
&= b_0(2\overline{a_0} + 1 - \overline{a_1}) + \sum_{n=1}^{\infty} b_n(-\overline{a_{n-1}} + 2\overline{a_n} - \overline{a_{n+1}}).
\end{aligned}$$

Set

$$h(z) = (2a_0 + 1 - a_1) + \sum_{n=1}^{\infty}(-a_{n-1} + 2a_n - a_{n+1})z^n.$$

Then $h \in H^2$, since

$$\begin{aligned}
||h(z)||^2 &= |2a_0 + 1 - a_1|^2 + \sum_{n=1}^{\infty} |-a_{n-1} + 2a_n - a_{n+1}|^2 \\
&\leq |2a_0 + 1 - a_1|^2 + \sum_{n=1}^{\infty}(|a_{n-1}|^2 + 2|a_n|^2 + |a_{n+1}|^2) \\
&\leq |2a_0 + 1 - a_1|^2 + 4\sum_{n=1}^{\infty} |a_{n-1}|^2 \\
&= |2a_0 + 1 - a_1|^2 + 4||f||^2 < \infty.
\end{aligned}$$

Now h has the property that

if $g \in H^2$ with $g \perp y$, then $g \perp h$.

Therefore there exists a number α such that $y = \alpha \cdot h$. In fact, more is true:

$$\alpha = \delta,$$

where δ is found as in Theorem A. (This is apparent on examining the proof of Theorem A.)

Thus

$$y(z) = \sum_{n=0}^{\infty} a_n z^n = \delta(2a_0 + 1 - a_1) + \delta\sum_{n=1}^{\infty}(-a_{n-1} + 2a_n - a_{n+1})z^n.$$

Equating coefficients, we have

$$\begin{aligned} a_0 &= \delta(2a_0 + 1 - a_1), \\ a_1 &= \delta(-a_0 + 2a_1 - a_2), \\ a_2 &= \delta(-a_1 + 2a_2 - a_3), \\ &\vdots \\ a_n &= \delta(-a_{n-1} + 2a_n - a_{n+1}), \end{aligned}$$

or

$$\begin{aligned} a_1 &= \left(\frac{2\delta - 1}{\delta}\right) a_0 + 1, \\ a_2 &= \left(\frac{2\delta - 1}{\delta}\right) a_1 - a_0, \\ &\vdots \\ a_n &= \left(\frac{2\delta - 1}{\delta}\right) a_n - a_{n-1}. \end{aligned}$$

□

We now characterize the solutions to this infinite system of equations.

Theorem 3. *Let $(a_0, a_1, \dots) \in \ell^2$. If $(a_0, a_1, \dots)$ are solutions to the infinite system of equations,*

$$\begin{aligned} a_0 &= \delta(2a_0 + 1 - a_1), \\ a_1 &= \delta(-a_0 + 2a_1 - a_2), \\ a_2 &= \delta(-a_1 + 2a_2 - a_3), \\ &\vdots \\ a_n &= \delta(-a_{n-1} + 2a_n - a_{n+1}) \end{aligned}$$

(δ any nonzero real number), then $(a_0, a_1, \dots)$ are the Taylor coefficients of some H^2 function g of the form

$$g(z) = \frac{B}{(z - b)},$$

where b is the solution to the quadratic equation

$$z^2 + \left(\left(\frac{1}{\delta}\right)^n - 2\right) z + 1 = 0$$

found outside the unit disk. The converse is also true provided δ is as in Theorem A.

Proof. Suppose that $(a_0, a_1, \dots)$ are solutions to the infinite system of equations. Define a function $g \in H^2$ by

$$g(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Then it is easy to verify that (1.2) holds for any δ, i.e., for any δ,

$$(T^*)^n T^n g(z) - \frac{1}{\delta} g(z) = 1,$$

for all $z \in \{z : |z| < 1\}$. (Here T is the operator of multiplication by $(z-1)$ on H^2.) Then, exactly as in the proof of Theorem 1, we can again conclude that g is of the form

$$g(z) = \frac{B}{(z-b)}.$$

The converse follows directly from Theorem 1 and Proposition 2. □

We conclude by stating an analogous result for T^n.

Theorem 4. *Let $(a_0, a_1, a_2, \dots) \in l_2$. Fix $n \geq 1$. If $(a_0, a_1, a_2, \dots)$ are solutions to the infinite system of equations,*

$$a_0 = \delta\left(\binom{2n}{n} a_0 - \binom{2n}{n+1} a_1 + \cdots + (-1)^n \binom{2n}{2n} a_n + (-1)^{n+1}\right),$$

$$a_1 = \delta\left(-\binom{2n}{n-1} a_0 + \binom{2n}{n} a_1 + \cdots + (-1)^{n+1} \binom{2n}{2n-1} a_n + (-1)^n \binom{2n}{2n} a_{n+1}\right),$$

$$\vdots$$

$$a_{n-1} = \delta\left((-1)^{n+1}\binom{2n}{1} a_0 + (-1)^n \binom{2n}{2} a_1 + (-1)^{n+1}\binom{2n}{3} a_2 + \cdots + (-1)^n \binom{2n}{2n} a_{2n-1}\right),$$

$$a_j = \delta\left((-1)^n \binom{2n}{0} a_{j-n} + (-1)^{n+1}\binom{2n}{1} a_{j-(n+1)} + \cdots + (-1)^n \binom{2n}{2n} a_{j+n}\right), \quad \textit{for all } j \geq n$$

(δ any arbitrary nonzero real number), then $(a_0, a_1, a_2, \dots,)$ are the Taylor coefficients of an H^2 function g of the form

$$g(z) = \frac{B}{(z-b_1)\cdots(z-b_n)},$$

where $b_1, b_2, \dots, b_n$ are the n solutions to the quadratic equation

$$z^2 + \left(\left(\frac{1}{\delta}\right)^n - 2\right) z + 1 = 0$$

found outside the unit disk. The converse is also true provided δ is as in Theorem A.

Remark 5. Using Proposition 2, it is possible to show that the minimal vector y_1 for T is of the form

$$y_1(z) = \frac{A}{(z-a)}.$$

However, it is not clear how to generalize this technique to show that the minimal vector y_n for T^n $(n > 1)$ is of the form

$$y_n(z) = \frac{A}{(z-a_1)(z-a_2)\cdots(z-a_n)}.$$

References

1. S. Ansari and P. Enflo. Extremal vectors and invariant subspaces. *Transactions of the American Mathematical Society.* To appear. CMP 96:17

Department of Mathematics, University of the Witwatersrand, PO Box 3, Wits 2050, South Africa

Current address: Department of Mathematics and Computer Science, Kent State University, Kent, Ohio 44242

E-mail address: angie@mcs.kent.edu