UNIQUENESS IN THE CAUCHY PROBLEMS FOR HIGHER ORDER ELLIPTIC DIFFERENTIAL OPERATORS

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(Communicated by J. Marshall Ash)

Abstract. In this note, we study the uniqueness in Cauchy problems for a class of higher order elliptic differential operators with Lipschitz coefficients. In particular, we prove the uniqueness under assuming the potentials being $L^{r_j}_{\text{loc}}$ with certain correct numbers $r_j$’s.

Notation. Let $\Omega$ be a domain in $\mathbb{R}^d$. Suppose $P(x, D) = \sum_{|\alpha| = m} a_\alpha(x) D^\alpha$ is a differential operator of degree $m$ with real functions $a_\alpha(x)$ on $\Omega$. We denote by $P = P(x, \cdot + ik)$ the symbol of $P(x, D)$ and by $N^P(x, k)$ the zero set of $P(x, \cdot + ik)$ for any $(x, k) \in \Omega \times \mathbb{R}^d$. Let’s define a subset in $\Omega \times S^{d-1}$:

$$\Sigma_P = \{(x, k) \in \Omega \times S^{d-1} : \sum \frac{d}{dz_j} P(x, \xi + ik) \cdot k_j \neq 0, \det \text{Hess}_C P(x, \xi + ik) \neq 0 \forall \xi \in N^P_{(x, k)} \}$$

where $\text{Hess}_C P = \left( \frac{d^2 P}{dz_j dz_l} \right)$ is the complex Hessian matrix of $P$, and $z = \xi + ik \in C^d$.

If $u$ is a function on $\Omega$, we define its normal support $N(\text{supp} u)$ as a subset of $\Omega \times S^{d-1}$. Say $(x, k) \in N(\text{supp} u)$ if there is a neighborhood $V$ of $x$ such that $\psi(y) \leq \psi(x)$ for all $y \in V \cap \text{supp} u$ and $d\psi(x) = \pm k$, where $\psi$ is some smooth function.

Let $s = \frac{2(d+1)}{d+3}$ be the restriction number and $s'$ be its conjugate number. We let $W^{m, 2}_{\text{loc}}$ be the Sobolev space of functions whose derivatives up to order $m$ belong to $L^2$. We have the following theorem.

**Theorem.** Suppose $P(x, D)$ is an elliptic differential operator with real Lipschitz functions $a_\alpha$ as coefficients on $\Omega$ and is of order $m < \frac{d}{2}$. If a function $u \in W^{m, 2}_{\text{loc}}(\Omega)$ satisfies

$$|Pu(x)| \leq \sum_{0 < \mu \leq m} V_\mu |\nabla^{m-\mu} u|$$

with $V_\mu \in L^\frac{d}{2}_{\text{loc}}(\Omega)$, then $N(\text{supp} u) \subset \Sigma_P$.

Remarks. (1) Actually we will prove that $N(\text{supp} u) \subset \Lambda_P'$ where $\Lambda_P$ is the set of $(x, k) \in \Omega \times S^{d-1}$ such that $N^P_{(x, k)}$ is locally contained in a smooth hypersurface with nonzero Gaussian curvature, which is smaller than $\Sigma_P$. In other words, we

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may replace the assumptions in $\Sigma_P$ by directly assuming some curvature condition for $N_{x,k}^P$. $\Sigma_P$ is a natural condition and is easy to verify. But the proof of $\Sigma_P \subset \Lambda_P$ is nontrivial which is essentially shown in Lemma 1 below. For more details, see [3].

(2) When coefficients are constants, this theorem was proved by the author in [3]. When $P(x,D)$ is hyperbolic, under some other curvature assumption for $N_{x,k}^P$, Sogge proves the same result in the case where $V_\mu = 0$ for all $\mu \leq m - 1$; see [2]. In general, if we don’t care about the optimal condition for the potentials, this is an old theorem by Calderon. See [1], [4].

Calderon’s theorem is actually equivalent to the following uniqueness theorem in the Cauchy problem.

Theorem 1. Suppose $P(x,D)$ is an elliptic differential operator with real Lipschitz functions $a_\alpha$ as coefficients on a domain $\Omega$ which contains $\mathbb{R}^d \setminus B(-e_d, \frac{1}{2})$ and satisfies the conditions

$$\frac{dP}{dz_d}(0, \xi + ie_d) \neq 0,$$

$$\det \text{Hess}_c P(0, \xi + ie_d) \neq 0$$

for all $\xi \in N_{(0,e_d)}^P$, where $\text{Hess}_c P$ is the complex Hessian matrix of $P$. Then for any function $u \in W^{m,2}_{loc}(\Omega)$ satisfying (1) for some $V_\mu \in L^\frac{2}{m}(\Omega)$, $u$ vanishes in a neighborhood of 0 if $u$ vanishes outside $B(-e_d, 1)$.

Let’s first prove our Theorem by assuming Theorem 1.

Proof of the Theorem. Let $(x^0, k^0) \in N(\text{suppu})$. Suppose $(x^0, k^0) \in \Sigma_P$. By the definition of $N(\text{suppu})$, there is a little ball $B$ such that $x^0 \in \partial B$ and $u = 0$ in $B$. Then there is a map $F$ which is the composition of translation, rotation, dilation and Kelvin transformation with respect to $x^0$ and $B$ such that $F(x^0) = 0$ and $F(k^0) = e_d$. Moreover $u \circ F^{-1} = 0$ outside $B(-e_d, 1)$ and $u \circ F^{-1}$ is defined on a domain $\Omega$ which contains $\mathbb{R}^d \setminus B(-e_d, \frac{1}{2})$. Let $v(y) = u \circ F^{-1}(y)$. Then $v$ satisfies the following differential inequality by (1):

$$|Q(y, D)v(y)| \leq \sum_{0 < \mu \leq m} V_\mu^1(y) |\nabla^{m-\mu} v(y)|$$

where $Q(y, \eta) = P(F^{-1}(y), (\frac{1}{2}DF^{-1}(y))^{-1} \eta)$ and $V_\mu^1(y) = V_\mu \circ F^{-1}(y)$ plus some bounded functions. So one may check that $(0, e_d) \in \Sigma_Q$ which means the assumptions in Theorem 1 are satisfied. So applying Theorem 1 to $Q$ and $v$, we have $v = 0$ in a neighborhood of 0. Pull back $v$ to $u$ by $F$. We have $u = 0$ in a neighborhood of $x^0$. This is a contradiction with $x^0 \in \text{suppu}$.

In order to prove Theorem 1, we need several lemmas. Let’s first study the differential operator with real constants coefficients. We denote by $A$ the vector $(a_\alpha)_{|\alpha|=m} \in \mathbb{R}^M$ for some number $M$ determined by $m$ and $P_A(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$, and denote by $N_{(A,k)}$ the zero set of $P_A(\cdot + ik)$. We are always interested in the case that $P_A$ is elliptic. Let’s introduce some functions as follows:

$$S(A, \xi, k) = \sum_j \frac{dP_A}{dz_j}(\xi + ik)k_j,$$
\[ H(A, \xi, k) = \left| \det \left( \frac{d^2 P_A}{dz_j dz_l} (\xi + ik) \right) \right|, \]
\[ L(A, \xi, k) = \sum_{(j, t)} \left| \det \left( \frac{\partial^2 P_A}{\partial^2 z_j} (\xi + ik) \right) \right|. \]

We notice that the assumption in Theorem 1 says that when \( A = (a_\alpha(0)) \) and \( k = e_d \), the first two of the above functions are positive on \( N_{(0, e_d)}^P \). By the Cauchy-Riemann equation and the transversality theorem, we proved that \( L(A, \xi, k) \) is also positive on \( N_{(0, e_d)}^P \). See [3].

**Lemma 1.** Suppose for some \( A \in \mathbb{R}^M \) and \( k_0 \in S^{d-1} \) the above three functions are positive on \( N_{(A, k_0)} \). Then there are some positive numbers \( c_0, b, \epsilon, \) an integer \( J \), a neighborhood \( K \) of \( k_0 \) in \( S^{d-1} \) and finite small balls \( \{ B_j(\epsilon) \}_{j=1}^J \) such that for any \( B \in \mathbb{R}^M \) with \( \| B - A \| \leq b \) and any \( k \in K \) there are finite hypersurfaces \( \{ S_j \}_{j=1}^J \) for which the following properties hold:

1. \( N(B, k) \cap B_j(\epsilon) \subset S_j \cap B_j(\epsilon) \);
2. \( N(B, k) \subset \bigcup_{j=1}^J B_j(\frac{\epsilon}{2}) \);
3. \( S_j \cap B_j(\epsilon) \) is a piece of hypersurface with nonzero Gaussian curvature which is bounded by \( c_0 \) from below for all \( j \).

Moreover for each such \( (B, k) \), there is a diffeomorphism \( G(B, k) : \bigcup_{j=1}^J B_j(\epsilon) \to D(\epsilon) \times N(B, k) \) such that \( |G'(B, k)| \) is bounded by \( c_0 \) from below.

**Proof.** We will prove this lemma in several steps as follows.

**Step 1:** There are positive constants \( c, b, \epsilon, J \) and a neighborhood \( K \) of \( k_0 \) in \( S^{d-1} \) and an \( \epsilon \) neighborhood \( U \) of \( N_{(A, k_0)} \) such that for any \( B \in \mathbb{R}^M \) with \( \| B - A \| \leq b \) and any \( k \in K \),

\[ N(B, k) \subset \frac{1}{2} U, \quad \min (S(B, \xi, k), H(B, \xi, k), L(B, \xi, k)) \geq c \]

for all \( \xi \in U \).

**Proof of Step 1.** Since \( P_A \) is an elliptic polynomial, the set \( N_{(A, k_0)} \) is a compact boundaryless submanifold of codim 2 by assumption. Functions \( S, H \) and \( L \) are continuous in three variables \( A, \xi \) and \( k \). So by assumption and compact argument and the \( \epsilon \) neighborhood theorem, Step 1 is proved.

**Step 2:** There are \( \epsilon \) and finite small balls such that for any \( B \) and \( k \) as in Step 1 there are finite hypersurfaces as in Lemma 1. (1), (2) and (3) of Lemma 1 hold.

**Proof of Step 2.** Since \( S(A, \xi, k_0) \) and \( H(A, \xi, k_0) \) are positive functions, Proposition 0.1 of [3] implies that there are finite \( \epsilon \) balls \( \{ B_j(\epsilon) \}_{j=1}^J \) with centers \( \{ \xi_j \} \subset N_{(A, k_0)} \) such that

\[ N_{(A, k_0)} \subset \bigcup_{j=1}^J B_j(\frac{\xi_j}{4}). \]

Moreover there are also finite real numbers \( t_j \) and vectors \( \{ x_j \}_{j=1}^J \subset \mathbb{R}^d \) such that if we define functions \( f_j(A, \xi, k_0) \) by

\[ \text{re} P_A(\xi + ik_0) + t_j \text{im} P_A(\xi + ik) + \langle x_j, (\xi - \xi_j) \rangle (t_j \text{re} P_A(\xi + ik_0) - \text{im} P_A(\xi + ik)) \]
\[ - \langle x_j, (\xi - \xi_j) \rangle (\text{re} P_A(\xi + ik_0) + t_j \text{im} P_A(\xi + ik)) \]
\[ \times \frac{\langle t_j \nabla \text{re} P_A(\xi_j + ik_0) - \nabla \text{im} P_A(\xi_j + ik_0), \nabla \text{re} P_A(\xi_j + ik_0) + t_j \nabla \text{im} P_A(\xi_j + ik_0) \rangle}{\langle \nabla \text{re} P_A(\xi_j + ik_0) + t_j \nabla \text{im} P_A(\xi_j + ik_0), \nabla \text{re} P_A(\xi_j + ik_0) + t_j \nabla \text{im} P_A(\xi_j + ik_0) \rangle}, \]
then $f_j(A, k_0)\circ (0)$ is a hypersurface with Gaussian curvature bounded by $2c_0$ from below in $B_j(\epsilon)$ for some constant $c_0$ which depends only on $A$ and $k_0$. Now let’s fix a $B$ and a $k$ as in Step 1. When $b$ and $K$ are small enough, $N_{(B, k)} \subset \bigcup_{j=1}^t B_j(\frac{\epsilon}{2})$. Choose $\eta_j \in B_j(\frac{\epsilon}{2})$ with $P_B(\eta_j + ik) = 0$. Replace $A$, $k_0$ and $\xi_j$ by $B$, $k$ and $\eta_j$ in the function $f_j$ for each $j$. Then once again when $b$ and $K$ are small enough, $f_j(B, k)\circ (0)\cap B_j(\epsilon)$ is a piece of hypersurface with Gaussian curvature bounded by $c_0$ from below for all $j$. This proves Step 2 with $S_j = f_j(B, k)\circ (0)$.

Step 3: The last part in Lemma 1 holds when $\epsilon$ is small and $J$ is large.

Proof of Step 3. By the $\epsilon$ neighborhood theorem, when $\epsilon$ is small and $J$ is large, there is a diffeomorphism $G_{(A, k_0)} : \bigcup B_j(2\epsilon) \to D(2\epsilon) \times N_{(A, k_0)}$ where $D(2\epsilon)$ is a 2-dimensional ball of radius $2\epsilon$. In fact $G_{(A, k_0)}$ may be defined by extending $N_{(B, k)}$ along the normal directions, which we may choose as $\nabla \text{re} P_A(\xi + ik_0) + t_j \nabla \text{im} P_A(\xi + ik_0)$ and $t_j \nabla \text{re} P_A(\xi + ik_0) - \nabla \text{im} P_A(\xi + ik_0) - v$ where $v$ is the projection of $t_j \nabla \text{re} P_A(\xi + ik_0) - \nabla \text{im} P_A(\xi + ik_0)$ in the $\nabla \text{re} P_A(\xi + ik_0) + t_j \nabla \text{im} P_A(\xi + ik_0)$ direction in each $B_j(\frac{\epsilon}{2})$. Since $P_B(\xi + ik)$ are smooth in $(B, \xi, k)$ and $L(B, \xi, k) \geq c$ by the assumption, for each $B$ closing $A$ and each $k$ closing $k_0$, there is a diffeomorphism $G_{(B, k)} : \bigcup B_j(\epsilon) \to D(\epsilon) \times N_{(B, k)}$ such that $|G_{(B, k)}|^{\epsilon}$ is bounded by $\frac{1}{2} |G_{(A, k_0)}|$ from below. This proves Step 3.

Finally if we let $c_0$ be a new constant decided by Step 2 and Step 3, we prove Lemma 1.

Let $\Gamma$ be the open cone such that $\Gamma \cap S^{d-1} = k$ which is as in Lemma 1. If $E$ is a compact convex set with interior, then we define $g_E(x) = \min(T \geq 1 : x \in TE)$. Fix once and for all $t > d$, and define $|u|_{p, E} = ||g_E u||_p$. Then by the Holder inequality we have

$$(2) \quad ||u||_p \leq C||u||_{q, E}|E|^\frac{1}{p} - \frac{1}{q}$$

for any $q > p$, where $C$ depends only on $t$ and $d$.

Lemma 2. Suppose $P_A$ is as in Lemma 1 and is of order $m < \frac{d}{r}$. Let $b$ and $\Gamma$ be as before or as in Lemma 1. Then there is a constant $C_A$ such that for all $B \in \mathbb{R}^M$ with $|B - A| \leq b$ and any $k \in \Gamma$ and all compact convex subsets $E \subset \mathbb{R}^d$ with $|E| \geq |k|^{-d}$, we have

$$(3) \quad ||e^{k-x} \nabla^{m - \mu} f||_q \leq C_A(|k|^d |E|)^{\frac{1}{2}} ||e^{k-x} P_B(D)f||_{2, E}$$

for all $f \in W^{m,2}$ with compact support and all integers $0 < \mu \leq m$, where $q_\mu$ are the real numbers satisfying $\frac{1}{2} - \frac{1}{q_\mu} = \frac{1}{4}$. When $\mu = 0$, we have the following inequality:

$$(4) \quad ||e^{k-x} \nabla f||_2 \leq C_A(|k| |\text{diam} E|) ||e^{k-x} P_B(D)f||_{2, E}.$$ 

Proof. Let $a = (\frac{1}{2}, 0)$, $b = (1, 0)$, $c = (1, \frac{1}{2})$ and $d = (\frac{1}{2}, \frac{1}{4})$. Let $Q$ be a subset of $R^2$ consisting of the quadrilateral $abcd$ and two sides $ad$ and $be$. Let $B$ and $k$ with $|k| = 1$ be as in Lemma 2. So the conclusions of Lemma 1 hold for this $(B, k)$. First let $0 < \mu \leq m$.

The inequality (3) is equivalent to

$$(5) \quad ||(m \hat{v})^\mu||_{q_\mu} \leq C_A(|k|^d |E|)^{\frac{1}{2}} ||v||_{2, E}$$

with $m(\xi) = \frac{|\xi + ik|^{m-\mu}}{P_A(\xi + ik)}$ for all $v \in \mathbb{C}^\infty$. 

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Let $U_{\frac{1}{2}} = \bigcup_{j=1}^{d} B_j(\frac{1}{2})$ and $U_{1} = \bigcup_{j=1}^{d} B_j(\epsilon)$ which are in Lemma 1. Let $\phi$ be a smooth cutoff function taking 1 on $U_{\frac{1}{2}}$ and 0 on $U_{\frac{1}{2}}^c$. Write $m = m_1 + m_2$ with $m_1 = m\phi$ and $m_2 = m(1 - \phi)$. By Lemma 1, the exact proof of Lemma 2.1 in [3] shows that

$$||(m_1v)^{q}||_q \leq CA||v||_{p}$$

(6)

for all $\left(\frac{1}{p}, \frac{1}{q}\right) \in Q$, where $C_A$ is some constant which depends only on $A$, $k_0$ and $d$. Since $m_2(\xi) \leq (1 + |\xi|)^{-\mu}$, by the Bessel potential theory, we have

$$||(m_2v)^{q}||_q \leq CA||v||_{p}$$

(7)

for all $\left(\frac{1}{p} - \frac{1}{q}, \frac{1}{q}\right) \in Q$. Let $q_\mu$ be such that $\frac{1}{s} - \frac{1}{q_\mu} = \frac{\mu}{d}$, and let $q_1^s$ be such that $\frac{1}{s} - \frac{1}{q_1^s} = \frac{\mu}{d}$ if $\mu \geq 2$, $q_1^s$ is sufficiently close to $s'$ and is bigger than $s'$. Then for any compact convex set $|E| \geq 1$, since $q_1^s < q_\mu$ and $m_1$ has compact support, we have by using (6) and (2)

$$||(m_1v)^{q_\mu}||_{q_\mu} \leq ||(m_1v)^{q_1^s}||_{q_1^s} \leq C_A||v||_{s} \leq C_A|E|^{\frac{\mu}{d} - \frac{1}{q_1^s}}||v||_{2,E}$$

(8)

which is bounded by $C_A|E|^\frac{\mu}{d}||v||_{2,E}$ since $|E| \geq 1$. Combining (8) and (7) we prove (5) and hence (3) with $|k| = 1$. After a scaling we prove Lemma 2 with $\mu \geq 1$. Finally when $\mu = 0$, the inequality (4) was already showed in [4] without using any curvature property in Lemma 1. So this proves Lemma 2.

**Lemma 3.** Suppose $f$ is supported in a ball $B$. Let $D(a,N)$ be a fixed ball in $R^d$. Then there is a pairwise disjoint compact convex subset $\{E_k\}$ with $\{k_j\} \subset D(a,N)$ such that

$$||e^{k_j}x f \cdot g_{E_{k_j}}||_{1,E_{k_j}} \leq C_0^2 ||e^{k_j}x f||_{L^1(E_{k_j})},$$

(9)

$$\sum |E_{k_j}|^{-1} \geq C^{-1}N^d, \forall s \geq 1,$$

(10)

$$\text{diam } E_{k_j} \leq C_0 N^{-\frac{1}{2}},$$

(11)

$E_{k_j}$ contains a ball of radius $(C_0 N)^{-1}$,

$$E_{k_j} \subset 2B$$

where $C_0$ is a universal constant depending only on $d$.

**Proof.** This is a special case of Wolff’s measure lemma in [4].

Now let’s start to prove Theorem 1. First we claim that we may assume the Lipschitz norm of $a_\alpha(x)$ is less than a small number $\rho$ which will be chosen later. In fact let $F_1(x) = \delta^{-1} x$, $F_2(x) = (x, -x_d)$, $F_3(x) = \frac{x + e_d}{|x + e_d|}$ and $F = F_3 \circ F_2 \circ F_1$. Then if $\delta$ is small enough, the function $v = u \circ F^{-1}$ is defined on a domain which contains $R^d \setminus B(-e_d, \frac{1}{2})$ and $v = 0$ outside $B(-e_d, 1)$. Moreover $v$ satisfies the following differential inequality:

$$|P_\delta(y, D)v(y)| \leq \sum_{0<\mu \leq m} V_\mu(y) |\nabla^{m-\mu} v(y)|$$

(12)

where $V_\mu(y)$ has the same properties as before, $P_\delta(y, D) = \sum_{|\alpha|=m} a_\alpha^\delta(y) D^\alpha$ with $a_\alpha^\delta(0) = a_\alpha(0)$ and $||a_\alpha||_{L^p} \leq \delta||a_\alpha||_{L^p}$. Let $\rho$ be this number. On the other hand, if we let $A = (a_\alpha^\delta(0)) = (a_\alpha(0))$ and $b$, $\Gamma$ be as in Lemma 2 or Lemma 1 with
\( k_0 = e_d \), then when \( \delta \) is small enough for any \( y \in B(0, \frac{1}{2}) \) with \( B = (a^\delta_\mu(y)) \) the inequalities (3) and (4) hold for all small \( \delta \).

Let’s assume \( 0 \notin \text{supp} \eta \). Let \( S \) be the convex hull of \( \text{supp} \eta \cap \{y \in R^d : y_d \geq -\frac{1}{10}\} \) and \( \phi \) be a smooth cutoff function such that \( \phi = 0 \) when \( y_d \leq -\frac{1}{d} \), \( \chi = 1 \) in a neighborhood of \( \partial S \) and \( \sum_{0 < \mu \leq m} ||V_\mu||_{L^4_H(supp \phi)} \leq \beta \) with a small constant \( \beta \) to be chosen later. Let \( w = v \phi \). Then by (4)

\[
\tag{13}
|P_\delta(y, D)w(y)| \leq \sum_{0 < \mu \leq m} V_\mu(y) |\nabla^{m-\mu} w(y)| + \chi
\]

where \( \chi \in L^2 \) and \( \text{supp} \chi \subset A^1 \cup A_2 \); here \( A_2 = \{y \in B(-e_d, 1) : -\frac{1}{10} \geq y_d \geq -\frac{1}{d}\} \) and \( A_1 \) is a compact subset of \( S \). Let \( r \leq \frac{1}{32} \) be a fixed small number so that the cone \( \Gamma_r = \{k \in R^d : k_d > r^{-1} \sqrt{|k|^2 - k_d^2}\} \) is contained in \( \Gamma \) which is as in Lemma 2 for \( PA \). So \( r \) is independent of \( \rho \).

**Lemma 4.** If \( \tau > 0 \), then there is an \( L_0 \) such that if \( k \in \Gamma_r \) and \( |k| \geq L_0 \), then

\[
\tag{14}
||e^{k \cdot y} \chi \cdot g_E||_{2,E} \leq ||e^{k \cdot y} \sum_{0 < \mu \leq m} V_\mu |\nabla^{m-\mu} w||_2
\]

for all \( E \subset B(0, \frac{1}{2}) \) with \( E \) containing a ball of radius \( \tau |k|^{-1} \).

**Proof.** Since \( \Gamma_r \) has conjugate cone \( \{k \in R^d : (k, k') \leq 0 \forall k' \in \Gamma_r\} \) which contains \( B(-e_d, 1) \cap \{y : y_d \leq \frac{1}{d}\} \supset A_2 \), the rest of the proof is exactly the same as the proof of Lemma 7.1 of [4]. So we are done.

**Proof of Theorem 1.** Let \( L \geq L_0 \) be a large number. We will apply Lemma 3 to the function

\[
f = \left( \sum_{0 < \mu \leq m} V_\mu |\nabla^{m-\mu} w| + \rho L^{-\frac{1}{2}} |\nabla^m w| \right)^2
\]

and the ball \( B(Le_d, \frac{1}{2}rL) \) with \( a = Le_d \) and \( N = \frac{1}{2}rL \). So \( \frac{1}{2}L \leq |k_j| \leq 2L \). Let \( Y_j = E_{k_j} \cap \text{supp} \nu \), let \( y_j \) be the center of the convex set \( E_{k_j} \) and let \( B_j = (a^0_\mu(y_j)) \). So we have \( ||B_j - A|| \leq b \) and the inequalities (3) and (2) in Lemma 2. Then by using Hölder’s inequality, (3), (4), and (11)

\[
||e^{k_j \cdot y} \left( \sum_{0 < \mu \leq m} V_\mu |\nabla^{m-\mu} w| + \rho L^{-\frac{1}{2}} |\nabla^m w| \right) ||_{L^2(E_{k_j})}
\]

\[
\leq \sum_{0 < \mu \leq m} ||V_\mu||_{L^2(Y_j)} ||e^{k_j \cdot y} |||_{|\mu|, \alpha} + \rho L^{-\frac{1}{2}} ||e^{k_j \cdot y} \nabla^m w||_2
\]

\[
\leq C_A \left( \sum_{0 < \mu \leq m} \langle |k_j|^{id} E_{k_j} \rangle \frac{2}{\alpha} ||V_\mu||_{L^2(Y_j)} + \rho L^{-\frac{1}{2}} |k_j| (\text{diam} E_{k_j}) \right) ||e^{k_j \cdot y} P_{B_j}(D)w||_{2,E_{k_j}}
\]

\[
\leq 2C_A \left( \sum_{0 < \mu \leq m} (L^d |E_{k_j}|)^{\frac{2}{\alpha}} ||V_\mu||_{L^2(Y_j)} + C_0 r^{-1} \rho \right) ||e^{k_j \cdot y} P_{B_j}(D)w||_{2,E_{k_j}}.
\]
On the other hand, since $a^\delta$ is Lipschitz continuous it follows that $|a^\delta(x) - a^\delta(y)| \leq \rho \cdot |x - y| \leq \rho \text{diam}E_k \cdot \mathcal{E}_{kj} \leq C_0 r^{-1} \rho L^{-\frac{1}{2}} \mathcal{E}_{kj}$ by (11). So

$$|P_{B_j}(D)w(y)| \leq |P_S(y, D)w(y)| + C_0 r^{-1} \rho L^{-\frac{1}{2}} \mathcal{E}_{kj} |\nabla^m w|$$

and hence by (13)

$$|P_{B_j}(D)w(y)| \leq \sum_{0 < \mu \leq m} V_\mu |\nabla^{m-\mu} w| + C_0 r^{-1} \rho L^{-\frac{1}{2}} \mathcal{E}_{kj} |\nabla^m w| + \chi.$$ 

Because of (14), we may ignore the term $\chi$ in the following process. Now by using (9) we have

\begin{equation}
|e^{kj} P_{B_j}(D)w(y)|_{2,E_kj} \leq 2 C_0 r^{-1} |e^{kj} \left( \sum_{0 < \mu \leq m} V_\mu |\nabla^{m-\mu} w| + \rho L^{-\frac{1}{2}} |\nabla^m w| \right) \mathcal{E}_{kj} |w|_{2,E_kj} \leq 2 C_0^2 r^{-1} |e^{kj} \left( \sum_{0 < \mu \leq m} V_\mu |\nabla^{m-\mu} w| + \rho L^{-\frac{1}{2}} |\nabla^m w| \right) |w|_{L^2(E_kj)} |w|.
\end{equation}

So combining (15) and (16), we have

\begin{equation}
1 \leq 2 C_0^2 r^{-1} \cdot 2 C_A \left( \sum_{0 < \mu \leq m} (L_d |E_kj|)^{\frac{d}{2}} |V_\mu|_{L^2(Y_j)} |w|_{L^2(Y_j)} + C_0 r^{-1} \rho \right).
\end{equation}

Remember the constants $r$, $C_0$ and $C_A$ are independent of $\rho$, i.e., $\delta$. So after making $\delta$ and hence $\rho$ small, (17) implies

$$\sum_{0 < \mu \leq m} (L_d |E_kj|)^{\frac{d}{2}} |V_\mu|_{L^2(Y_j)} \geq C$$

and hence

\begin{equation}
\max_{0 < \mu \leq m} \left\{ |V_\mu|_{L^2(Y_j)} \right\} \geq C (L_d |E_kj|)^{-1}
\end{equation}

for some constant $C$ depending only on $d$ and $A$. Summing up over $j$ for (18), (10) implies that

$$\sum_{0 < \mu \leq m} |V_\mu|_{L^2(Y_j)} \geq C_0^{-1} C,$$

which is a contradiction if $\beta$ is small enough. This proves Theorem 1.

\section*{ACKNOWLEDGMENTS}

I would like to thank Professor Tom Wolff for his constant encouragement while I was considering this problem. I would also like to thank the Alfred P. Sloan Foundation for a Doctoral Dissertation Fellowship, 1992-93.
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