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ENDOMORPHISMS OF FINITE FULL TRANSFORMATION SEMIGROUPS

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ABSTRACT. We describe all endomorphisms of finite full transformation semigroups and count their number.

A *full transformation semigroup* $\mathcal{T}_X$ on a set X is the set X^X of all transformations (i.e., self-maps) $X \to X$ of X with composition of transformations as multiplication. This is an important object in semigroup theory, combinatorics, many-valued logic, etc. Various properties of $\mathcal{T}_X$ are known. In particular, Schreier [4] proved in 1936 that automorphisms of $\mathcal{T}_X$ are inner: for every automorphism α there exists a uniquely determined element $g \in \mathcal{G}_X \subset \mathcal{T}_X$ of the symmetric group $\mathcal{G}_X$ on X such that $\alpha(t) = gtg^{-1}$ for all $t \in \mathcal{T}_X$. Here the juxtaposition gt stands for the composition $g \circ t$ and the composition acts from the right to the left: $g \circ t(x) = g(t(x))$ for every $x \in X$. Thus the automorphism group of $\mathcal{T}_X$ is naturally isomorphic to $\mathcal{G}_X$.

Surprisingly, no one has considered endomorphisms of $\mathcal{T}_X$. Our paper seems to be the first attempt at filling that gap. We consider the finite case only, that is, X is a finite set of cardinality n for $n \geq 0$.

We introduce a few notations and terms. Endomorphisms that are not automorphisms are called *proper*. The kernel congruence $\ker(\varepsilon)$ of an endomorphism ε is defined by $(s,t) \in \ker(\varepsilon) \Leftrightarrow \varepsilon(s) = \varepsilon(t)$ for any $s, t \in \mathcal{T}_X$. Δ_A is the identity relation on a set A. If $\varepsilon' = \varepsilon_{|\mathcal{G}_X}$ is the restriction of ε to $\mathcal{G}_X$, then $\mathbf{ker}(\varepsilon')$ also stands for the corresponding normal subgroup of $\mathcal{G}_X$. The *second projection* (also called the *range*) of $t \in \mathcal{T}_X$ is the set $\mathrm{pr}_2\, t = t(X)$. In particular, $\mathrm{pr}_2(st) \subset \mathrm{pr}_2\, s$. The *rank* of t is the cardinality $|\mathrm{pr}_2(t)|$ of $\mathrm{pr}_2(t)$.

We can assume that $X = \{1, 2, \dots, n\}$ and write $\mathcal{T}_n$ instead of $\mathcal{T}_X$. Analogously, $\mathcal{G}_n$ stands for $\mathcal{G}_X$. We consider $\mathcal{G}_n$ as a subgroup of $\mathcal{G}_{n+1}$ consisting of all permutations that fix the point $n+1$. Also, $\mathcal{A}_X$ denotes the alternating group on X. For $n = 3$ or $n \geq 5$, $\mathcal{A}_n$ is the only nontrivial normal subgroup of $\mathcal{G}_n$, while $\mathcal{G}_4$ contains another nontrivial normal subgroup $\mathcal{K}$, Klein's four-group. For every $x \in X$, c_x denotes the constant transformation in $\mathcal{T}_X$ that maps all elements of X onto x. For example, $c_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix}$ in $\mathcal{T}_4$.

Our main results are the following Theorem and Corollary. Their proof is followed by a Proposition that is another corollary to our main theorem.

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Theorem. (A) *Choose a permutation g of X and define $\alpha^g(t) = gtg^{-1}$ for all $t \in \mathcal{T}_X$. Then α^g is an automorphism of $\mathcal{T}_X$.*

(E) *Choose $\beta, \gamma \in \mathcal{T}_X$ such that $\beta^3 = \beta$ and $\beta\gamma = \gamma\beta = \gamma^2 = \gamma$. If*

$$\varepsilon_{\beta,\gamma}(t) = \begin{cases} \beta, & \text{for } t \in \mathcal{G}_X \backslash \mathcal{A}_X, \\ \beta^2, & \text{for } t \in \mathcal{A}_X, \\ \gamma, & \text{for } t \in \mathcal{T}_X \backslash \mathcal{G}_X, \end{cases}$$

then $\varepsilon_{\beta,\gamma}$ is an endomorphism of $\mathcal{T}_X$.

If $\beta^2 = \beta = \gamma$, then $\varepsilon_{\beta,\gamma}$ is a constant endomorphism that maps $\mathcal{T}_X$ onto a trivial semigroup $\{\gamma\}$. If $\beta^2 = \beta \neq \gamma$, then $\varepsilon_{\beta,\gamma}$ is an endomorphism of rank 2 that maps $\mathcal{T}_X$ onto a two-element semilattice $\{\beta, \gamma\}$ with $\gamma < \beta$. If $\beta \neq \beta^2 \neq \gamma$, then $\varepsilon_{\beta,\gamma}$ is an endomorphism of rank 3 that maps $\mathcal{T}_X$ onto a three-element semigroup $\{\beta, \beta^2, \gamma\}$, where γ is a zero element and $\{\beta, \beta^2\}$ a two-element subgroup.

Conversely, every automorphism of $\mathcal{T}_X$ has the form (A) *and every proper endomorphism the form* (E)*, except that $\mathcal{T}_4$ has 24 additional endomorphisms σ^g, $g \in \mathcal{G}_4$, defined as follows:*

Each of the six cosets of $\mathcal{K}$ in $\mathcal{G}_4$ contains exactly one element of $\mathcal{G}_3$. If $t \in \mathcal{G}_4$, let $\sigma(t)$ be that (only) element of $\mathcal{K}t \cap \mathcal{G}_3$, and if $t \in \mathcal{T}_4 \backslash \mathcal{G}_4$, let $\sigma(t) = c_4$. Then $\sigma^g = \alpha^g\sigma$, that is, $\sigma^g(t) = g\sigma(t)g^{-1}$. In particular, $\sigma = \sigma^e$, where e is the identity element of $\mathcal{G}_4$.

Corollary. *Every proper endomorphism of $\mathcal{T}_n$ has rank 1, 2, or 3, except that $\mathcal{T}_4$ also has additional endomorphisms of rank 7. There are*

$$n! \sum_{m=1}^{n} \frac{m^{n-m}}{(n-m)!m!}$$

endomorphisms of rank 1,

$$n! \sum_{m=2}^{n} \sum_{r=1}^{m-1} \frac{m^{n-m} r^{m-r}}{(n-m)!(m-r)!r!}$$

endomorphisms of rank 2, and

$$n! \sum_{m=3}^{n} \sum_{k=1}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{r=1}^{m-2k} \frac{m^{n-m} r^{m-k-r}}{2^k (n-m)!(m-2k-r)!k!r!}$$

endomorphisms of rank 3.

For $n = 4$, there are 24 endomorphisms of rank 7.

Thus $\mathcal{T}_n$ has

$$n!\left[1 + \sum_{m=1}^{n} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{r=1}^{m-2k} \frac{m^{n-m} r^{m-k-r}}{2^k (n-m)!(m-2k-r)!k!r!}\right]$$

endomorphisms for $n > 1, n \neq 4$ and

$$n!\left[2 + \sum_{m=1}^{n} \sum_{k=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{r=1}^{m-2k} \frac{m^{n-m} r^{m-k-r}}{2^k (n-m)!(m-2k-r)!k!r!}\right] = 345$$

endomorphisms for $n = 4$.

In particular, the orders of $\mathcal{T}_n$ and $\operatorname{End}(\mathcal{T}_n)$ *for $1 \leq n \leq 9$ are:*

$n =$	1	2	3	4	5	6	7	8	9
$\lvert\mathcal{T}_n\rvert = n^n =$	1	4	27	256	3,125	46,656	823,543	16,777,216	387,420,489
$\lvert\operatorname{End}(\mathcal{T}_n)\rvert =$	1	7	40	345	3,226	38,503	529,614	8,219,025	141,633,028

Problem. Find an asymptotic estimate for $|\operatorname{End}(\mathcal{T}_n)|_{n\to\infty}$, where $\operatorname{End}(\mathcal{T}_n)$ is the semigroup of all endomorphisms of $\mathcal{T}_n$. It seems that the estimate can be found by the Laplace method and arguments from [2]. Is it true that $\frac{|\operatorname{End}(\mathcal{T}_n)|}{|\mathcal{T}_n|}$ converges monotonically to 0? It appears to be so.

Constant (that is, of rank 1) endomorphisms of $\mathcal{T}_n$ are in one-to-one correspondence with idempotents of $\mathcal{T}_n$. Harris and Schoenfeld (see [3]) found an asymptotic estimate for the number of idempotents in $\mathcal{T}_n$.

The proofs of the Theorem and Corollary follow from Lemmas 1–5.

Lemma 1. *Let $G \subset \mathcal{T}_X$ be a subgroup of $\mathcal{T}_X$ for a finite or infinite X. All elements $g \in G$ have the same range $Y \subset X$, and the mapping $g \mapsto \bar{g} = g_{|Y}$, where $g_{|Y}$ is the restriction of g to Y, is an isomorphism of G onto a group of permutations of Y.*

Proof. Here "permutation" means a bijection of Y onto itself. Lemma 1 is folklore, and we give its brief proof for completeness' sake.

For every $g, h \in G$ there exists $x \in G$ such that $gx = h$, and hence $\operatorname{pr}_2 h = \operatorname{pr}_2 gx \subset \operatorname{pr}_2 g$. Analogously, $\operatorname{pr}_2 g \subset \operatorname{pr}_2 h$, so that $\operatorname{pr}_2 h = \operatorname{pr}_2 g$.

Let e be the identity element of G. Then $\operatorname{pr}_2 \bar{g} \subset Y = \operatorname{pr}_2 ge = g(\operatorname{pr}_2 e) = g(Y) = \operatorname{pr}_2 \bar{g}$, whence $\operatorname{pr}_2 \bar{g} = Y$. Thus $\bar{g}$ is a permutation of Y, and so $g\Delta_Y = \bar{g} = \Delta_Y \bar{g}$. It follows that $\bar{h}\bar{g} = h\Delta_Y\bar{g} = hg\Delta_Y = \overline{hg}$ and $g \mapsto \bar{g}$ is a homomorphism of G onto a group of permutations of Y. This homomorphism is an isomorphism because $\bar{g} = \bar{e} \Rightarrow g^2 = \bar{g}g = \bar{e}g = eg = g \Rightarrow g = e$. □

Lemma 2. *Every endomorphism of $\mathcal{T}_n$ injective on $\mathcal{G}_n$ is an automorphism.*

Proof. If ε is an endomorphism injective on $\mathcal{G}_n$, then $\mathcal{G}_n$ is isomorphic to its image $G = \varepsilon(\mathcal{G}_n)$ under ε, and so $|G| = n!$. By Lemma 1, G is isomorphic to a group of permutations of a subset $Y \subset X$, where $Y = \operatorname{pr}_2 e$ for e the identity element of G. Thus $n! = |G| \le |Y|!$. It follows that $Y = X$, and hence $G = \mathcal{G}_n$.

Since $c_x g = c_x$ for all $g \in \mathcal{G}_n$, we obtain $\varepsilon(c_x)\varepsilon(g) = \varepsilon(c_x g) = \varepsilon(c_x)$. Here $\varepsilon(g)$ may be an arbitrary element of $\mathcal{G}_n$, and hence $\varepsilon(c_x)h = \varepsilon(c_x)$ for all $h \in \mathcal{G}_n$. This is possible only if $\varepsilon(c_x)$ is a constant endomorphism, say, $\varepsilon(c_x) = c_{f(x)}$, where $f(x)$ is a suitable element of X.

Obviously, $tc_x = c_{t(x)}$ for all $t \in \mathcal{T}_n$. Thus $c_{f(t(x))} = \varepsilon(c_{t(x)}) = \varepsilon(tc_x) = \varepsilon(t)\varepsilon(c_x) = \varepsilon(t)c_{f(x)} = c_{\varepsilon(t)(f(x))}$, so that $f(t(x)) = \varepsilon(t)(f(x))$ for all $x \in X$. Therefore, $ft = \varepsilon(t)f$ and, for $x, y \in X$ and $g \in \mathcal{G}_X$, $f(x) = f(y) \Rightarrow f(g(x)) = \varepsilon(g)(f(x)) = \varepsilon(g)(f(y)) = f(g(y))$, and hence $f(x) = f(y) \Rightarrow f(g(x)) = f(g(y))$ for every $g \in \mathcal{G}_n$. It follows that either $x = y$, and hence f is injective, or $f(x) = f(y)$ for all $x, y \in X$, and hence f is a constant transformation. In the latter case $f = c_a$ for some $a \in X$, and our formula $f(t(x)) = \varepsilon(t)(f(x))$ becomes $a = \varepsilon(t)(a)$ for every $t \in \mathcal{T}_n$. Thus a is a fixed point of $\varepsilon(t)$ for every t, and hence for every $g \in \mathcal{G}_n$, which is possible only if $X = \{a\}$. Therefore, f is injective for every finite X, and so $f, f^{-1} \in \mathcal{G}_n$. Now the formula $ft = \varepsilon(t)f$ can be rewritten as $\varepsilon(t) = ftf^{-1}$, which shows that ε is an automorphism of $\mathcal{T}_n$. □

Recall that every ideal of $\mathcal{T}_n$ is of the form $I_k = \{t \in \mathcal{T}_n : \text{rank}(t) < k\}$ with $2 \le k \le n+1$ (see [1]). Thus $\mathcal{T}_n$ is a disjoint union of its group of units $\mathcal{G}_n$ and the maximal ideal I_n.

Lemma 3. *Every proper endomorphism ε of $\mathcal{T}_n, n \neq 4$, maps $\mathcal{G}_n$ into elements $\alpha, \beta \in \mathcal{T}_n$ with $\alpha = \beta^2$ and $\beta^3 = \beta$. Also, ε maps $I_n = \mathcal{T}_n \backslash \mathcal{G}_n$ into a single idempotent $\gamma \in \mathcal{T}_X$ such that $\beta\gamma = \gamma\beta = \gamma$. Thus $\text{pr}_2\,\varepsilon$ has cardinality 1, 2, or 3. Respectively, $\text{pr}_2\,\varepsilon$ is a trivial semigroup $\{\gamma\}$, a semilattice $\{\alpha, \beta\}$of order 2, or a two-element group $\{\alpha, \beta\}$ with zero γ adjoined.*

Proof. By Lemma 2, ε is not injective on $\mathcal{G}_n$, and so $\ker(\varepsilon)$ does not induce an identity congruence on $\mathcal{G}_n$. It follows from the description of congruence relations on $\mathcal{T}_n$ (see [1]) that one of the equivalence classes of $\ker(\varepsilon)$ is either I_n or I_{n+1}. In the latter case $I_{n+1} = \mathcal{T}_n$, and hence ε maps $\mathcal{T}_n$ onto the trivial semigroup $\{\gamma\}$ for some idempotent $\gamma \in \mathcal{T}_n$.

If $\mathbf{ker}(\varepsilon) = I_n$, then ε maps $I_n = \mathcal{T}_n \backslash \mathcal{G}_n$ into an idempotent γ. Also, $\ker(\varepsilon)$ decomposes $\mathcal{G}_n$ into cosets modulo a nontrivial normal subgroup N of $\mathcal{G}_n$. If $n \neq 4$, then $N = \mathcal{G}_n$ or $N = \mathcal{A}_n$, the alternating group. In the former case ε maps $\mathcal{G}_n$ into an idempotent $\alpha \in \mathcal{T}_n$, and $\varepsilon(\mathcal{T}_n)$ is a two-element semilattice $\{\alpha, \gamma\}$ with γ as the zero element. In the latter case $\mathcal{G}_n/N$ is a two-element group, and hence ε maps N into an idempotent α and $\mathcal{G}_n \backslash N$ into β such that $\beta^2 = \alpha$ and $\beta^3 = \alpha\beta = \beta$. $\square$

It remains to describe proper endomorphisms of $\mathcal{T}_4$.

Lemma 4. *All proper endomorphisms of $\mathcal{T}_4$ are either endomorphisms of ranks 1, 2 or 3 described in Lemma 3 or endomorphisms ε of rank 7, where $\text{pr}_2\,\varepsilon$ is isomorphic to the symmetric group $\mathcal{G}_3$ with zero adjoined.*

There are 24 endomorphisms σ^g of rank 7; they correspond to 24 permutations $g \in \mathcal{G}_4$ of the symmetric group $\mathcal{G}_4$ and have the structure described in the Theorem.

Proof. The only difference with our proof of Lemma 3 is that $\mathcal{G}_4$ contains another nontrivial normal subgroup $N = \mathcal{K}$, which is Klein's four-group. It is easy to see that the factor group $\mathcal{G}_n/\mathcal{K}$ is isomorphic to $\mathcal{G}_3$, the symmetric group of degree 3. Obviously, $\mathcal{K}$ has index 6 in $\mathcal{G}_4$ and the complement I_4 of $\mathcal{G}_4$ in $\mathcal{T}_4$ forms a congruence class modulo $\mathbf{ker}(\varepsilon)$, where ε is our endomorphism. Thus the rank of ε is 7 and $\text{pr}_2\,\varepsilon$ is isomorphic to $\mathcal{G}_3^0$ (the group $\mathcal{G}_3$ with zero adjoined).

The rest of the proof is based on two facts: (1) $\mathcal{T}_4$ contains exactly four subsemigroups S isomorphic to $\mathcal{G}_3^0$, and (2) each of these four semigroups has exactly six automorphisms.

(1) We know that S is a group G with zero adjoined. Let t be that zero. Then t is an idempotent of $\mathcal{T}$ of rank r, where $1 \le r \le 4$. Also, $gt = t$ for every $g \in G$, that is, each of the r elements of $\text{pr}_2\,t$ is a fixed point of each g. Applying Lemma 1 we see that G is isomorphic to a group $\bar{G}$ of permutations of a subset $Y \subset \{1, 2, 3, 4\}$, and r elements of $\text{pr}_2\,t$ are fixed points of all elements $\bar{g} \in \bar{G}$. Thus the elements of $\bar{G}$ can actually permute only $|Y| - r$ points, and so $3! = |G| \le (|Y| - r)! \le (4-1)!$. It follows that $|Y| - r = 3$, and hence $r = 1$ and $|Y| = 4$, which implies $Y = X$. Thus $t = c_x$ for some $x \in X$. Since $|G| = 6$, G is the group of *all* permutations of X with a fixed point x. We have four choices for x, which give us four choices for S.

(2) Assume that $x = 4$. Then elements of G actually permute only the three elements of $\{1, 2, 3\}$, and, since we identified $\mathcal{G}_3$ with an appropriate subgroup of

$\mathcal{G}_4$, we see that $S = \mathcal{G}_3 \cup \{c_4\}$. Every automorphism of S leaves c_4 fixed and induces an automorphism of $\mathcal{G}_3$. Since $\mathcal{G}_3$ has precisely six (inner) automorphisms, S has six automorphisms too.

Now it is easy to see that all endomorphisms of rank 7 have the form $\varepsilon^g = \alpha^g \varepsilon$ for $g \in \mathcal{G}_4$, so that $\varepsilon^g(t) = \varepsilon(t)^g = g\varepsilon(t)g^{-1}$ for every $t \in \mathcal{T}_4$. Clearly, we obtain different endomorphisms for different g. If we choose $\varepsilon = \sigma^e$ as described in the Theorem, all other endomorphisms of rank 7 are $(\sigma^e)^g = \sigma^g$. □

Our Theorem follows from Lemmas 3 and 4. □

Now we are ready to count the number of endomorphisms of $\mathcal{T}_n$. Since each automorphism of $\mathcal{T}_n$ is inner, their total number is $n!$ and it remains to find the number of proper endomorphisms.

By Lemma 3, each proper endomorphism ε, $n \neq 4$, is determined by its range. The range is determined by two elements β and γ such that $\beta^3 = \beta$ and $\beta\gamma = \gamma\beta = \gamma^2 = \gamma$. These relations between β and γ are characterized in the following lemma.

Lemma 5. *Three (not necessarily distinct) transformations β, $\alpha = \beta^2$ and γ are the range of an endomorphism of $\mathcal{T}_n$ if and only if*

(i) *the restriction $\bar{\beta} = \beta_{|Y}$ of β to $Y = \mathrm{pr}_2\,\beta$ is an involution, that is, $\bar{\beta}$ is a permutation of Y such that $\bar{\beta}^2 = \bar{\beta}$;*

(ii) *every element x of $\mathrm{pr}_2\,\gamma$ is a fixed point of both γ and β;*

(iii) *if $x \notin Y$, then $\gamma(x) = \gamma(\beta(x))$;*

(iv) *if (x, y) is a transposition in $\bar{\beta}$, that is, $\beta(x) = y$ and $\beta(y) = x$ for $x \neq y$, then $\gamma(x) = \gamma(y)$.*

Proof. Necessity. (i) It follows from $\beta^3 = \beta$ that $\{\beta, \beta^2\}$ is a two-element subgroup of $\mathcal{T}_n$. By Lemma 1, $\bar{\beta} = \beta_{|Y}$ is a permutation of Y and $\bar{\beta}^2 = \Delta_Y$, that is, $\bar{\beta}$ is an involution of Y. We do not exclude the case when $\bar{\beta} = \Delta_Y$.

(ii) Let $x \in \mathrm{pr}_2\,\gamma$. If $x = \gamma(y)$ for some $y \in X$ and $\delta\gamma = \gamma$ for $\delta \in \mathcal{T}_n$, then $\delta(x) = \delta\gamma(y) = \gamma(y) = x$. Thus $\beta\gamma = \gamma\gamma = \gamma$ implies $\gamma(x) = \beta(x) = x$.

(iii) This condition follows from $\gamma = \gamma\beta$.

(iv) If (x, y) is a transposition of $\bar{\beta}$, then $\gamma(x) = \gamma(\beta(y)) = \gamma(y)$.

Sufficiency. Suppose that (i)–(iv) hold. Then (i) implies $\beta^3 = \bar{\beta}^2\beta = \Delta_Y\beta = \beta$. Since $\gamma(x) \in \mathrm{pr}_2\,\gamma$, (ii) implies $\gamma^2(x) = \gamma(\gamma(x)) = \gamma(x)$ for all $x \in X$, that is, $\gamma^2 = \gamma$. Also by (ii), $\beta(\gamma(x)) = \gamma(x)$ for all $x \in X$, and hence $\beta\gamma = \gamma$.

It remains to prove that $\gamma\beta = \gamma$ or, equivalently, $\gamma(\beta(x)) = \gamma(x)$ for all $x \in X$. By (iv) this is so for $x \notin Y$. If $x \in Y$, then, by (i), x is either a fixed point of $\bar{\beta}$ (and hence of β) or a part of a transposition (x, y) of $\bar{\beta}$. In the former case $\gamma(\beta(x)) = \gamma(x)$. In the latter case, by (iv), $\gamma(\beta(x)) = \gamma(y) = \gamma(x)$. Thus $\gamma\beta = \gamma$. □

Proof of the Corollary. Proper endomorphisms of $\mathcal{T}_n$ are in one-to-one correspondence with the pairs $\{\beta, \gamma\}$ that satisfy the conditions of Lemma 5. We count the number of these pairs in the following way.

First we classify these pairs for a given $Y = \mathrm{pr}_2\,\beta$. If Y is an m-element subset of X, then $1 \leq m \leq n$. We will choose $\bar{\beta}$ with $\mathrm{pr}_2\,\bar{\beta} = Y$, then extend $\bar{\beta}$ to β, and then choose an appropriate γ. Then we calculate the number of choices and add these numbers for all possible choices of Y.

By Lemma 5.(i), $\bar{\beta}$ is a permutation of Y whose cycles are either fixed points or transpositions. Thus $\bar{\beta}$ is completely determined by its transpositions. If there are k transpositions, they move $2k$ elements of Y. Let Z be the set of these $2k$

elements. There are $\binom{m}{2k}$ choices for Z. Obviously, $k \geq 0$. By Lemma 5.(ii) and by $\mathrm{pr}_2\,\gamma \subset Y$, we see that $2k < m$. Thus $0 \leq k \leq \lfloor \frac{m-1}{2} \rfloor$, where $\lfloor r \rfloor$ denotes the integral part of r.

To calculate the number of choices for k transpositions in Z, split Z into k disjoint *ordered* pairs of distinct elements. Each ordered pair has two components, and there are $\binom{2k}{k}$ ways of choosing the set of k first components for these pairs. To form ordered pairs, bijectively map k first components onto k remaining elements of Z. This can be done in $k!$ ways. Therefore, there are $\binom{2k}{k}k!$ different ways of choosing these k ordered pairs.

Each two-element subset can be turned into an ordered pair in two ways, and hence k disjoint transpositions can be turned into 2^k different sets of ordered pairs. Thus there are

$$\binom{m}{2k}\frac{(2k)!}{2^k k!} = \frac{m!}{2^k(m-2k)!k!}$$

ways to choose $\bar{\beta}$ of rank m.

To extend $\bar{\beta}$ to β we have to define $\beta(x) \in Y$ for all $x \in X \backslash Y$. Since $X \backslash Y$ contains $n-m$ elements, there are m^{n-m} ways of extending each $\bar{\beta}$. Thus, given Y, there are $\frac{m^{n-m}m!}{2^k(m-2k)!k!}$ choices for β. We can choose Y in $\binom{n}{m}$ different ways. It follows that there are

$$\binom{n}{m}\frac{m^{n-m}m!}{2^k(m-2k)!k!} = \frac{m^{n-m}n!}{2^k(n-m)!(m-2k)!k!}$$

choices for β of rank m.

Given β of rank m with k transpositions, we now choose γ. Let $\mathrm{pr}_2\,\gamma = W \subset Y$, with W containing r elements. By Lemma 4.(ii), W consists of fixed points of β. There are $m-2k$ fixed points, so that $1 \leq r \leq m-2k$, and there are $\binom{m-2k}{r}$ choices for W.

By Lemma 5.(ii), $\gamma(x) = x$ for every $x \in W$. It remains to define $\gamma(x) \in W$ for $x \notin W$. By Lemma 5.(iii), we need to define $\gamma(x)$ for $x \in Y \backslash W$ only. The set $Y \backslash W$ contains $m-r$ elements and includes our $2k$ transposed elements. To define γ for these $2k$ elements, by Lemma 5.(iv), we have to define γ only for k of these elements. It follows that we can define $\gamma(x)$ arbitrarily only for $m-r-k$ values of x. Thus, given W, there are r^{m-k-r} choices for γ.

It follows that there are $\binom{m-2k}{r}r^{m-k-r}$ choices for γ of rank r. Varying r between 1 and $m-2k$, we obtain $\sum_{r=1}^{m-2k}\binom{m-2k}{r}r^{m-k-r}$ choices for γ and

$$\begin{aligned}\frac{m^{n-m}n!}{2^k(n-m)!(m-2k)!k!}&\sum_{r=1}^{m-2k}\binom{m-2k}{r}r^{m-k-r}\\ &= \sum_{r=1}^{m-2k}\frac{m^{n-m}r^{m-k-r}n!}{2^k(n-m)!(m-2k-r)!k!r!}\end{aligned}$$

choices for $\{\beta, \gamma\}$ with β of rank m.

It follows that the total number of proper endomorphisms of $\mathcal{T}_n$ is

$$\sum_{m=1}^{n}\sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor}\sum_{r=1}^{m-2k}\frac{m^{n-m}r^{m-k-r}n!}{2^k(n-m)!(m-2k-r)!k!r!}.$$

This and part (A) prove the last two claims of the Corollary.

Endomorphisms of rank 1 are characterized by the condition $\beta = \gamma$, that is, $k = 0$ and $r = m$, which yields the first claim of the Corollary. The number of endomorphisms of rank 1 coincides with the number of idempotents of $\mathcal{T}_n$, and thus we recover a result of Tainiter (see [5]).

Endomorphisms of rank 2 are characterized by the condition $\beta^2 = \beta \neq \gamma$, that is, $k = 0$ and $r < m$. Thus $m \geq 2$, which produces the second claim of the Corollary.

Endomorphisms of rank 3 are characterized by the condition $k \neq 0$, which gives us the third claim of the Corollary and completes its proof. □

Remark. The Theorem and Corollary make it possible to classify and count various special endomorphisms. For example, an endomorphism ε is called a *retraction* if it is idempotent (that is, $\varepsilon^2 = \varepsilon$). The image of a retraction is called a *retract*. We can easily describe retractions and retracts of $\mathcal{T}_n$.

Proposition. *An endomorphism ε of $\mathcal{T}_n$ is a retraction if and only if it is one of the following:*

(i) *ε is an endomorphism of rank 1;*

(ii) *ε maps elements of $\mathcal{G}_n$ into the identity element e of $\mathcal{G}_n$, and elements of $\mathcal{T}_n \backslash \mathcal{G}_n$ into any idempotent γ different from e;*

(iii) *ε maps elements of the alternating group $\mathcal{A}_n$ into e, the remaining permutations of $\mathcal{G}_n \backslash \mathcal{A}_n$ into an odd permutation $\beta \in \mathcal{G}_n$, which is an involution (that is, $\beta^2 = e$) with fixed points, and ε maps elements of $\mathcal{T}_n \backslash \mathcal{G}_n$ into $\gamma \in \mathcal{T}_n$ such that $\gamma^2 = \beta\gamma = \gamma\beta = \gamma$.*

Also, $\mathcal{T}_4$ has four additional retractions. They are endomorphisms σ^g of rank 7 with $g \in \mathcal{K}$.

The number of retractions of rank 1 is $n! \sum_{m=1}^{n} \frac{m^{n-m}}{(n-m)!m!}$. The number of retractions of rank 2 is $n! \sum\limits_{m=1}^{n-1} \frac{m^{n-m}}{(n-m)!m!}$. The number of retractions of rank 3 is

$$n! \sum_{k=1}^{\lfloor \frac{n+1}{4} \rfloor} \sum_{r=1}^{n-4k-2} \frac{r^{n-k-r}}{2^{2k+1}(n-4k-r-2)!(2k+1)!r!}.$$

The number of retractions of rank 7 for $n = 4$ is 4.

Proof. Necessity. It is clear that endomorphisms described in (i) and (ii) are retractions. If ε is an endomorphism of type (iii), then $\beta \neq \beta^2 = e$ because β is odd and e even. Also, $\gamma \neq e$ because $\gamma\beta = \gamma$ and $e\beta = \beta \neq \gamma$. Thus ε has rank 3. Since β is an odd permutation, it does not belong to $\mathcal{A}_n$, and hence $\varepsilon(\beta) = \beta$. Also, $\varepsilon(e) = e$, and $\varepsilon(\gamma) = \gamma$ because $\gamma \notin \mathcal{G}_n$. Thus $\varepsilon^2 = \varepsilon$ and ε is a retraction.

It remains to check that, if $n = 4$, σ^g are retractions for $g \in \mathcal{K}$. For every $t \in I_4$, $\sigma^g(t) = g\sigma(t)g^{-1} = gc_4g^{-1} = c_{g(4)} \in I_4$, and hence $\sigma^g(\sigma^g(t)) = \sigma^g(c_{g(4)}) = \sigma^g(t)$. If $t \in \mathcal{G}_4$ and $g \in \mathcal{K}$, then $t(tg^{-1})^{-1} = tgt^{-1} \in \mathcal{K}$, whence $\mathcal{K}t = \mathcal{K}tg^{-1}$. By our definition of σ, $\sigma(t) \in \mathcal{K}t$, and hence $\sigma^g(t) = g\sigma(t)g^{-1} \in g\mathcal{K}tg^{-1} = g\mathcal{K}t = \mathcal{K}t$, because $g\mathcal{K} = \mathcal{K}$. It follows that $\mathcal{K}\sigma^g(t) = \mathcal{K}t$ for every $t \in \mathcal{G}_4$, and so $\mathcal{K}\sigma^g(\sigma^g(t)) = \mathcal{K}\sigma^g(t)$, which means that $\sigma^g(\sigma^g(t)) = \sigma^g(t)$, because $\mathcal{K}$ is the kernel of the group homomorphism $\sigma^g_{|\mathcal{G}_4} : \mathcal{G}_4 \to \mathcal{G}_4$. Thus $(\sigma^g)^2 = \sigma^g$ and σ^g is a retraction of $\mathcal{T}_4$.

Sufficiency. Let ε be a retraction of $\mathcal{T}_n$. If $\operatorname{rank}(\varepsilon) = 1$, then ε belongs to class (i).

Let rank$(\varepsilon) = 2$. By the Theorem the image of ε is $\{\beta, \gamma\}$, where β and γ are distinct idempotents. If $\beta \in I_n$, then $\gamma = \beta\gamma \in I_n$, and hence $\beta = \varepsilon(e) = \varepsilon^2(e) = \varepsilon(\varepsilon(e)) = \varepsilon(\beta) = \gamma$, which is a contradiction. Thus $\beta \in \mathcal{G}_n$. The only idempotent of $\mathcal{G}_n$ is e, and so $\beta = e$ and ε belongs to class (ii).

Let rank$(\varepsilon) = 3$. By the Theorem the image of ε is $\{\beta, \beta^2, \gamma\}$. If $\beta \in I_n$, then $\beta^2 \in I_n$ and we obtain a contradiction as in the case of ε of rank 2. Thus $\beta \in \mathcal{G}_n$, and hence $\alpha = \beta^2 = e$ because α is an idempotent element of $\mathcal{G}_n$. It follows that β is an involution in $\mathcal{G}_n$. By Lemma 5.(ii), β has fixed points. If it is an even permutation, then $\beta \in \mathcal{A}_n$, and hence $\varepsilon(\beta) = \beta$. Thus, for every $t \in \mathcal{G}_n \backslash \mathcal{A}_n$, $\beta = \varepsilon(t) = \varepsilon^2(t) = \varepsilon(\beta) = \alpha$, which is a contradiction. Therefore, β is odd, and so ε belongs to class (iii).

Let rank$(\varepsilon) = 7$ (and hence $n = 4$ and $\varepsilon = \sigma^g$ for some $g \in \mathcal{G}_4$). For every $t \in \mathcal{G}_4$, $\sigma^g(\sigma^g(t)) = \sigma^g(t)$, which means $g\sigma(\sigma^g(t))g^{-1} = g\sigma(t)g^{-1}$, and this implies $\sigma(\sigma^g(t)) = \sigma(t)$. In the proof of necessity we saw that $\sigma^2 = \sigma$, and so $\sigma(t) = \sigma(\sigma^g(t)) = \sigma(g\sigma(t)g^{-1}) = \sigma(g)\sigma^2(t)\sigma(g^{-1}) = \sigma(g)\sigma(t)\sigma(g^{-1}) = \sigma(gtg^{-1})$. Thus $\sigma(gtg^{-1}t^{-1}) = \sigma(gtg^{-1})\sigma(t)^{-1} = e$. Therefore, $gtg^{-1}t^{-1} \in \mathbf{ker}(\sigma_{|\mathcal{G}_4}) = \mathcal{K}$ for all $t \in \mathcal{G}_4$.

Suppose that g is a transposition $(i\ j)$. If t is a 3-cycle $(i\ j\ k)$, then

$$(i\ j\ k) = (i\ j)(i\ j\ k)(j\ i)(i\ k\ j) = gtg^{-1}t^{-1} \in \mathcal{K}$$

(recall that we apply the factors in the product $gtg^{-1}t^{-1}$ from the right to the left). If g is a 3-cycle $(i\ j\ k)$, choose $t = (i\ j)$. We obtain

$$(i\ k\ j) = (i\ j\ k)(i\ j)(i\ k\ j)(j\ i) = gtg^{-1}t^{-1} \in \mathcal{K}.$$

If g is a 4-cycle $(i\ j\ k\ l)$ (where $\{i, j, k, l\} = \{1, 2, 3, 4\}$), choose $t = (i\ j)$. We obtain

$$(i\ k\ j) = (i\ j\ k\ l)(i\ j)(i\ l\ k\ j)(j\ i) = gtg^{-1}t^{-1} \in \mathcal{K}.$$

It follows that if g is a 2-, 3-, or 4-cycle, then $\mathcal{K}$ contains a 3-cycle, which is not true. Thus either $g = e$ or g is a product of two disjoint transpositions, i.e., $g \in \mathcal{K}$.

It is clear that the number of retractions of rank 1 is the number of idempotents of $\mathcal{T}_n$, and the number of retractions of rank 2 is the number of idempotents less 1 (because retractions of rank 2 are in one-to-one correspondence with idempotents $\gamma \in I_n$). We skip a proof that the number of retractions of rank 3 is indeed given by the formula in the Proposition. The number of retractions of rank 7 is the order of Klein's four-group $\mathcal{K}$, that is, 4. □

A description of all retracts of $\mathcal{T}_n$ easily follows from the Proposition.

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