

## EIGENVALUE PINCHING THEOREMS ON COMPACT SYMMETRIC SPACES

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ABSTRACT. We prove two first eigenvalue pinching theorems for Riemannian symmetric spaces (Theorems 1 and 2). As their application, we answer negatively a question raised by Elworthy and Rosenberg, who proposed to show that for every compact simple Lie group  $G$  with a bi-invariant Riemannian metric  $h$  on  $G$  with respect to  $-\frac{1}{2}B$ ,  $B$  being the Killing form of the Lie algebra  $\mathfrak{g}$ , the first eigenvalue  $\lambda_1(h)$  would satisfy

$$\sum_{j=1}^2 \sum_{\ell=3}^n |[v_j, v_\ell]|^2 > n(2\lambda_1(h) - 1),$$

for all orthonormal bases  $\{v_j\}_{j=1}^n$  of tangent spaces of  $G$  (cf. Corollary 3). This problem arose in an attempt to give a spectral geometric proof that  $\pi_2(G) = 0$  for a Lie group  $G$ .

### §1. EIGENVALUE PINCHING THEOREMS

Lichnerowicz and Obata's celebrated theorem says (cf. [2]) that if the Ricci curvature of an  $n$ -dimensional compact Riemannian manifold  $(M, g)$  satisfies

$$\text{Ric}_g \geq n - 1,$$

then the first eigenvalue  $\lambda_1(g)$  of the Laplacian  $\Delta_g$  satisfies

$$\lambda_1(g) \geq n,$$

and equality holds if and only if  $(M, g)$  is the standard unit sphere  $(S^n, g_0)$  of curvature 1.

It is known (cf. [8], [7], [3], [1]) that the following eigenvalue pinching theorem holds:

**Theorem.** *Let  $(M, g)$  be a compact  $n$ -dimensional Riemannian manifold with sectional curvature  $K_g \geq 1$ . Then, there exists a constant  $C(n) > 1$ , depending only on  $n$ , such that, if  $C(n)n \geq \lambda_1(g) \geq n$ ,  $M$  is homeomorphic to  $S^n$ .*

It would be a difficult problem to determine such a constant  $C(n)$  explicitly. We would propose the conjecture that

$$C(n) \geq \left(\frac{4}{n} + 1\right) \frac{n-1}{n} > 1.$$

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Indeed, we prove under a strong symmetry condition,

**Theorem 1.** *Let  $(M, g)$  be a compact  $n$ -dimensional Riemannian manifold with Ricci curvature*

$$\text{Ric}_g \geq n - 1.$$

*Assume that*

$$\left(\frac{4}{n} + 1\right)(n - 1) \geq \lambda_1(g) \geq n,$$

*and  $(M, g)$  is a simply connected, irreducible, Riemannian symmetric space. Then  $(M, g)$  is the standard sphere  $(S^n, g_0)$ .*

*Proof.* All simply connected compact irreducible Riemannian symmetric spaces are divided into two classes: type I or type II (cf. [5]). The spaces of type II are of the form  $G/K$  with a compact simple Lie group  $G$  and some closed subgroup  $K$ , and the invariant metric is given by  $g = \frac{1}{2(n-1)}h$ , where  $h$  is a  $G$ -invariant Riemannian metric on  $M = G/K$  induced from the Killing form of the Lie algebra  $\mathfrak{g}$  of  $G$ . Then the Ricci tensor  $\text{Ric}_h$  satisfies

$$\text{Ric}_h = \frac{1}{2}h.$$

All the first eigenvalues of  $(M, h)$  are listed in the tables in [9]. A space of type I is a compact simply connected simple Lie group  $G$ , and  $g = \frac{1}{4(n-1)}g'$ . Here  $g'$  is a bi-invariant metric on  $G$  induced from the Killing form of  $\mathfrak{g}$  whose Ricci tensor satisfies

$$\text{Ric}_{g'} = \frac{1}{4}g',$$

and one can also find in [9] a list of the first eigenvalues of  $(G, g')$ . Due to these tables, one can see that all the first eigenvalues of  $(M, g)$ , except when  $(M, g)$  is the unit sphere  $(S^n, g_0)$ , are bigger than  $\left(\frac{4}{n} + 1\right)(n - 1)$ .  $\square$

## §2. ISOMETRIC IMMERSIONS

In [6], Lawson and Simons proved the following theorem.

**Theorem.** *Let  $(M, g)$  be an  $n$ -dimensional compact Riemannian manifold isometrically immersed in the unit sphere  $S^N(1)$  with second fundamental form  $\beta$ . Assume that*

$$\sum_{j=1}^q \sum_{\ell=q+1}^n (2|\beta(v_j, v_\ell)|^2 - \langle \beta(v_j, v_j), \beta(v_\ell, v_\ell) \rangle) < q(n - q),$$

*for every orthonormal base  $\{v_j\}_{j=1}^n$  of  $T_x M$  ( $x \in M$ ). Then the homology groups of  $M$  vanish,  $H_q(M, \mathbf{Z}) = H_{n-q}(M, \mathbf{Z}) = 0$ .*

In [4], Elworthy and Rosenberg proved

**Theorem.** *Let  $(M, g)$  be an  $n$ -dimensional compact Riemannian manifold isometrically minimally immersed in  $S^N(1)$ . Assume that*

$$\sum_{j=1}^q \sum_{\ell=q+1}^n K_g(v_j, v_\ell) > \frac{q(n - q)}{2},$$

for every orthonormal base  $\{v_j\}_{j=1}^n$  of  $T_xM$ ,  $x \in M$ . Then

$$H_q(M, \mathbf{Z}) = H_{n-q}(M, \mathbf{Z}) = 0.$$

In particular, if  $K_g(v_j, v_\ell) > \frac{1}{2}$  for all  $v_j, v_\ell \in T_xM$  ( $x \in M$ ),  $j < \ell$ , then  $M$  is a homotopy sphere. Here  $K_g(X, Y)$ , for an orthonormal pair  $X, Y \in T_xM$ , means the sectional curvature of  $(M, g)$  for a plane in  $T_xM$  defined by  $X$  and  $Y$ .

We show:

**Theorem 2.** *Let  $(M, h)$  be a compact simply connected irreducible Riemannian symmetric space and  $h$  an invariant Riemannian metric induced from the Killing form of the Lie algebra of the isometry group of  $(M, h)$ . Assume that*

$$(\#) \quad \sum_{j=1}^2 \sum_{\ell=3}^n K_h(v_j, v_\ell) > \frac{n}{4}(2\lambda_1(h) - 1),$$

for all orthonormal bases  $\{v_j\}_{j=1}^n$  of  $T_xM$ ,  $x \in M$ . Then  $(M, h)$  is the unit sphere  $(S^n, 2(n-1)g_0)$  of constant curvature  $\frac{1}{2(n-1)}$ .

*Proof.* In general, for a Riemannian manifold  $(M, g)$ ,

$$\begin{aligned} \sum_{j=1}^2 \sum_{\ell=3}^n K_g(v_j, v_\ell) &= \sum_{j=1}^2 \sum_{\ell=1}^n K_g(v_j, v_\ell) - \sum_{j,\ell=1}^2 K_g(v_j, v_\ell) \\ &= \text{Ric}_g(v_1) + \text{Ric}_g(v_2) - 2K_g(v_1, v_2) \\ &\geq 2(\inf \text{Ric}_g - \max K_g). \end{aligned}$$

For our case, since  $(M, h)$  satisfies  $\text{Ric}_h = \frac{1}{2}h$ ,

$$\sum_{j=1}^2 \sum_{\ell=3}^n K_h(v_j, v_\ell) = 1 - 2K_h(v_1, v_2),$$

which yields

$$0 \leq \inf \sum_{j=1}^2 \sum_{\ell=3}^n K_h(v_j, v_\ell) = 1 - 2 \max K_h \leq 1.$$

Thus, the inequality  $(\#)$  implies  $1 \geq \frac{n}{4}(2\lambda_1(h) - 1)$ .

But, if we put  $g = \frac{1}{2(n-1)}h$ , then  $\text{Ric}_g = (n-1)g$ , and the first eigenvalue  $\lambda_1(g)$  of  $g$  satisfies

$$\left(\frac{4}{n} + 1\right)(n-1) \geq \lambda_1(g).$$

Therefore, we have Theorem 2, because of Theorem 1. □

Theorem 2 immediately yields

**Corollary 3.** *Let  $G$  be a compact simply connected simple Lie group, and  $h$  a bi-invariant Riemannian metric  $G$  induced from the Killing form  $B$  of the Lie algebra  $\mathfrak{g}$  of  $G$  by identifying  $G \cong G \times G/\Delta G$  as a symmetric space. That is,*

$$h_x(X_x, Y_x) = -\frac{1}{2}B(X, Y), \quad X, Y \in \mathfrak{g},$$

where  $X_x, Y_x \in T_x G$ ,  $x \in G$ . Assume that

$$\sum_{j=1}^2 \sum_{\ell=3}^n |[v_j, v_\ell]|^2 > n(2\lambda_1(h) - 1),$$

for every orthonormal base  $\{v_j\}_{j=1}^n$  of  $\mathfrak{g}$  with respect to  $-\frac{1}{2}B$ . Then  $(M, h)$  is  $(SU(2), h)$ .

*Remark 4.* (1) In [4, pp. 74-75], Elworthy and Rosenberg proposed to give a spectral geometric proof that  $\pi_2(G) = 0$  for a Lie group  $G$ . The problem has been reduced to a question as to whether the above inequality in Corollary 3 would hold for all compact simply connected simple Lie group. But our Corollary 3 answers their question negatively.

(2) Let  $(M, h)$  be a compact simply connected irreducible Riemannian symmetric space. Let

$$\Phi_1 : (M, \frac{\lambda_1}{n}h) \rightarrow S^N(1)$$

be an isometric minimal immersion into the unit sphere defined by the first eigenfunctions of  $(M, h)$ . Let  $\beta$  be the second fundamental form and  $\tilde{h} = \frac{\lambda_1}{n}h$ . The inequality (#) in Theorem 2 is equivalent to

$$\sum_{j=1}^2 \sum_{\ell=3}^n K_{\tilde{h}}(v'_j, v'_\ell) > \frac{|\beta|^2}{2} + \frac{n}{2},$$

for every orthonormal base  $\{v'_j\}_{j=1}^n$  of  $T_x M$  ( $x \in M$ ) relative to  $\tilde{h}$ .

(3) Note that

$$\max K_h = \max\{|\alpha|^2; \alpha \in \Sigma^+(G, K)\},$$

where  $\Sigma^+(G, K)$  is the set of all positive roots  $\alpha$  of  $G$  relative to a Cartan subalgebra  $\mathfrak{t}$  of  $\mathfrak{g}$  whose restrictions  $\alpha|_{\mathfrak{a}}$  to  $\mathfrak{a}$  do not vanish. Here  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$  is the Cartan decomposition of  $\mathfrak{g}$ , and the Cartan subalgebra  $\mathfrak{t}$  is by definition a maximal abelian subalgebra of  $\mathfrak{g}$  containing a maximal abelian subalgebra  $\mathfrak{a}$  of  $\mathfrak{g}$  contained in  $\mathfrak{m}$ .

(4) If a Riemannian symmetric space  $(M, g)$  satisfies the inequality (#), then there might exist a  $q$ -th moment stable equation for  $q \geq \frac{\dim M}{2}$  (cf. [4, p. 59]).

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