THE FUNDAMENTAL GROUP
OF A COMPACT METRIC SPACE

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Abstract. We give a forcing free proof of a conjecture of Mycielski that the fundamental group of a connected locally connected compact metric space is either finitely generated or has the power of the continuum.

Shelah \[S\], using models, absoluteness and Cohen’s forcing method, gives a proof of the following conjecture of Mycielski.

Theorem. Suppose that \(X\) is a compact metric space, which is connected and locally connected. Then the fundamental group of \(X\) is either finitely generated or has the power of the continuum.

Thus in particular the group of rationals can’t be the fundamental group of a connected locally connected compact metric space.

We can’t drop ‘compact’ in the theorem — any countably generated group can be realized as the fundamental group of a Polish space, see \[Sp\]. We also can’t drop ‘locally connected’ — the fundamental group of the ‘tail of the peacock’ is free with a countable infinity of generators. It seems to be open, however, whether a finitely generated group can be realized as the fundamental group of a compact metric space.

We present a forcing free proof of Mycielski’s conjecture.

Definitions. Let \(X\) be a metric space. A path from \(x_0\) to \(x_1\) inside \(V \subseteq X\) is a continuous function \(f : [0, 1] \rightarrow V\) with \(f(0) = x_0\) and \(f(1) = x_1\). \(f\) is a loop at \(x\) if \(f(0) = f(1) = x\). The reversal of \(f\), denoted by \(f^{-1}\), is a path from \(f(1)\) to \(f(0)\) defined by \(f^{-1}(t) = f(1 - t)\). The diameter of \(f\) is the diameter of the set \(\{f(t) : t \in [0, 1]\}\).

A path \(f\) is homotopic to another path \(g\), \(f \sim g\), if there is a homotopy from \(f\) to \(g\), i.e., a continuous function \(F : [0, 1] \times [0, 1] \rightarrow X\) such that \(F(t, 0) = f(t)\), \(F(t, 1) = g(t)\), \(F(0, s) = f(0) = g(0)\) and \(F(1, s) = f(1) = g(1)\) (thus endpoints are kept constant; this is usually called ‘a homotopy relative to \(\{0, 1\}\)’). If \(f\) is a path from \(x_0\) to \(x_1\) and \(g\) a path from \(x_1\) to \(x_2\), then \(f \ast g\) is the concatenation of \(f\) and \(g\), i.e., a path from \(x_0\) to \(x_2\) defined by \((f \ast g)(t) = f(2t)\) for \(t \in [0, 1/2]\) and \((f \ast g)(t) = g(2t - 1)\) for \(t \in [1/2, 1]\).

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\( g(2t - 1) \) for \( t \in [1/2, 1] \). The iterated concatenation \( f_0 \ast (f_1 \ast \cdots \ast f_n) \) is written as \( f_0 \ast \cdots \ast f_n \).

The relation \( \sim \) is an equivalence relation. If \( X \) is path connected, the set of equivalence classes of loops at a given point \( x \in X \), with multiplication and inverse defined from * and \(^{-1}\), is a group called the fundamental group of \( X \). This group doesn’t depend on the choice of \( x \).

Let \( \mathbb{N} \) be the set of nonnegative integers. Let \( x \in X \) and suppose that \( f_n (n \in \mathbb{N}) \) are loops at \( x \) whose diameters converge to 0. Then \( (f_0 \ast f_1 \ast \cdots) \) is the pointwise limit of \( f_0 \ast \cdots \ast f_n \). The limit exists and is continuous, so it is a loop at \( x \). In fact, if \( t \in [1 - 2^{-n}, 1 - 2^{-(n+1)}] \), then

\[
(f_0 \ast f_1 \ast \cdots)(t) = f_n((t - (1 - 2^{-n})) \cdot 2^{n+1}).
\]

Note also that

\[
(f_0 \ast f_1 \ast \cdots) = (f_0 \ast \cdots \ast f_n) \ast (f_{n+1} \ast \cdots).
\]

From now on we assume that \( X \) is a connected locally connected compact metric space. Then \( X \) is also path connected and locally path connected (a theorem of Mazurkiewicz, see [K]).

**Lemma 1.** Suppose that the fundamental group of \( X \) is not finitely generated. Then there exists \( x \in X \) such that for each \( n \in \mathbb{N} \) there exists a loop \( f_n \) at \( x \) which is of diameter \(< 2^{-n}\) and which is not homotopic to the constant loop at \( x \).

**Proof.** Suppose otherwise. Then for each \( n \in X \) there exists \( n(x) \in \mathbb{N} \) such that every loop at \( x \) which has diameter \(< 2^{-n}\) is homotopic to the constant loop at \( x \). By compactness there exists a cover of \( X \) by path connected sets \( V_i \ (i < k) \) and there exist points \( x_i \in V_i \ (i < k) \) such that for every \( i \) the diameter of \( V_i \) is \(< 2^{-(n(x_i)+1)}\) and any loop at \( x_i \) which has diameter \(< 2^{-n(x_i)}\) is homotopic to the constant loop at \( x_i \).

Fix a path \( g_i \) from \( x_0 \) to \( x_1 \ (i < k) \); \( g_0 \) = the constant loop at \( x_0 \). For \( i \) and \( j \) such that \( V_i \cap V_j \neq \emptyset \) fix a path \( h_{ij} \) in \( V_i \cup V_j \) going from \( x_i \) to \( x_j \). Note that any path \( s \) from \( x_i \) to \( x_j \) which is contained in \( V_i \cup V_j \) is homotopic to \( h_{ij} \). Indeed, suppose that \( n(x_i) \leq n(x_j) \). Then \( s \ast h_{ij}^{-1} \) is a loop at \( x_i \) and has diameter \(< 2^{-n(x_i)}\), so \( s \ast h_{ij}^{-1} \) is homotopic to the constant loop at \( x_i \). An elementary manipulation of this homotopy gives a homotopy from \( s \) to \( h_{ij} \). If \( n(x_i) > n(x_j) \), consider \( h_{ij}^{-1} \ast s \), a loop at \( x_j \).

We shall show that the fundamental group of \( X \) is generated by the (homotopy classes of) loops \( \widetilde{h}_{ij} = g_i \ast h_{ij} \ast g_j^{-1} \). To this end, suppose that a loop \( s \) at \( x_0 \) is given. By a change of scale \( s \sim s_0 \ast \cdots \ast s_l \), where each \( s_i \) is a path inside one piece of our cover. Say \( s_i \) goes from \( y_i \) to \( y_{i+1} \) inside \( V_{\phi(i)} \); \( y_0 = y_{l+1} = x_0 \); \( \phi(0) = \phi(l) = 0 \). For \( i = 0, \ldots, l \), fix inside \( V_{\phi(i)} \) a path \( t_i \) from \( y_{i+1} \) to \( x_{\phi(i)} \); \( t_0 = s_0^{-1} \), \( t_l = \) the constant loop at \( x_0 \). Let \( \widetilde{s}_i = t_{i+1}^{-1} \ast s_i \ast t_i \ (i = 1, \ldots, l) \). Clearly

\[
\widetilde{s}_1 \ast \cdots \ast \widetilde{s}_l \sim s.
\]

Also, each \( \widetilde{s}_i \), being a path from \( x_{\phi(i-1)} \) to \( x_{\phi(i)} \) inside \( V_{\phi(i-1)} \cup V_{\phi(i)} \), must be homotopic to \( h_{\phi(i-1)\phi(i)} \). Thus

\[
s \sim h_{\phi(0)\phi(1)} \ast \cdots \ast h_{\phi(l-1)\phi(l)}.
\]

and hence also

\[
s \sim \widetilde{h}_{\phi(0)\phi(1)} \ast \cdots \ast \widetilde{h}_{\phi(l-1)\phi(l)}.
\]
For the sequel suppose that the fundamental group of \( X \) is not finitely generated and let \( x \) and \( f_n \) (\( n \in \mathbb{N} \)) be as claimed by Lemma 1. We shall find a set of size of the continuum of mutually non-homotopic loops. For \( \alpha \in \{0, 1\}^\mathbb{N} \) let \( f_n^\alpha = \) the constant loop at \( x \) if \( \alpha(n) = 0 \), and let \( f_n^\alpha = f_n \) otherwise. Define a loop \( f_n \) at \( x \) as \((f_0^\alpha \ast f_1^\beta \ast \cdots)\). Write \( \alpha \approx \beta \) if \( f_\alpha \sim f_\beta \). Then \( \approx \) is an equivalence relation in \( \{0, 1\}^\mathbb{N} \). It is enough to prove that \( \approx \) has continuum many equivalence classes.

**Lemma 2.** Suppose that \( \alpha \) and \( \beta \) from \( \{0, 1\}^\mathbb{N} \) differ exactly at one point. Then \( \alpha \not\approx \beta \).

**Proof.** Suppose that \( f_\alpha \sim f_\beta \). Let \( n \) be the unique point at which \( \alpha \) and \( \beta \) are different. Then, for \( m \neq n \) we have \( f_m^\alpha = f_m^\beta \), hence \((f_0^\alpha \ast \cdots \ast f_{n-1}^\alpha) = (f_0^\beta \ast \cdots \ast f_{n-1}^\beta)\) and \((f_n^\alpha \ast \cdots) = (f_n^\beta \ast \cdots)\). Thus from

\[
(f_0^\alpha \ast \cdots \ast f_{n-1}^\alpha \ast f_n^\alpha \ast f_{n+1}^\beta \ast \cdots) \sim (f_0^\beta \ast \cdots \ast f_{n-1}^\beta \ast f_n^\beta \ast f_{n+1}^\beta \ast \cdots),
\]

we get \( f_n^\alpha \sim f_n^\beta \), which is a contradiction. \( \square \)

**More definitions.** We recall some basic facts about Polish spaces (see \([K]\)). A Polish space is a completely metrizable separable space. Let \( Y \) be a Polish space. For a subset \( A \) of \( Y \): \( A \) is nowhere dense if its closure has empty interior, \( A \) is meager if it is a countable union of nowhere dense sets, \( A \) is comeager in an open set \( U \) if \( U \setminus A \) is meager. \( A \) has the Baire property if its symmetric difference with some open set is meager. A nonmeager set with the Baire property is comeager in some nonempty open set. The Baire category theorem implies that a set which is comeager in a nonempty open set is nonmeager. The Kuratowski-Ulam theorem implies that if \( A \subseteq Y \times Z \) is comeager in \( U \times V \), where \( U \) and \( V \) are open subsets of Polish spaces \( Y \) and \( Z \), then

\[
\{ y \in U : \{ z \in V : \langle y, z \rangle \in A \} \text{ is comeager in } V \}
\]

is comeager in \( U \).

A subset \( A \) of \( Y \) is analytic if there exist a Polish space \( Z \) and a closed set \( D \subseteq Y \times Z \) such that \( A \) is the projection of \( D \) into \( Y \). Analytic sets have the Baire property. Continuous preimages of analytic sets are analytic.

A subset \( P \) of \( Y \) is perfect if it is closed, nonempty, and has no isolated points. Perfect sets have the power of the continuum.

The set \( \{0, 1\}^\mathbb{N} \) becomes a Polish space (homeomorphic to the Cantor discontinuum) when viewed as the product of countably many copies of the two-point discrete space \( \{0, 1\} \). The canonical basis of \( \{0, 1\}^\mathbb{N} \) is the collection of all sets \([\sigma] = \{ \alpha : \sigma \subseteq \alpha \} \), where \( \sigma \) is a finite zero-one sequence, i.e., \( \sigma : \{0, 1, \ldots, n-1\} \mapsto \{0, 1\} \) for some \( n \).

**Lemma 3.** \( \approx \) has the Baire property as a subset of \( \{0, 1\}^\mathbb{N} \times \{0, 1\}^\mathbb{N} \).

**Proof.** Let \( H \) and \( H \) be respectively the spaces of all loops at \( x \) and all homotopies between them, endowed with the sup metric. Both spaces are Polish. The homotopy relation \( \sim \) restricted to \( H \) is an analytic subset of \( H \times H \). Indeed, it is the projection onto \( H \times H \) of

\[
\{ (\langle f, g \rangle, F) : F \text{ is a homotopy from } f \text{ to } g \},
\]
which is a closed subset of \((H \times H) \times \mathbb{H}\). Note also that the function from \(\{0, 1\}^N\) to \(H\) which takes \(\alpha\) to \(f_\alpha\) is continuous. It follows that \(\approx\) is analytic (as a continuous preimage of an analytic set), and thus has the Baire property.

\section*{Lemma 4.} If \(E \subseteq \{0, 1\}^N \times \{0, 1\}^N\) is an equivalence relation which has the Baire property and if \(\neg xEy\) whenever \(x\) and \(y\) differ by one coordinate only, then \(E\) is meager.

\section*{Proof.} Should \(E\) be nonmeager, it would be comeager in some basic neighbourhood \([\sigma] \times [\tau]\). By the Kuratowski-Ulam theorem,

\[ A = \{\alpha \in [\sigma] : \{\beta \in [\tau] : \alpha \approx \beta\} \text{ is comeager in } [\tau]\} \]

is comeager in \([\sigma]\). Let \(n > \text{the length of } \sigma\). Consider a map \(\Phi : [\sigma] \rightarrow [\sigma]\) defined by \(\Phi(\alpha)(n) = 1 - \alpha(n)\) and \(\Phi(\alpha)(i) = \alpha(i)\) for \(i \neq n\). \(\Phi\) is a homeomorphism and thus \(\Phi[A]\) is comeager in \([\sigma]\). Choose \(\alpha \in A \cap \Phi[A]\) and let \(\gamma = \Phi(\alpha)\). Then \(\alpha\) and \(\gamma\) differ only at \(n\), hence \(\neg \alpha E \gamma\). Also, by the definition of \(A\), we have in \([\tau]\) comeagerly many \(\beta\) with \(\alpha \approx \beta\). As \(\gamma \in A\), the same is true about \(\gamma\). Thus there exists \(\beta\) with \(\alpha \approx \beta\) and \(\gamma \approx \beta\). But then \(\alpha \approx \gamma\), which is a contradiction. \(\square\)

\section*{Remark.} Another way to see that \(E\) is meager might be as follows. Suppose for contradiction that \(E\) is nonmeager. Consider \(G = \{0, 1\}^N \times \{0, 1\}^N\) as a Polish group with coordinatewise addition mod 2. By the Baire category version of a theorem of Steinhaus (see [O]), if \(B \subseteq G\) has the Baire property and is nonmeager, then the difference set \(B - B = \{b_0 - b_1 : b_0, b_1 \in B\}\) contains a neighbourhood of the unit element \((0, 0)\) (here \(0 = (0, 0, \ldots)\)). So, for each \((\delta, \epsilon) \in G\), which is close enough to \((0, 0)\), there exist \((\alpha, \beta) \in B\) such that \((\alpha + \gamma, \beta + \delta) \in B\). For \(n \in \mathbb{N}\) let \(\epsilon_n \in \{0, 1\}^N\) be the function that takes value 1 at \(n\) and 0 elsewhere. Then \((\epsilon_n, 0) \rightarrow (0, 0)\), when \(n \rightarrow \infty\). So, for large enough \(n\) there exists \((\alpha, \beta) \in B\) with \((\alpha + \epsilon_n, \beta) \in B\). Applied to \(B = E\) this yields that for large enough \(n\) there exist \(\alpha\) and \(\beta\) such that \(\alpha \approx \beta\) and \(\alpha \approx \epsilon_n \approx \beta\), whence \(\alpha \approx \epsilon_n \approx \beta\). This contradicts Lemma 2.

Similar arguments show that if \(E\) is Lebesgue measurable then it must be null.

\section*{Corollary.} \(\approx\) is meager.

Recall now the following theorem of Mycielski [M].

\section*{Theorem (Mycielski).} Suppose that \(Y\) is a Polish space without isolated points and that \(R \subseteq Y \times Y\) is meager. Then there exists a perfect set \(P \subseteq Y\) such that if \(\alpha\) and \(\beta\) are distinct points of \(P\) then \((\alpha, \beta) \notin R\).

Applying this theorem to \(\approx\) and \(\{0, 1\}^N\) we get a perfect set of mutually \(\approx\) non-equivalent elements of \(\{0, 1\}^N\). The proof of Mycielski’s conjecture is complete.

A slight modification of the above proof gives the following theorem.

\section*{Theorem.} Let \(\kappa < 2^{8\kappa}\) be an infinite cardinal number. Suppose that \(X\) is a path connected locally path connected metric space which is \(\kappa\)-Lindelöf (i.e., every open cover of \(X\) has a subcover of size \(\leq \kappa\)). Then the power of the fundamental group of \(X\) is either \(\leq \kappa\) or \(2^{8\kappa}\).

\section*{References}


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