THE FUNDAMENTAL GROUP
OF A COMPACT METRIC SPACE

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(Communicated by Andreas R. Blass)

Abstract. We give a forcing free proof of a conjecture of Mycielski that the fundamental group of a connected locally connected compact metric space is either finitely generated or has the power of the continuum.

Shelah [S], using models, absoluteness and Cohen’s forcing method, gives a proof of the following conjecture of Mycielski.

Theorem. Suppose that $X$ is a compact metric space, which is connected and locally connected. Then the fundamental group of $X$ is either finitely generated or has the power of the continuum.

Thus in particular the group of rationals can’t be the fundamental group of a connected locally connected compact metric space.

We can’t drop ‘compact’ in the theorem — any countably generated group can be realized as the fundamental group of a Polish space, see [Sp]. We also can’t drop ‘locally connected’ — the fundamental group of the ‘tail of the peacock’ is free with a countable infinity of generators. It seems to be open, however, whether a finitely generated group can be realized as the fundamental group of a compact metric space.

We present a forcing free proof of Mycielski’s conjecture.

Definitions. Let $X$ be a metric space. A path from $x_0$ to $x_1$ inside $V \subseteq X$ is a continuous function $f : [0,1] \to V$ with $f(0) = x_0$ and $f(1) = x_1$. $f$ is a loop at $x$ if $f(0) = f(1) = x$. The reversal of $f$, denoted by $f^{-1}$, is a path from $f(1)$ to $f(0)$ defined by $f^{-1}(t) = f(1-t)$. The diameter of $f$ is the diameter of the set $\{f(t) : t \in [0,1]\}$.

A path $f$ is homotopic to another path $g$, $f \sim g$, if there is a homotopy from $f$ to $g$, i.e., a continuous function $F : [0,1] \times [0,1] \to X$ such that $F(t,0) = f(t)$, $F(t,1) = g(t)$, $F(0,s) = f(0) = g(0)$ and $F(1,s) = f(1) = g(1)$ (thus endpoints are kept constant; this is usually called ‘a homotopy relative to $\{0,1\}$’). If $f$ is a path from $x_0$ to $x_1$ and $g$ a path from $x_1$ to $x_2$, then $f \ast g$ is the concatenation of $f$ and $g$, i.e., a path from $x_0$ to $x_2$ defined by $(f \ast g)(t) = f(2t)$ for $t \in [0,1/2]$ and $(f \ast g)(t) = g(2t-1)$ for $t \in [1/2,1]$.

Received by the editors August 17, 1996 and, in revised form, February 26, 1997.

1991 Mathematics Subject Classification. Primary 03E15, 55Q05; Secondary 04A20, 55Q52.

Key words and phrases. Baire category, fundamental group, perfect set.

The author was partially supported by KBN grant 2 P03A 011 09. The author thanks J. Mycielski for introducing him to [S].

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Suppose that the fundamental group of \( M \) is not finitely generated. Then there exists \( x \in X \) such that for each \( n \in \mathbb{N} \) there exists a loop \( f_n \) at \( x \) which is of diameter \( < 2^{-n} \) and which is not homotopic to the constant loop at \( x \).

**Proof.** Suppose otherwise. Then for each \( x \in X \) there exists \( n(x) \in \mathbb{N} \) such that every loop at \( x \) which has diameter \( < 2^{-n(x)} \) is homotopic to the constant loop at \( x \). By compactness there exists a cover of \( X \) by path connected sets \( V_i \) \((i < k)\) and there exist points \( x_i \in V_i \) \((i < k)\) such that for every \( i \) the diameter of \( V_i \) is \( < 2^{-(n(x_i)+1)} \) and any loop at \( x_i \) which has diameter \( < 2^{-(n(x_i))} \) is homotopic to the constant loop at \( x_i \).

Fix a path \( g_i \) from \( x_0 \) to \( x_1 \) \((i < k)\); \( g_0 = \) the constant loop at \( x_0 \). For \( i \) and \( j \) such that \( V_i \cap V_j \neq \emptyset \) fix a path \( h_{ij} \) in \( V_i \cup V_j \) going from \( x_i \) to \( x_j \). Note that any path \( s \) from \( x_i \) to \( x_j \) which is contained in \( V_i \cup V_j \) is homotopic to \( h_{ij} \). Indeed, suppose that \( n(x_i) \leq n(x_j) \). Then \( s \cdot h_{ij}^{-1} \) is a loop at \( x_i \) and has diameter \( < 2^{-(n(x_i))} \), so \( s \cdot h_{ij}^{-1} \) is homotopic to the constant loop at \( x_i \). An elementary manipulation of this homotopy gives a homotopy from \( s \) to \( h_{ij} \). If \( n(x_i) > n(x_j) \), consider \( h_{ij}^{-1} \cdot s \), a loop at \( x_j \).

We shall show that the fundamental group of \( X \) is generated by the (homotopy classes of) loops \( h_{ij} \). To this end, suppose that a loop \( s \) at \( x_0 \) is given. By a change of scale \( s \sim s_{01} \cdots s_{1l} \), where each \( s_i \) is a path inside one piece of our cover. Say \( s_i \) goes from \( y_i \) to \( y_{i+1} \) inside \( V_{\phi(i)} \); \( y_0 = y_{l+1} = x_0, \phi(0) = \phi(l) = 0 \). For \( i = 0, \ldots, l \) fix inside \( V_{\phi(i)} \) a path \( t_i \) from \( y_{i+1} \) to \( x_{\phi(i)} \); \( t_0 = s_0^{-1}, t_l = \) the constant loop at \( x_0 \). Let \( \tilde{s}_i = t_i^{-1} \cdot s_i \cdot t_i \) \((i = 1, \ldots, l)\). Clearly
\[
\tilde{s}_1 \cdots \tilde{s}_l \sim s.
\]
Also, each \( \tilde{s}_i \), being a path from \( x_{\phi(i-1)} \) to \( x_{\phi(i)} \) inside \( V_{\phi(i-1)} \cup V_{\phi(i)} \), must be homotopic to \( h_{\phi(i-1)\phi(i)} \). Thus
\[
s \sim h_{\phi(0)\phi(1)} \cdots h_{\phi(l-1)\phi(l)}.
\]
and hence also
\[
s \sim \tilde{h}_{\phi(0)\phi(1)} \cdots \tilde{h}_{\phi(l-1)\phi(l)}. \quad \square
\]
For the sequel suppose that the fundamental group of $X$ is not finitely generated and let $x$ and $f_n$ ($n \in \mathbb{N}$) be as claimed by Lemma 1. We shall find a set of size of the continuum of mutually non-homotopic loops. For $\alpha \in \{0, 1\}^\mathbb{N}$ let $f_n^\alpha = \text{the constant loop at } x$ if $\alpha(n) = 0$, and let $f_n^\alpha = f_n$ otherwise. Define a loop $f_n$ at $x$ as $(f_0^\alpha * f_1^\alpha * \cdots )$. Write $\alpha \approx \beta$ if $f_\alpha \sim f_\beta$. Then $\approx$ is an equivalence relation in $\{0, 1\}^\mathbb{N}$. It is enough to prove that $\approx$ has continuum many equivalence classes.

**Lemma 2.** Suppose that $\alpha$ and $\beta$ from $\{0, 1\}^\mathbb{N}$ differ exactly at one point. Then $\alpha \not\approx \beta$.

**Proof.** Suppose that $f_\alpha \sim f_\beta$. Let $n$ be the unique point at which $\alpha$ and $\beta$ are different. Then, for $m \neq n$ we have $f_m^\alpha = f_m^\beta$, hence $(f_0^\alpha * \cdots * f_{n-1}^\alpha) = (f_0^\beta * \cdots * f_{n-1}^\beta)$ and $(f_{n+1}^\alpha * \cdots ) = (f_{n+1}^\beta * \cdots )$. Thus from

$$
(f_0^\alpha * \cdots * f_{n-1}^\alpha * f_n^\alpha * f_{n+1}^\alpha * \cdots ) \sim (f_0^\beta * \cdots * f_{n-1}^\beta * f_n^\beta * f_{n+1}^\beta * \cdots ),
$$

we get $f_n^\alpha \sim f_n^\beta$, which is a contradiction. 

**More definitions.** We recall some basic facts about Polish spaces (see [K]). A Polish space is a completely metrizable separable space. Let $Y$ be a Polish space. For a subset $A$ of $Y$: $A$ is nowhere dense if its closure has empty interior, $A$ is meager if it is a countable union of nowheredense sets, $A$ is comeager in an open set $U$ if $U \setminus A$ is meager, $A$ has the Baire property if its symmetric difference with some open set is meager. A nonmeager set with the Baire property is comeager in some nonempty open set. The Baire category theorem implies that a set which is comeager in a nonempty open set is nonmeager. The Kuratowski-Ulam theorem implies that if $A \subseteq Y \times Z$ is comeager in $U \times V$, where $U$ and $V$ are open subsets of Polish spaces $Y$ and $Z$, then

$$
\{y \in U : \{z \in V : (y, z) \in A\} \text{ is comeager in } V\}
$$

is comeager in $U$.

A subset $A$ of $Y$ is analytic if there exist a Polish space $Z$ and a closed set $D \subseteq Y \times Z$ such that $A$ is the projection of $D$ into $Y$. Analytic sets have the Baire property. Continuous preimages of analytic sets are analytic.

A subset $P$ of $Y$ is perfect if it is closed, nonempty, and has no isolated points. Perfect sets have the power of the continuum.

The set $\{0, 1\}^\mathbb{N}$ becomes a Polish space (homeomorphic to the Cantor discontinuum) when viewed as the product of countably many copies of the two-point discrete space $\{0, 1\}$. The canonical basis of $\{0, 1\}^\mathbb{N}$ is the collection of all sets $[\sigma] = \{\alpha : \sigma \subseteq \alpha\}$, where $\sigma$ is a finite zero-one sequence, i.e., $\sigma : \{0, 1, \ldots, n-1\} \mapsto \{0, 1\}$ for some $n$.

**Lemma 3.** $\approx$ has the Baire property as a subset of $\{0, 1\}^\mathbb{N} \times \{0, 1\}^\mathbb{N}$.

**Proof.** Let $H$ and $\mathbb{H}$ be respectively the spaces of all loops at $x$ and all homotopies between them, endowed with the sup metric. Both spaces are Polish. The homotopy relation $\sim$ restricted to $H$ is an analytic subset of $H \times H$. Indeed, it is the projection onto $H \times H$ of

$$
\{((f, g), F) : F \text{ is a homotopy from } f \text{ to } g\},
$$
which is a closed subset of $(H \times H) \times \mathbb{H}$. Note also that the function from $\{0,1\}^N$ to $H$ which takes $\alpha$ to $f_\alpha$ is continuous. It follows that $\approx$ is analytic (as a continuous preimage of an analytic set), and thus has the Baire property. \hfill \qed

Lemma 4. If $E \subseteq \{0,1\}^N \times \{0,1\}^N$ is an equivalence relation which has the Baire property and if $\neg xEy$ whenever $x$ and $y$ differ by one coordinate only, then $E$ is meager.

Proof. Should $E$ be nonmeager, it would be comeager in some basic neighbourhood $[\sigma] \times [\tau]$. By the Kuratowski-Ulam theorem,

$$A = \{ \alpha \in [\sigma] : \{ \beta \in [\tau] : \alpha E \beta \} \text{ is comeager in } [\tau] \}$$

is comeager in $[\sigma]$. Let $n$ be the length of $\sigma$. Consider a map $\Phi : [\sigma] \rightarrow [\sigma]$ defined by $\Phi(\alpha)(n) = 1 - \alpha(n)$ and $\Phi(\alpha)(i) = \alpha(i)$ for $i \neq n$. $\Phi$ is a homeomorphism and thus $\Phi[A]$ is comeager in $[\sigma]$. Choose $\alpha \in A \cap \Phi[A]$ and let $\gamma = \Phi(\alpha)$. Then $\alpha$ and $\gamma$ differ only at $n$, hence $\neg \alpha E \gamma$. Also, by the definition of $A$, we have in $[\tau]$ comeagerly many $\beta$ with $\alpha E \beta$. As $\gamma \in A$, the same is true about $\gamma$. Thus there exists $\beta$ with $\alpha E \beta$ and $\gamma E \beta$. But then $\alpha E \gamma$, which is a contradiction. \hfill \qed

Remark. Another way to see that $E$ is meager might be as follows. Suppose for contradiction that $E$ is nonmeager. Consider $G = \{0,1\}^N \times \{0,1\}^N$ as a Polish group with coordinatewise addition mod 2. By the Baire category version of a theorem of Steinhaus (see [O]), if $B \subseteq G$ has the Baire property and is nonmeager, then the difference set $B - B = \{ b_0 - b_1 : b_0, b_1 \in B \}$ contains a neighborhood of the unit element $(0,0)$ (here $0 = (0,0,\ldots)$). So, for each $\langle \delta, \epsilon \rangle \in G$, which is close enough to $(0,0)$, there exist $\langle \alpha, \beta \rangle \in B$ such that $\langle \alpha + \gamma, \beta + \delta \rangle \in B$. For $n \in \mathbb{N}$ let $\epsilon_n \in \{0,1\}^N$ be the function that takes value 1 at $n$ and 0 elsewhere. Then $\langle \epsilon_n, 0 \rangle \rightarrow (0,0)$ when $n \rightarrow \infty$. So, for large enough $n$ there exists $\langle \alpha, \beta \rangle \in B$ with $\langle \alpha + \epsilon_n, \beta \rangle \in B$. Applied to $B = E$ this yields that for large enough $n$ there exist $\alpha$ and $\beta$ such that $\alpha E \beta$ and $\alpha + \epsilon_n E \beta$, whence $\alpha E \alpha + \epsilon_n$. This contradicts Lemma 2.

Similar arguments show that if $E$ is Lebesgue measurable then it must be null.

Corollary. $\approx$ is meager.

Recall now the following theorem of Mycielski [M].

Theorem (Mycielski). Suppose that $Y$ is a Polish space without isolated points and that $R \subseteq Y \times Y$ is meager. Then there exists a perfect set $P \subseteq Y$ such that if $\alpha$ and $\beta$ are distinct points of $P$ then $\langle \alpha, \beta \rangle \notin R$.

Applying this theorem to $\approx$ and $\{0,1\}^N$ we get a perfect set of mutually $\approx$ non-equivalent elements of $\{0,1\}^N$. The proof of Mycielski’s conjecture is complete.

A slight modification of the above proof gives the following theorem.

Theorem. Let $\kappa < 2^{2^{n_0}}$ be an infinite cardinal number. Suppose that $X$ is a path connected locally path connected metric space which is $\kappa$-Lindelöf (i.e., every open cover of $X$ has a subcover of size $\leq \kappa$). Then the power of the fundamental group of $X$ is either $\leq \kappa$ or $2^{\kappa_0}$.

References


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