

## THE FUNDAMENTAL GROUP OF A COMPACT METRIC SPACE

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**ABSTRACT.** We give a forcing free proof of a conjecture of Mycielski that the fundamental group of a connected locally connected compact metric space is either finitely generated or has the power of the continuum.

Shelah [S], using models, absoluteness and Cohen's forcing method, gives a proof of the following conjecture of Mycielski.

**Theorem.** *Suppose that  $X$  is a compact metric space, which is connected and locally connected. Then the fundamental group of  $X$  is either finitely generated or has the power of the continuum.*

Thus in particular the group of rationals can't be the fundamental group of a connected locally connected compact metric space.

We can't drop 'compact' in the theorem — any countably generated group can be realized as the fundamental group of a Polish space, see [Sp]. We also can't drop 'locally connected' — the fundamental group of the 'tail of the peacock' is free with a countable infinity of generators. It seems to be open, however, whether a finitely generated group can be realized as the fundamental group of a compact metric space.

We present a forcing free proof of Mycielski's conjecture.

**Definitions.** Let  $X$  be a metric space. A path from  $x_0$  to  $x_1$  inside  $V \subseteq X$  is a continuous function  $f : [0, 1] \mapsto V$  with  $f(0) = x_0$  and  $f(1) = x_1$ .  $f$  is a loop at  $x$  if  $f(0) = f(1) = x$ . The reversal of  $f$ , denoted by  $f^{-1}$ , is a path from  $f(1)$  to  $f(0)$  defined by  $f^{-1}(t) = f(1 - t)$ . The diameter of  $f$  is the diameter of the set  $\{f(t) : t \in [0, 1]\}$ .

A path  $f$  is homotopic to another path  $g$ ,  $f \sim g$ , if there is a homotopy from  $f$  to  $g$ , i.e., a continuous function  $F : [0, 1] \times [0, 1] \mapsto X$  such that  $F(t, 0) = f(t)$ ,  $F(t, 1) = g(t)$ ,  $F(0, s) = f(0) = g(0)$  and  $F(1, s) = f(1) = g(1)$  (thus endpoints are kept constant; this is usually called 'a homotopy relative to  $\{0, 1\}$ '). If  $f$  is a path from  $x_0$  to  $x_1$  and  $g$  a path from  $x_1$  to  $x_2$ , then  $f * g$  is the concatenation of  $f$  and  $g$ , i.e., a path from  $x_0$  to  $x_2$  defined by  $(f * g)(t) = f(2t)$  for  $t \in [0, 1/2]$  and  $(f * g)(t) =$

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$g(2t - 1)$  for  $t \in [1/2, 1]$ . The iterated concatenation  $f_0 * (f_1 * (\dots (f_{n-1} * f_n) \dots))$  is written as  $f_0 * \dots * f_n$ .

The relation  $\sim$  is an equivalence relation. If  $X$  is path connected, the set of equivalence classes of loops at a given point  $x \in X$ , with multiplication and inverse defined from  $*$  and  $^{-1}$ , is a group called the fundamental group of  $X$ . This group doesn't depend on the choice of  $x$ .

Let  $\mathbb{N}$  be the set of nonnegative integers. Let  $x \in X$  and suppose that  $f_n$  ( $n \in \mathbb{N}$ ) are loops at  $x$  whose diameters converge to 0. Then  $(f_0 * f_1 * \dots)$  is the pointwise limit of  $f_0 * \dots * f_n$ . The limit exists and is continuous, so it is a loop at  $x$ . In fact, if  $t \in [1 - 2^{-n}, 1 - 2^{-(n+1)}]$ , then

$$(f_0 * f_1 * \dots)(t) = f_n((t - (1 - 2^{-n})) \cdot 2^{n+1}).$$

Note also that

$$(f_0 * f_1 * \dots) = (f_0 * \dots * f_n) * (f_{n+1} * \dots).$$

From now on we assume that  $X$  is a connected locally connected compact metric space. Then  $X$  is also path connected and locally path connected (a theorem of Mazurkiewicz, see [K]).

**Lemma 1.** *Suppose that the fundamental group of  $X$  is not finitely generated. Then there exists  $x \in X$  such that for each  $n \in \mathbb{N}$  there exists a loop  $f_n$  at  $x$  which is of diameter  $< 2^{-n}$  and which is not homotopic to the constant loop at  $x$ .*

*Proof.* Suppose otherwise. Then for each  $x \in X$  there exists  $n(x) \in \mathbb{N}$  such that every loop at  $x$  which has diameter  $< 2^{-n(x)}$  is homotopic to the constant loop at  $x$ . By compactness there exists a cover of  $X$  by path connected sets  $V_i$  ( $i < k$ ) and there exist points  $x_i \in V_i$  ( $i < k$ ) such that for every  $i$  the diameter of  $V_i$  is  $< 2^{-(n(x_i)+1)}$  and any loop at  $x_i$  which has diameter  $< 2^{-n(x_i)}$  is homotopic to the constant loop at  $x_i$ .

Fix a path  $g_i$  from  $x_0$  to  $x_i$  ( $i < k$ );  $g_0 =$  the constant loop at  $x_0$ . For  $i$  and  $j$  such that  $V_i \cap V_j \neq \emptyset$  fix a path  $h_{ij}$  in  $V_i \cup V_j$  going from  $x_i$  to  $x_j$ . Note that any path  $s$  from  $x_i$  to  $x_j$  which is contained in  $V_i \cup V_j$  is homotopic to  $h_{ij}$ . Indeed, suppose that  $n(x_i) \leq n(x_j)$ . Then  $s * h_{ij}^{-1}$  is a loop at  $x_i$  and has diameter  $< 2^{-n(x_i)}$ , so  $s * h_{ij}^{-1}$  is homotopic to the constant loop at  $x_i$ . An elementary manipulation of this homotopy gives a homotopy from  $s$  to  $h_{ij}$ . If  $n(x_i) > n(x_j)$ , consider  $h_{ij}^{-1} * s$ , a loop at  $x_j$ .

We shall show that the fundamental group of  $X$  is generated by the (homotopy classes of) loops  $\tilde{h}_{ij} = g_i * h_{ij} * g_j^{-1}$ . To this end, suppose that a loop  $s$  at  $x_0$  is given. By a change of scale  $s \sim s_0 * \dots * s_l$ , where each  $s_i$  is a path inside one piece of our cover. Say  $s_i$  goes from  $y_i$  to  $y_{i+1}$  inside  $V_{\phi(i)}$ ;  $y_0 = y_{l+1} = x_0$ ,  $\phi(0) = \phi(l) = 0$ . For  $i = 0, \dots, l$ , fix inside  $V_{\phi(i)}$  a path  $t_i$  from  $y_{i+1}$  to  $x_{\phi(i)}$ ;  $t_0 = s_0^{-1}$ ,  $t_l =$  the constant loop at  $x_0$ . Let  $\tilde{s}_i = t_{i-1}^{-1} * s_i * t_i$  ( $i = 1, \dots, l$ ). Clearly

$$\tilde{s}_1 * \dots * \tilde{s}_l \sim s.$$

Also, each  $\tilde{s}_i$ , being a path from  $x_{\phi(i-1)}$  to  $x_{\phi(i)}$  inside  $V_{\phi(i-1)} \cup V_{\phi(i)}$ , must be homotopic to  $h_{\phi(i-1)\phi(i)}$ . Thus

$$s \sim h_{\phi(0)\phi(1)} * \dots * h_{\phi(l-1)\phi(l)},$$

and hence also

$$s \sim \tilde{h}_{\phi(0)\phi(1)} * \dots * \tilde{h}_{\phi(l-1)\phi(l)}. \quad \square$$

For the sequel suppose that the fundamental group of  $X$  is not finitely generated and let  $x$  and  $f_n$  ( $n \in \mathbb{N}$ ) be as claimed by Lemma 1. We shall find a set of size of the continuum of mutually non-homotopic loops. For  $\alpha \in \{0, 1\}^{\mathbb{N}}$  let  $f_n^\alpha =$  the constant loop at  $x$  if  $\alpha(n) = 0$ , and let  $f_n^\alpha = f_n$  otherwise. Define a loop  $f_\alpha$  at  $x$  as  $(f_0^\alpha * f_1^\alpha * \dots)$ . Write  $\alpha \approx \beta$  if  $f_\alpha \sim f_\beta$ . Then  $\approx$  is an equivalence relation in  $\{0, 1\}^{\mathbb{N}}$ . It is enough to prove that  $\approx$  has continuum many equivalence classes.

**Lemma 2.** *Suppose that  $\alpha$  and  $\beta$  from  $\{0, 1\}^{\mathbb{N}}$  differ exactly at one point. Then  $\alpha \not\approx \beta$ .*

*Proof.* Suppose that  $f_\alpha \sim f_\beta$ . Let  $n$  be the unique point at which  $\alpha$  and  $\beta$  are different. Then, for  $m \neq n$  we have  $f_m^\alpha = f_m^\beta$ , hence  $(f_0^\alpha * \dots * f_{n-1}^\alpha) = (f_0^\beta * \dots * f_{n-1}^\beta)$  and  $(f_{n+1}^\alpha * \dots) = (f_{n+1}^\beta * \dots)$ . Thus from

$$(f_0^\alpha * \dots * f_{n-1}^\alpha * f_n^\alpha * f_{n+1}^\alpha * \dots) \sim (f_0^\beta * \dots * f_{n-1}^\beta * f_n^\beta * f_{n+1}^\beta * \dots),$$

we get  $f_n^\alpha \sim f_n^\beta$ , which is a contradiction. □

**More definitions.** We recall some basic facts about Polish spaces (see [K]). A Polish space is a completely metrizable separable space. Let  $Y$  be a Polish space. For a subset  $A$  of  $Y$ :  $A$  is nowhere dense if its closure has empty interior,  $A$  is meager if it is a countable union of nowhere dense sets,  $A$  is comeager in an open set  $U$  if  $U \setminus A$  is meager,  $A$  has the Baire property if its symmetric difference with some open set is meager. A nonmeager set with the Baire property is comeager in some nonempty open set. The Baire category theorem implies that a set which is comeager in a nonempty open set is nonmeager. The Kuratowski-Ulam theorem implies that if  $A \subseteq Y \times Z$  is comeager in  $U \times V$ , where  $U$  and  $V$  are open subsets of Polish spaces  $Y$  and  $Z$ , then

$$\{y \in U : \{z \in V : \langle y, z \rangle \in A\} \text{ is comeager in } V\}$$

is comeager in  $U$ .

A subset  $A$  of  $Y$  is analytic if there exist a Polish space  $Z$  and a closed set  $D \subseteq Y \times Z$  such that  $A$  is the projection of  $D$  into  $Y$ . Analytic sets have the Baire property. Continuous preimages of analytic sets are analytic.

A subset  $P$  of  $Y$  is perfect if it is closed, nonempty, and has no isolated points. Perfect sets have the power of the continuum.

The set  $\{0, 1\}^{\mathbb{N}}$  becomes a Polish space (homeomorphic to the Cantor discontinuum) when viewed as the product of countably many copies of the two-point discrete space  $\{0, 1\}$ . The canonical basis of  $\{0, 1\}^{\mathbb{N}}$  is the collection of all sets  $[\sigma] = \{\alpha : \sigma \subseteq \alpha\}$ , where  $\sigma$  is a finite zero-one sequence, i.e.,  $\sigma : \{0, 1, \dots, n-1\} \mapsto \{0, 1\}$  for some  $n$ .

**Lemma 3.**  *$\approx$  has the Baire property as a subset of  $\{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$ .*

*Proof.* Let  $H$  and  $\mathbb{H}$  be respectively the spaces of all loops at  $x$  and all homotopies between them, endowed with the sup metric. Both spaces are Polish. The homotopy relation  $\sim$  restricted to  $H$  is an analytic subset of  $H \times H$ . Indeed, it is the projection onto  $H \times H$  of

$$\{ \langle \langle f, g \rangle, F \rangle : F \text{ is a homotopy from } f \text{ to } g \},$$

which is a closed subset of  $(H \times H) \times \mathbb{H}$ . Note also that the function from  $\{0, 1\}^{\mathbb{N}}$  to  $H$  which takes  $\alpha$  to  $f_\alpha$  is continuous. It follows that  $\approx$  is analytic (as a continuous preimage of an analytic set), and thus has the Baire property.  $\square$

**Lemma 4.** *If  $E \subseteq \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$  is an equivalence relation which has the Baire property and if  $\neg xEy$  whenever  $x$  and  $y$  differ by one coordinate only, then  $E$  is meager.*

*Proof.* Should  $E$  be nonmeager, it would be comeager in some basic neighbourhood  $[\sigma] \times [\tau]$ . By the Kuratowski-Ulam theorem,

$$A = \{\alpha \in [\sigma] : \{\beta \in [\tau] : \alpha E \beta\} \text{ is comeager in } [\tau]\}$$

is comeager in  $[\sigma]$ . Let  $n >$  the length of  $\sigma$ . Consider a map  $\Phi : [\sigma] \mapsto [\sigma]$  defined by  $\Phi(\alpha)(n) = 1 - \alpha(n)$  and  $\Phi(\alpha)(i) = \alpha(i)$  for  $i \neq n$ .  $\Phi$  is a homeomorphism and thus  $\Phi[A]$  is comeager in  $[\sigma]$ . Choose  $\alpha \in A \cap \Phi[A]$  and let  $\gamma = \Phi(\alpha)$ . Then  $\alpha$  and  $\gamma$  differ only at  $n$ , hence  $\neg \alpha E \gamma$ . Also, by the definition of  $A$ , we have in  $[\tau]$  comeagerly many  $\beta$  with  $\alpha E \beta$ . As  $\gamma \in A$ , the same is true about  $\gamma$ . Thus there exists  $\beta$  with  $\alpha E \beta$  and  $\gamma E \beta$ . But then  $\alpha E \gamma$ , which is a contradiction.  $\square$

*Remark.* Another way to see that  $E$  is meager might be as follows. Suppose for contradiction that  $E$  is nonmeager. Consider  $G = \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^{\mathbb{N}}$  as a Polish group with coordinatewise addition mod 2. By the Baire category version of a theorem of Steinhaus (see [O]), if  $B \subseteq G$  has the Baire property and is nonmeager, then the difference set  $B - B = \{b_0 - b_1 : b_0, b_1 \in B\}$  contains a neighbourhood of the unit element  $\langle \bar{0}, \bar{0} \rangle$  (here  $\bar{0} = \langle 0, 0, \dots \rangle$ ). So, for each  $\langle \delta, \epsilon \rangle \in G$ , which is close enough to  $\langle \bar{0}, \bar{0} \rangle$ , there exist  $\langle \alpha, \beta \rangle \in B$  such that  $\langle \alpha + \gamma, \beta + \delta \rangle \in B$ . For  $n \in \mathbb{N}$  let  $\epsilon_n \in \{0, 1\}^{\mathbb{N}}$  be the function that takes value 1 at  $n$  and 0 elsewhere. Then  $\langle \epsilon_n, \bar{0} \rangle \rightarrow \langle \bar{0}, \bar{0} \rangle$ , when  $n \rightarrow \infty$ . So, for large enough  $n$  there exists  $\langle \alpha, \beta \rangle \in B$  with  $\langle \alpha + \epsilon_n, \beta \rangle \in B$ . Applied to  $B = E$  this yields that for large enough  $n$  there exist  $\alpha$  and  $\beta$  such that  $\alpha E \beta$  and  $\alpha + \epsilon_n E \beta$ , whence  $\alpha E \alpha + \epsilon_n$ . This contradicts Lemma 2.

Similar arguments show that if  $E$  is Lebesgue measurable then it must be null.

**Corollary.**  $\approx$  is meager.

Recall now the following theorem of Mycielski [M].

**Theorem** (Mycielski). *Suppose that  $Y$  is a Polish space without isolated points and that  $R \subseteq Y \times Y$  is meager. Then there exists a perfect set  $P \subseteq Y$  such that if  $\alpha$  and  $\beta$  are distinct points of  $P$  then  $\langle \alpha, \beta \rangle \notin R$ .*

Applying this theorem to  $\approx$  and  $\{0, 1\}^{\mathbb{N}}$  we get a perfect set of mutually  $\approx$  non-equivalent elements of  $\{0, 1\}^{\mathbb{N}}$ . The proof of Mycielski's conjecture is complete.

A slight modification of the above proof gives the following theorem.

**Theorem.** *Let  $\kappa < 2^{\aleph_0}$  be an infinite cardinal number. Suppose that  $X$  is a path connected locally path connected metric space which is  $\kappa$ -Lindelöf (i.e., every open cover of  $X$  has a subcover of size  $\leq \kappa$ ). Then the power of the fundamental group of  $X$  is either  $\leq \kappa$  or  $2^{\aleph_0}$ .*

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