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NOTE ON THE TOPOLOGICAL DEGREE OF THE SUBDIFFERENTIAL OF A LOWER SEMI-CONTINUOUS CONVEX FUNCTION

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ABSTRACT. The purpose of the present paper is to prove that the topological degree of the subdifferential of a coercive lower semi-continuous function on a sufficiently large ball in a reflexive Banach space is equal to one.

A well known result of Krasnoselskii [7] says that the degree of the gradient of a coercive potential mapping on a sufficiently large ball in R^N is equal to one. As Nirenberg pointed out in [8], this result seems intuitively clear, but the proof is not. Many authors had new views of this theorem. See [1], [9]–[11]. The aim of this note is to give a generalization of the degree of a maximal monotone mapping and prove that the topological degree of the subdifferential of a coercive convex function is equal to one on a sufficiently large ball. The main result of this paper is Theorem 5. Its proof is very simple and elementary.

Let us recall the properties of the degree of mappings of class (S_+), as introduced by Browder [4]. See also [3], [6]. Let E be a real reflexive Banach space of dual E^*. We can always assume that both E and E^* are locally uniformly convex [12]. The duality pairing between E^* and E is denoted by $(.,.)$. Let θ^* be the zero element of E^*, $J : E \to E^*$ the duality mapping, $\Omega \subset E$ an open bounded subset, and $T : \overline{\Omega} \to E^*$ a demi-continuous operator of class (S_+) (see [4] for a definition); if $\theta^* \notin T(\partial\Omega)$, then there is a topological degree $deg(T, \Omega, \theta^*)$ satisfying

Theorem 1. (a)

$$deg(J, \Omega, \theta^*) = \begin{cases} 1, \ if \theta^* \in J(\Omega), \\ 0, \ if \theta^* \notin J(\overline{\Omega}). \end{cases}$$

(b) *If $deg(T, \Omega, \theta^*) \neq 0$, then $Tx = \theta^*$ has a solution in Ω.*

(c) *If Ω_1, Ω_2 are two disjoint open subsets of Ω, then*

$$deg(T, \Omega_1, \theta^*) + deg(T, \Omega_2, \theta^*) = deg(T, \Omega_1 \cup \Omega_2, \theta^*).$$

(d) *If $\{T_t\}_{[0,1]}$ is a homotopy of class (S_+), and $\theta^* \notin T_t(\partial\Omega)$, then $deg(T, \Omega, \theta^*)$ doesn't depend on $t \in [0, 1]$.*

Now, suppose that $A : D(A) \subseteq E \to 2^{E^*}$ is a maximal monotone operator. Let $A_\lambda = (\lambda J^{-1} + A^{-1})^{-1}$, $R_\lambda = I - \lambda J^{-1} A_\lambda$ be its Yosida approximation and resolvent

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respectively, and $T : E \to E^*$ be a demi-continuous bounded operator of class (S_+). We introduce $deg(A, D(A) \cap \Omega, \theta^*)$ by means of several lemmas. In what follows "$\rightharpoonup$" stands for weak convergence.

Lemma 2. *If $x_j \in D(A)$, with $x_j \rightharpoonup x_0$, and ϵ_j is a positive sequence converging to 0 such that $\theta^* \in (A + \epsilon_j T)x_j$, then $x_j \to x_0$, and $\theta^* \in Ax_0$.*

Proof. By assumption, we have

$$(\epsilon_j Tx_j - \epsilon_i Tx_i, x_j - x_i) \le 0, \forall i, j \ge 1.$$

Letting $i \to 0$, we get

$$\epsilon_j (Tx_j, x_j - x_0) \le 0.$$

But T is an operator of class (S_+), so $x_j \to x_0$, and $Tx_j \rightharpoonup Tx_0$. Hence, we get $x_0 \in D(A)$, and $\theta^* \in Ax_0$. This completes the proof.

Now, suppose that $\Omega \cap D(A) \ne \varnothing$, and $\theta^* \notin Ax$, $\forall x \in \partial\Omega \cap D(A)$; then by using Lemma 2, we know that there exists $\epsilon_0 > 0$, such that $\theta^* \notin (A + \epsilon T)(\partial\Omega)$ for $\epsilon \in (0, \epsilon_0)$. By adapting the technique in [4], one can show that there exists $\lambda_0(\epsilon) > 0$, such that

$$\theta^* \notin (A_\lambda + \epsilon T)(\partial\Omega), \forall \lambda \in (0, \lambda_0(\epsilon)).$$

Note that $A_\lambda + \epsilon T$ is a demi-continuous operator of class (S_+), so that the degree $deg(A_\lambda + \epsilon T, \Omega, \theta^*)$ is well defined. We show that this degree doesn't depend on ϵ, λ, and T. Specifically we have:

Lemma 3. *Let T_1, T_2 be two demi-continuous operators of class (S_+), and $\lambda_1,\lambda_2 > 0$, $\epsilon_1, \epsilon_2 > 0$. Then $\{A_{t\lambda_1+(1-t)\lambda_2} + t\epsilon_1 T_1 + (1-t)\epsilon_2 T_2\}_{t\in[0,1]}$ is a homotopy of class (S_+).*

The proof of this lemma relies on the following result.

Lemma 4 (cf. Proposition 3.56 in [2])**.** *If $\lambda_j \to \lambda$, then $A_{\lambda_j} x \to A_\lambda x$, for any $x \in E$.*

Proof of Lemma 3. Let $t_j \to t_0$, $x_j \rightharpoonup x_0$ be such that

$$\overline{\lim}_{j\to\infty} (A_{t_j\lambda_1+(1-t_j)\lambda_2} x_j + t_j \epsilon_1 T_1 x_j + (1-t_j)\epsilon_2 T_2 x_j, x_j - x_0) \le 0.$$

By the monotonicity of A_λ and Lemma 4, we know that

$$\underline{\lim}_{j\to\infty} (A_{t_j\lambda_1+(1-t_j)\lambda_2} x_j, x_j - x_0) \ge 0.$$

Consequently

$$\overline{\lim}_{j\to\infty} (t_j \epsilon_1 T_1 x_j + (1-t_j)\epsilon_2 T_2 x_j, x_j - x_0) \le 0.$$

Since $\{t\epsilon_1 T_1 + (1-t)\epsilon_2 T_2\}_{t\in[0,1]}$ is a homotopy of class (S_+) (see [4]), it follows that $x_j \to x_0$, $T_1 x_j \rightharpoonup T_1 x_0, T_2 x_j \rightharpoonup T_2 x_0, A_{t_j\lambda_1+(1-t_j)\lambda_2} x_j \rightharpoonup A_{t_0\lambda_1+(1-t_0)\lambda_2} x_0$. The proof is complete.

The above arguments show that $deg(A_\lambda + \epsilon T, \Omega, \theta^*)$ is constant for sufficiently small $\epsilon, \lambda > 0$ and doesn't depend on T, so we can define the topological degree $deg(A, D(A) \cap \Omega, \theta^*)$ as the common value of $deg(A_\lambda + \epsilon T, \Omega, \theta^*)$. In particular the theory applies to the case when $A = \partial\varphi$, the subdifferential of a proper, convex, and lower semi-continuous function $\varphi : E \to (-\infty, +\infty]$.

Our main result is the following

Theorem 5. *Let $\varphi : D(\varphi) \subseteq E \to (-\infty, +\infty]$ be a proper lower-semicontinuous convex function such that $\lim_{\|x\|\to\infty} \varphi(x) = \infty$. Then there exists $r_0 > 0$, such that*

$$deg(\partial\varphi, D(\partial\varphi) \cap B(0,r), \theta^*) = 1,$$

for any $r > r_0$.

Proof. Since $\lim_{\|x\|\to\infty} \varphi(x) = +\infty$, there exists $x_0 \in E$, such that $\theta^* \in \partial\varphi(x_0)$. Let $r_0 > 0$ be sufficiently large such that $\theta^* \notin \partial\varphi(x)$ for any $x \in D(\partial\varphi) \cap \partial B(0,r)$, and any $r \geq r_0$.

There exists an $\epsilon_0 > 0$, such that

$$\theta^* \notin \cup_{t\in[0,1]}\big(t\partial\varphi(x) + t\epsilon J(x - x_0) + (1-t)\epsilon J(x - x_0)\big),$$
$$\forall x \in D(\partial\varphi) \cap \partial B(0,r), \epsilon \in (0, \epsilon_0).$$

For each $\epsilon \in (0, \epsilon_0)$, one can prove by contradiction that there exists $\lambda_0 > 0$ with the property

$$\theta^* \notin \cup_{t\in[0,1]}\big(t\partial\varphi_\lambda + t\epsilon J(. - x_0) + (1-t)\epsilon J(. - x_0)\big)\partial B(0,r), \forall \lambda \in (0, \lambda_0).$$

By (a), (d) of Theorem 1, we know that

$$deg\big(\partial\varphi_\lambda + \epsilon J(. - x_0), B(0,r), \theta^*\big) = deg\big(J(. - x_0), B(0,r), \theta^*\big) = 1.$$

So $deg\big(\partial\varphi, D(\partial\varphi \cap B(0,r)), \theta^*\big) = 1$, as desired.

A more general form of Theorem 5 is

Theorem 6. *Let $A : D(A) \subseteq E \to 2^{E^*}$ be a maximal monotone operator, and $\Omega \subset E$ be an open bounded subset with $\Omega \cap D(A) \neq \varnothing$. Suppose that there exists $\overline{x} \in \Omega \cap D(A)$, such that $(f, x - \overline{x}) \geq 0$, $\forall x \in D(A) \cap \partial\Omega, f \in Ax$, and $\theta^* \notin Ax, \forall x \in D(A) \cap \partial\Omega$. Then $deg(A, \Omega \cap D(A), \theta^*) = 1$.*

The proof of this theorem is similar to that of Theorem 5, and we omit the details.

References

[1] H. Amann, A note on degree theory for gradient mappings, Proc. Amer. Math. Soc., 85(1982),591-595. MR **83i:**47069

[2] H. Attouch, Variational Convergence for Functions and Operators, Pitman,Boston,1984. MR **86f:**49002

[3] J. Berkovits and V. Mustonen, On the degree for mappings of monotone type, Nonlinear Anal., 12(1986),1373-1383. MR **88b:**47073

[4] F.E.Browder, Fixed point theory and nonlinear problems, Bull.Amer.Math. Soc., 1(1983),1-39. MR **84h:**58027

[5] F.E.Browder, Degree theory for nonlinear mappings, Proc. Symp.Pure Math., Vol.45,pp.203-226, Amer.Math.Soc., Providence,R.I., 1986. MR **87g:**47108

[6] Y.Q. Chen and S. S. Zhang, Degree theory for multivalued (S) type mappings and fixed point theorems, Applied Math.Mech., English Ed., 11(1990),441-454. MR **91h:**47065

[7] M.A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations,Macmillan, New York,1964. MR **28:**2414

[8] L. Nirenberg, Variational and topological methods in nonlinear problems, Bull. Amer. Math. Soc., 4(1981),267-302. MR **83e:**58015

[9] P. H. Rabinowitz, A note on topological degree for potential operators, J. Math. Anal. Appl., 51(1975),483-492. MR **57:**10518

[10] E. H. Rothe, A relation between the type numbers of a critical point and the index of the corresponding field of gradient vectors, Math. Nachr., 4(1950-51), 12-17. MR **12:**720c
[11] J. C. Scovel, A simple intuitive proof of a theorem in degree theory for gradient mappings, Proc. Amer. Math. Soc., 93(1985), 751-753. MR **86g:**55001
[12] S. L. Trojansky, On locally uniformly convex and differentiable norms in certain nonseparable Banach spaces, Studia Math., 37(1971), 173-180.

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