

NONCOMPLETE LINEAR SYSTEMS  
ON ELLIPTIC CURVES AND ABELIAN VARIETIES:  
ADDENDUM TO A PAPER BY CH. BIRKENHAKE

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ABSTRACT. Here we give a result on the postulation (i.e. the 2-normality) of nonlinearly normal embeddings of Abelian varieties. This result improves some of the results proved in a recent paper by Ch. Birkenhake.

In [Bi1], [Bi2] and [Bi3], Ch. II, Ch. Birkenhake considered the postulation and the minimal free resolution of noncomplete embeddings into  $\mathbf{P}^N$ , respectively of Abelian varieties, projective spaces and curves. In [Bi1], §1 and §2, a general set-up for the study of the minimal free resolution of noncomplete embeddings of algebraic varieties into  $\mathbf{P}^N$  was given. A cursory reading of [Bi1] shows that a key point of the proofs was the reduction to the case of a product of elliptic curves and then to the case (via Künneth formula) to the case of an embedding of an elliptic curve. In [Bi1] only the good properties of general projections into  $\mathbf{P}^{m-1}$  of a complete embedding into  $\mathbf{P}^m$  of an elliptic curve were used. However, there are similar results for general projections into  $\mathbf{P}^s$  with  $s$  much smaller than  $m-1$  (see [BE1], [BE2], [BE3] and the statement here of 1.1). Here we want to show how to use these results to improve some of the results of [Bi1]. Our result is the following theorem which improves [Bi1], Cor. 4.3.

**Theorem 0.1.** *Suppose  $(X, L)$  is a general complex Abelian variety of type  $(d_1, \dots, d_g)$ . Fix an integer  $n \geq 3$  such that  $nd_g \geq 6$ . Let  $w$  be the largest integer such that  $(nd_g - w)(nd_g - w + 1) \geq 4nd_g$ . Fix an integer  $c$  with  $1 \leq c \leq wn^{g-1}$ . Then the general vector subspace  $V \subseteq H^0(X, L^{\otimes n})$  with  $\text{codim}(V) = c$  is 2-normal, i.e. the restriction map  $S^2(V) \rightarrow H^0(X, L^{\otimes 2n})$  is surjective.*

For the postulation of general projections of Hirzebruch surfaces and Veronese embeddings of  $\mathbf{P}^2$ , see [BE4].

1. THE PROOFS

As in [Bi1] we work in characteristic 0. Let  $k \geq 1$  be an integer. Recall that with the terminology of [Bi1] a closed subscheme  $Z$  of  $\mathbf{P}(V^*)$  is  $k$ -normal if the canonical map  $S^k(V) \rightarrow H^0(Z, \mathcal{O}_Z(k))$  is surjective. The following result was proved in [BE1] and it is a very particular case of the particular case “ $g = 1$ ” of [BE2] (projections into  $\mathbf{P}^3$ ) and [BE3], Th. I (projections into  $\mathbf{P}^N$ ,  $N \geq 4$ ).

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**Theorem 1.1.** *Fix integers  $d, N$  with  $d > N \geq 3$  and  $(N + 2)(N + 1)/2 \geq 2d$ . Fix an elliptic curve  $Y$  and  $L \in \text{Pic}^d(Y)$  and let  $X \subset \mathbf{P}^{d-1}$  be the linearly normal embedding of  $Y$  determined by  $H^0(Y, L)$ . Then the general projection of  $X$  into  $\mathbf{P}^N$  is  $k$ -normal for all integers  $k \geq 2$ .*

*Remark 1.2.* Note that since  $h^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(2)) = (N+2)(N+1)/2$  and  $h^0(Y, L^{\otimes 2}) = 2d$ , the statement of Theorem 1.1 is sharp.

*Remark 1.3.* Note that in the statement of Theorem 1.1 we do not need the assumption that  $L$  is a square, i.e. that  $d$  is even, made in [Bi1], Cor. 4.2. Hence in our improvement 0.1 of [Bi1], Cor. 4.3, we do not need the assumption that  $nd_g$  is even.

*Proof of Theorem 0.1.* Theorem 0.1 follows from the proof of [Bi1], Cor. 4.3, using 1.1 instead of [Bi1], Cor. 4.2.  $\square$

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