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DUALITY FOR FULL CROSSED PRODUCTS OF C^*-ALGEBRAS BY NON-AMENABLE GROUPS

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Abstract. Let (A, G, δ) be a cosystem and (A, G, α) be a dynamical system. We examine the extent to which induction and restriction of ideals commute, generalizing some of the results of Gootman and Lazar (1989) to full crossed products by non-amenable groups. We obtain short, new proofs of Katayama and Imai-Takai duality, the faithfulness of the induced regular representation for full coactions and actions by amenable groups. We also give a short proof that the space of dual-invariant ideals in the crossed product is homeomorphic to the space of invariant ideals in the algebra, and give conditions under which the restriction mapping is open.

Introduction

Let a locally compact group G coact, or act, on a C^*-algebra A and denote the full crossed product by $A \times G$. There are three well known maps between the ideal spaces of A and $A \times G$. Every ideal I of A generates an ideal in $A \times G$, denoted by $\mathrm{Ext}(I)$. Every representation of $A \times G$ restricts to a representation of A; this is well-defined on ideals, so for an ideal K in $A \times G$ we obtain the ideal $\mathrm{Res}(K)$ in A. A representation of A can be induced to a representation of $A \times G$ and it turns out that this gives a well-defined process on ideals, so for an ideal I in A, there is an ideal $\mathrm{Ind}(I)$ in $A \times G$.

In §2, we examine the extent to which the maps Res and Ind commute. Our results generalize some of the results of Gootman and Lazar [1] to full crossed products by coactions and actions of non-amenable groups. From these results we obtain the following corollaries. We give a short, self-contained proof of the faithfulness of the induced regular representation for full coactions [10, Corollary 2.7] (Corollary 2.4). It is suprising to find such a simple proof of Quigg's result, since the proofs in [10, Corollary 2.7] and [12, Theorem 4.1(2)] seem quite deep. We give a short proof of the analogous result for full crossed products by actions of amenable groups. Also we provide short, elegant proofs of both Katayama and Imai-Takai duality for full crossed products [5, Theorem 4], [3, Theorem 3.6], [11, Theorem 6] (Corollaries 2.6 and 2.12).

In §3 we give key properties of the maps $\mathrm{Res}, \mathrm{Ext}$ and Ind for both coactions and actions. In particular, we show that the dual-invariant ideals of $A \times G$ are exactly the ideals induced from A (Propositions 3.1(iv) and 3.3(iii)). In the coaction case,

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this is a generalization (to non-amenable G) of a version of Mansfield's imprimitivity theorem for induction from the trivial subgroup [6, Theorem 28]. It follows easily from this that Res is a homeomorphism from the dual-invariant ideals of $A \times G$ onto the invariant ideals of A (Corollaries 3.2(i) and 3.4(i)). Although these results may be well known in certain cases, our emphasis here is to provide self-contained proofs where the key ideas work for both actions and coactions.

Then we examine the openness of Res and Res Ind. For coactions, Res is open in general but we need to assume G is amenable in order to prove that Res Ind is open; on the other hand, for actions the reverse is the case: Res Ind is open in general, and we assume amenability in order to prove that Res is open (Corollaries 3.2(ii),(iii) and 3.4(ii),(iii)).

Our main aim in this work has been to present coactions and actions in a unified framework, so as to make clear their similarities and differences, which can only be examined when G is allowed to be non-amenable. Gootman and Lazar do not investigate the issue of amenability, choosing rather to assume amenability for ideal theoretic results [1, §3]. Furthermore, we have avoided the technicalities of the imprimitivity bimodule techniques employed in [4], [6]. Our overall approach has similarities to the work of Nakagami and Takesaki on duality of von Neumann algebras [7, Chapter 1, §2].

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1. Definitions

Throughout A is a C^*-algebra and G is a locally compact group. We denote the full group C^*-algebra by $C^*(G)$ with canonical embedding i_G, and $C_0(G)$ is the continuous functions on G disappearing at infinity. All representations σ are nondegenerate, tensor products $\otimes$ are minimal, and identity maps are denoted by id. Given a nondegenerate homomorphism ϕ into the multiplier algebra $M(C)$, there is nondegenerate homomorphism $\phi \otimes \mathrm{id}\colon A \otimes B \to M(C \otimes B)$. Let $\mathcal{I}(A)$ denote the space of closed ideals of A, with the topology which has subbasic open sets of the form $O_J = \{I \colon I \not\supseteq J\}$, where J is a closed ideal in A. The relative topology on the space of primitive ideals $\operatorname{Prim} A$ in $\mathcal{I}(A)$ is the usual hull-kernel topology.

We now recall some definitions from [9, Lemma 1.1]. Let B be a C^*-algebra and $\iota\colon A \to M(B)$ be a homomorphism. Define

$$\begin{aligned} \mathrm{Res}_\iota\colon \quad & \mathcal{I}(B) \to \mathcal{I}(A) \quad \text{by} \quad \mathrm{Res}_\iota(\ker\sigma) = \ker(\bar{\sigma}\circ\iota), \text{ and} \\ \mathrm{Ext}_\iota\colon \quad & \mathcal{I}(A) \to \mathcal{I}(B) \quad \text{by} \quad \mathrm{Ext}_\iota(I) = \overline{B\iota(I)B}. \end{aligned}$$

We showed there that Res_ι is continuous and containment preserving, and that Res_ι is open onto its range if and only if $\mathrm{Ext}_\iota|_{\mathrm{Im}\,\mathrm{Res}}$ is continuous. Also $I \subseteq \mathrm{Res}\,\mathrm{Ext}(I)$ since $I \subseteq \mathrm{Res}(K) \Leftrightarrow \mathrm{Ext}(K) \subseteq I$. These are key elements in this work.

The map in $C_b(G, M(C^*(G)))$ defined by $s \mapsto i_G(s)$ is unitary. When this element is considered to be in $M(C_0(G) \otimes C^*(G))$, we denote it by w_G; as an element of $M(C^*(G) \otimes C_0(G))$, it will be denoted by v_G.

The map $\delta_G\colon C^*(G) \to M(C^*(G) \otimes C^*(G))$ is the unique nondegenerate homomorphism such that $\delta_G(i_G(s)) = i_G(s) \otimes i_G(s)$. It satisfies the relation $(\delta_G \otimes \mathrm{id}) \circ \delta_G = (\mathrm{id} \otimes \delta_G) \circ \delta_G$, as maps of $C^*(G)$ into $M(C^*(G) \otimes C^*(G) \otimes C^*(G))$.

Define $\alpha_G\colon C_0(G) \to M(C_0(G) \otimes C_0(G))$ by $\alpha_G(f)(s,t) = f(st)$. It satisfies the relation $(\alpha_G \otimes \mathrm{id}) \circ \alpha_G = (\mathrm{id} \otimes \alpha_G) \circ \alpha_G$.

We use the same definitions for full crossed products by coaction as in [8] (minimal tensor products and full group C^*-algebras). A covariant homomorphism of a cosystem (A, G, δ) into $M(B)$ is a pair (ϕ, ψ) of nondegenerate homomorphisms $\phi\colon A \to M(B)$ and $\psi\colon C_0(G) \to M(B)$, such that

$$\phi \otimes \mathrm{id}(\delta(a)) = \psi \otimes \mathrm{id}(w_G)(\phi(a) \otimes 1)\psi \otimes \mathrm{id}(w_G^*),$$

as elements of $M(B \otimes C^*(G))$. A covariant representation is a pair (π, μ) of representations into $B(H) = M(K(H))$, so that the covariance condition holds in $M(K(H) \otimes C^*(G))$.

An action α of a locally compact group G on a C^*-algebra A can be considered to be a nondegenerate injection $\alpha\colon A \to M(A \otimes C_0(G))$ which satisfies:

$$\alpha(A)(1 \otimes C_0(G)) = A \otimes C_0(G), \text{ and } (\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \alpha_G) \circ \alpha.$$

Then a covariant homomorphism of a dynamical system (A, G, α) into $M(B)$ is a pair (ϕ, ψ) of nondegenerate homomorphisms $\phi\colon A \to M(B)$ and $\psi\colon C^*(G) \to M(B)$, such that

$$\phi \otimes \mathrm{id}(\alpha(a)) = \psi \otimes \mathrm{id}(v_G)(\phi(a) \otimes 1)\psi \otimes \mathrm{id}(v_G^*),$$

as elements of $M(B \otimes C_0(G))$. A covariant representation is a covariant homomorphism (π, U) into $B(H) = M(K(H))$, so that the covariance condition holds in $M(K(H) \otimes C_0(G))$.

Let λ and ρ be the left and right regular representations of G on $L^2(G)$, M be the representation of $C_0(G)$ on $L^2(G)$ as multiplication operators, and define a representation M^{-1} by $M^{-1}(f)(\xi)(s) = f(s^{-1})\xi(s)$. There are many covariance identities which are readily verified: for example,

$$\lambda \otimes \mathrm{id}(\delta_G(s)) = [M \otimes \mathrm{id}(w_G)]\,[\lambda(s) \otimes 1]\,[M \otimes \mathrm{id}(w_G^*)], \tag{1}$$

$$\mathrm{id} \otimes \lambda(\delta_G(s)) = [\mathrm{id} \otimes M^{-1}(v_G^*)]\,[1 \otimes \lambda(s)]\,[\mathrm{id} \otimes M^{-1}(v_G)], \text{and} \tag{2}$$

$$\mathrm{id} \otimes M(\alpha_G(f)) = [\mathrm{id} \otimes \lambda(w_G^*)]\,[1 \otimes M(f)]\,[\mathrm{id} \otimes \lambda(w_G)]. \tag{3}$$

Let (A, G, δ) be a cosystem and $\pi\colon A \to B(H)$ be a representation. Then the *induced regular representation* $\mathrm{Ind}_\delta\,\pi\colon A \times_\delta G \to B(H \otimes L^2(G))$ is the unique nondegenerate homomorphism arising from the covariant pair $(\pi \otimes \lambda \circ \delta, 1 \otimes M)$.

An ideal $I = \ker \pi$ is δ*-invariant* if

$$\ker(\pi \otimes \lambda \circ \delta) = \ker \pi.$$

It will follow from Proposition 2.1 that invariance is independent of the choice of representation. Denote by $\mathcal{I}_\delta(A)$, the subspace of δ-invariant ideals of A with the relative topology. The *dual action* $\hat{\delta}\colon A \times_\delta G \to M((A \times_\delta G) \otimes C_0(G))$ satisfies

$$\hat{\delta}(j_A(a)\, j_{C_0(G)}(f)) = [j_A(a) \otimes 1]\,[j_{C_0(G)} \otimes \mathrm{id}(\alpha_G(f))].$$

Let (A, G, α) be a dynamical system and let $\pi\colon A \to B(H)$ be a representation. Then the *induced regular representation*

$$\mathrm{Ind}_\alpha\,\pi\colon A \times_\alpha G \to B(H \otimes L^2(G))$$

is the unique nondegenerate homomorphism arising from the covariant pair $(\pi \otimes M^{-1} \circ \alpha, 1 \otimes \lambda)$. The ideal I in A is α*-invariant* if $\ker(\pi \otimes M^{-1} \circ \alpha) = \ker \pi$.

Denote the space of α-invariant ideals of A, with the relative topology, by $\mathcal{I}_\alpha(A)$. The *dual coaction* $\hat{\alpha}\colon A \times_\alpha G \to M((A \times_\alpha G) \otimes C^*(G))$ satisfies

$$\hat{\alpha}(k_A(a)\, k_G(s)) = [k_A(a) \otimes 1]\, [k_G \otimes \mathrm{id}(\delta_G(s))].$$

Remark 1. If π is a faithful representation of A, then $\pi \otimes M^{-1} \circ \alpha$ is also faithful, and this implies that the canonical embedding k_A of A into the crossed product $A \times_\alpha G$ is faithful. For coactions, however, the situation is different because faithfulness of π is not enough to ensure the faithfulness of $\pi \otimes \lambda \circ \delta$. Thus, in general, we cannot assume that the embedding j_A of A into $A \times_\delta G$ is faithful. A cosystem where j_A is faithful is called *normal* [10, Definition 2.1]. Notice, if G is amenable then $A \times_\delta G$ is normal.

We note that the ideal $\ker j_A$ is δ-invariant. To see this, take a faithful representation ϕ of $A \times_\delta G$; then $(\pi, \mu) = (\phi \circ j_A, \phi \circ j_{C_0(G)})$ is a covariant representation of (A, G, δ). The covariance condition says

$$\pi \otimes \mathrm{id}(\delta(a)) = \mu \otimes \mathrm{id}(w_G)\pi(a) \otimes 1\mu \otimes \mathrm{id}(w_G^*),$$

as elements of $M(K(H) \otimes C^*(G))$. Since $\mu \otimes \lambda(w_G)$ is unitary, by applying $\mathrm{id} \otimes \lambda$ to both sides we have that $\ker(\pi \otimes \lambda \circ \delta) = \ker \pi$. Thus $\ker \pi$ is invariant, and since $\pi = \phi \circ j_A$ and ϕ is faithful, $\ker j_A$ is invariant.

2. Restriction and induction of ideals

Let (A, G, δ) be a cosystem. Given the canonical embedding j_A of A into $A \times_\delta G$, we have $\mathrm{Res}_{j_A}\colon \mathcal{I}(A \times_\delta G) \to \mathcal{I}(A)$ (defined in §1). But we choose to denote this map by Res_δ. Also we have $\mathrm{Ext}_\delta\colon \mathcal{I}(A) \to \mathcal{I}(A \times_\delta G)$. Similarly for a dynamical system (A, G, α) we have $\mathrm{Res}_\alpha\colon \mathcal{I}(A \times_\alpha G) \to \mathcal{I}(A)$ and $\mathrm{Ext}_\alpha\colon \mathcal{I}(A) \to \mathcal{I}(A \times_\alpha G)$.

Define $\mathrm{Ind}_\delta\colon \mathcal{I}(A) \to \mathcal{I}(A \times_\delta G)$ by $\mathrm{Ind}_\delta(\ker \pi) = \ker(\mathrm{Ind}_\delta \pi)$, where $\mathrm{Ind}_\delta \pi$ is the induced regular representation. It will follow from our first proposition that Ind_δ is well-defined on ideals. Define $\mathrm{Ind}_\alpha\colon \mathcal{I}(A) \to \mathcal{I}(A \times_\alpha G)$ similarly. Now define

$$\begin{aligned} k_A\colon A \to M(A \otimes K(L^2(G))) &\quad \text{by} \quad k_A(a) = \mathrm{id} \otimes \lambda(\delta(a)), \\ k_{C_0(G)}\colon C_0(G) \to M(A \otimes K(L^2(G))) &\quad \text{by} \quad k_{C_0(G)}(f) = 1 \otimes M(f), \text{ and} \\ k_G\colon G \to M(A \otimes K(L^2(G))) &\quad \text{by} \quad k_G(s) = 1 \otimes \rho(s). \end{aligned}$$

One can check using standard covariance relations that $(k_A, k_{C_0(G)})$ is covariant for δ, $(k_A \times k_{C_0(G)}, k_G)$ is covariant for $\hat{\delta}$, and that

$$\phi := (k_A \times k_{C_0(G)}) \times k_G\colon (A \times_\delta G) \times_{\hat{\delta}} G \to A \otimes K(L^2(G)),$$

is a well-defined surjection. Let h be the homeomorphism $h\colon \mathcal{I}(A) \to \mathcal{I}(A \otimes K(L^2(G)))$ satisfying $h(\ker \pi) = \ker(\pi \otimes \mathrm{id})$. The following diagram is a useful summary of the maps involved.

$$\begin{tikzcd} I(A \times_\delta G) \arrow[d, shift left, "\mathrm{Res}_\delta"] \arrow[drr, shift left, "\mathrm{Res}_{\hat{\delta}}"'] & & \\ \mathcal{I}(A) \arrow[u, shift left, "\mathrm{Ind}_\delta"] \arrow[rr, "\mathrm{Res}_\psi \circ h"'] & & \mathcal{I}(A \times_\delta G \times_{\hat{\delta}} G) \arrow[ull, shift left, "\mathrm{Ind}_{\hat{\delta}}"'] \end{tikzcd} \tag{4}$$

In this section we examine the extent to which this diagram commutes. Our results are analagous to Gootman and Lazar, which are for spatial crossed products by coaction (compare [1, Corollary 2.12, Theorem 2.9, Corollary 2.10]). Our techniques lead to new self-contained proofs of the faithfulness of the induced regular representation for full coactions, the faithfulness of the induced regular representation for actions by amenable groups, and both Katayama and Imai-Takai duality.

Proposition 2.1. *Let (A, G, δ) be a cosystem and I be an ideal in A. Then*

$$\operatorname{Ind}_\delta(I) = \operatorname{Res}_{\hat\delta} \operatorname{Res}_\phi \circ h(I).$$

Proof. Using the definitions:

$$\begin{aligned} \operatorname{Res}_{\hat\delta} \operatorname{Res}_\phi \circ h(\ker\pi) &= \operatorname{Res}_{\hat\delta} \operatorname{Res}_\phi(\ker(\pi\otimes\mathrm{id})) = \operatorname{Res}_{\hat\delta}(\ker(\pi\otimes\mathrm{id})\circ\phi) \\ &= \ker((\pi\otimes\mathrm{id})\circ\phi\circ l_{A\times G}) = \ker(\pi\otimes\mathrm{id}(k_A\times k_{C_0(G)})) \\ &= \ker(\pi\otimes\mathrm{id}((\mathrm{id}\otimes\lambda\circ\delta)\times 1\otimes M)) \\ &= \ker(\pi\otimes\lambda\circ\delta\times 1\otimes M). \qquad \square \end{aligned}$$

Remark 2. It follows from this proposition that $\operatorname{Ind}_\delta$ is well-defined. It is continuous, and containment preserving, since all Res maps have these properties (§1).

Proposition 2.2. *Let (A, G, δ) be a cosystem and K be an ideal in $A\times_\delta G$. Then*

$$\operatorname{Ind}_{\hat\delta}(K) = \operatorname{Res}_\phi \circ h \circ \operatorname{Res}_\delta(K).$$

Proof. Let $\sigma\colon A\times_\delta G\to B(H)$ be a representation, $\pi=\sigma\circ j_A$ and $\mu=\sigma\circ j_{C_0(G)}$ so that $\sigma=\pi\times\mu$. Then $\operatorname{Ind}_{\hat\delta}\sigma(l_{A\times_\delta G}(j_A(a)))=\pi(a)\otimes 1$,

$$\operatorname{Ind}_{\hat\delta}\sigma(l_{A\times_\delta G}(j_{C_0(G)}(f))) = \mu\otimes M^{-1}(\alpha_G(f)), \text{ and } \operatorname{Ind}_{\hat\delta}\sigma(l_G(s)) = 1\otimes\lambda(s).$$

Also $\operatorname{Res}_\phi\circ h\circ\operatorname{Res}_\delta(\ker\sigma) = \operatorname{Res}_\phi\circ h(\ker\pi) = \operatorname{Res}_\phi(\ker(\pi\otimes\mathrm{id})) = \ker((\pi\otimes\mathrm{id})\circ\phi)$. On the generators, $(\pi\otimes\mathrm{id})\circ\phi$ does the following:

$$(\pi\otimes\mathrm{id})\circ\phi(l_{A\times_\delta G}(j_A(a))) = \pi\otimes\lambda(\delta(a)),$$

$$(\pi\otimes\mathrm{id})\circ\phi(l_{A\times_\delta G}(k_{C_0(G)}(f))) = 1\otimes M(f), \text{ and } (\pi\otimes\mathrm{id})\circ\phi(l_G(s)) = 1\otimes\rho(s).$$

We want to know that $\operatorname{Ind}_{\hat\delta}\sigma$ has the same kernel as $(\pi\otimes\mathrm{id})\circ\phi$. We claim that

$$\operatorname{Ad}(\mu\otimes\lambda(w_G)\,1\otimes S)\circ\operatorname{Ind}_{\hat\delta}\sigma = (\pi\otimes\mathrm{id})\circ\phi,$$

where S is the operator in $B(L^2(G))$ such that $S(\xi)(t)=\xi(t^{-1})$. Using the covariance of (π,μ) we obtain $\operatorname{Ad}(\mu\otimes\lambda(w_G)\ 1\otimes S)\operatorname{Ind}_{\hat\delta}\sigma(l_{A\times_\delta G}(j_A(a))) = \pi\otimes\lambda(\delta(a))$. Using equation (3) one can verify that

$$\operatorname{Ad}(\mu\otimes\lambda(w_G)\,1\otimes S)\operatorname{Ind}_{\hat\delta}\sigma(l_{A\times_\delta G}(j_{C_0(G)}(f))) = 1\otimes M(f).$$

Since the image of λ commutes with the image of ρ we have:

$$\operatorname{Ad}(\mu\otimes\lambda(w_G)\,1\otimes S)\operatorname{Ind}_{\hat\delta}\sigma(l_G(s)) = 1\otimes\rho(s). \qquad \square$$

Remark 3. These calculations are essentially those referred to in [5, Theorem 4] (cf. working in [1, Proposition 2.11]).

Corollary 2.3. *Let (A, G, δ) be a cosystem and K be an ideal in $A\times_\delta G$. Then*

$$\operatorname{Res}_{\hat\delta}\operatorname{Ind}_{\hat\delta}(K) = \operatorname{Ind}_\delta\operatorname{Res}_\delta(K).$$

Proof. $\operatorname{Res}_{\hat\delta}\operatorname{Ind}_{\hat\delta}(K) = \operatorname{Res}_{\hat\delta}\operatorname{Res}_\phi\circ h\circ\operatorname{Res}_\delta(K) = \operatorname{Ind}_\delta\operatorname{Res}_\delta(K)$. $\square$

Corollary 2.4. *Let (A, G, δ) be a cosystem and $\pi\colon A\to B(H)$ be a representation such that $\ker\pi\subseteq\ker j_A$. Then the induced regular representation $\operatorname{Ind}_\delta\pi\colon A\times_\delta G\to B(H\otimes L^2(G))$ is faithful.*

Proof. Since $\hat{\delta}$ is an action, 0 is a $\hat{\delta}$-invariant ideal in $A \times_\delta G$. Thus by Corollary 2.3, $0 = \mathrm{Res}_{\hat{\delta}} \mathrm{Ind}_{\hat{\delta}}(0) = \mathrm{Ind}_\delta \mathrm{Res}_\delta(0) = \mathrm{Ind}_\delta(\ker j_A)$. Since Ind_δ is containment preserving and $\ker \pi \subseteq \ker j_A$, $\mathrm{Ind}_\delta(\ker \pi) \subseteq \mathrm{Ind}_\delta(\ker j_A) = 0$. □

Remark 4. Corollary 2.4 was first proved by Quigg [10, Corollary 2.7], an extension of a result of Raeburn [12, Theorem 4.1(2)]; our method is competely new and significantly shorter.

It follows from Corollary 2.4 that $\ker j_A = \ker(\mathrm{id} \otimes \lambda \circ \delta)$. To see this, let π be a faithful representation of A, so that $\ker(\mathrm{id} \otimes \lambda \circ \delta) = \ker(\pi \otimes \lambda \circ \delta)$. Corollary 2.4 says that $\mathrm{Ind}_\delta(\pi)$ is faithful, so $\ker j_A = \mathrm{Res}_\delta(0) = \mathrm{Res}_\delta(\mathrm{Ind}_\delta(\ker \pi)) = \ker(\mathrm{id} \otimes \lambda \circ \delta)$.

Corollary 2.5. *Let (A, G, δ) be a cosystem and I be a δ-invariant ideal in A. Then*

$$\mathrm{Ind}_{\hat{\delta}} \mathrm{Ind}_\delta(I) = \mathrm{Res}_\phi \circ h(I).$$

Proof. Since I is δ-invariant, $\mathrm{Res}_\delta \mathrm{Ind}_\delta(I) = I$ by definition. By Proposition 2.2,

$$\mathrm{Ind}_{\hat{\delta}} \mathrm{Ind}_\delta(I) = \mathrm{Res}_\phi \circ h(\mathrm{Res}_\delta \mathrm{Ind}_\delta(I)) = \mathrm{Res}_\phi \circ h(I). \qquad \square$$

Corollary 2.6 (Katayama Duality)**.** *Let (A, G, δ) be a normal cosystem. Define*

$$\begin{aligned} k_A &\colon A \to M(A \otimes K(L^2(G))) \quad \textit{by} \quad k_A(a) = \mathrm{id} \otimes \lambda(\delta(a)), \\ k_{C_0(G)} &\colon C_0(G) \to M(A \otimes K(L^2(G))) \quad \textit{by} \quad k_{C_0(G)}(f) = 1 \otimes M(f), \ \textit{and} \\ k_G &\colon G \to M(A \otimes K(L^2(G))) \quad \textit{by} \quad k_G(s) = 1 \otimes \rho(s). \end{aligned}$$

Then $\phi := (k_A \times k_{C_0(G)}) \times k_G \colon (A \times_\delta G) \times_{\hat{\delta}} G \to A \otimes K(L^2(G))$ is a surjection, which defines an injection on the reduced crossed product $(A \times_\delta G) \times_{\hat{\delta},r} G$.

Proof. Since (A, G, δ) is normal, $\ker j_A = 0$ is invariant and Corollary 2.5 implies $\mathrm{Ind}_{\hat{\delta}} \mathrm{Ind}_\delta(0) = \ker \phi$. By Corollary 2.4, $\mathrm{Ind}_\delta(0) = 0$ and so $\ker \phi$ is equal to the kernel of the regular representation. □

Remark 5. One can verify that ϕ intertwines the double dual coaction $\hat{\hat{\delta}}$ of G on $(A \times_\delta G) \times_{\hat{\delta},r} G$ with a coaction $\tilde{\delta}$ on $A \otimes K(L^2(G))$ defined by

$$\tilde{\delta} = \mathrm{Ad}(1 \otimes M \otimes \mathrm{id}(w_G^*))[\mathrm{id} \otimes \Sigma(\delta \otimes \mathrm{id})],$$

where Σ is the flip isomorphism of $\Sigma \colon C^*(G) \otimes K \to K \otimes C^*(G)$ (use equation 1).

It follows from Corollary 2.6 and $A \times_\delta G \cong (A/\ker(j_A)) \times_{\delta^{\ker(j_A)}} G$ [8, remark after Theorem 2.3] that for any cosystem $\big(A \times_\delta G\big) \times_{\hat{\delta},r} G \cong (A/\ker j_A) \otimes K(L^2(G))$.

We now state the analagous results for the case of crossed products by actions. There is no need to repeat the arguments, since they are very similar (that's the whole point of our approach). The comparable results of Gootman and Lazar for reduced crossed products by actions are [1, Corollary 2.7, Theorem 2.4, Corollary 2.5]. Let (A, G, α) be a dynamical system, and define

$$\begin{aligned} j_A &\colon A \to M(A \otimes K(L^2(G))) \quad \text{by} \quad j_A(a) = \mathrm{id} \otimes M^{-1}(\alpha(a)), \\ j_G &\colon C_0(G) \to M(A \otimes K(L^2(G))) \quad \text{by} \quad j_G(f) = 1 \otimes \lambda(s), \quad \text{and} \\ j_{C_0(G)} &\colon G \to M(A \otimes K(L^2(G))) \quad \text{by} \quad j_{C_0(G)}(f) = 1 \otimes M(f). \end{aligned}$$

One can show that (j_A, j_G) is covariant for α, $(j_A \times j_G, j_{C_0(G)})$ is covariant for $\hat{\alpha}$, and that $\psi := (j_A \times j_G) \times j_{C_0(G)} \colon (A \times_\alpha G) \times_{\hat{\alpha}} G \to A \otimes K(L^2(G))$ is a well-defined surjection.

Proposition 2.7. *Let (A, G, α) be a dynamical system and I be an ideal in A. Then* $\operatorname{Ind}_\alpha(I) = \operatorname{Res}_{\hat\alpha} \operatorname{Res}_\psi \circ h(I)$.

Proposition 2.8. *Let (A, G, α) be a dynamical system and K be an ideal in $A \times_\alpha G$. Then* $\operatorname{Ind}_{\hat\alpha}(K) = \operatorname{Res}_\psi \circ h \circ \operatorname{Res}_\alpha(K)$.

Proof. The key equation to check is $\operatorname{Ad}(U \otimes M^{-1}(v_G)) \circ \operatorname{Ind}_{\hat\alpha} \sigma = (\pi \otimes \mathrm{id}) \circ \psi$ (use equation 2). □

Corollary 2.9. *Let (A, G, α) be a dynamical system and K an ideal in $A \times_\alpha G$. Then* $\operatorname{Res}_{\hat\alpha} \operatorname{Ind}_{\hat\alpha}(K) = \operatorname{Ind}_\alpha \operatorname{Res}_\alpha(K)$.

Corollary 2.10. *Let (A, G, α) be a dynamical system and I be an α-invariant ideal in A. Then* $\operatorname{Ind}_{\hat\alpha} \operatorname{Ind}_\alpha(I) = \operatorname{Res}_\psi \circ h(I)$.

Corollary 2.11. *Let (A, G, α) be a dynamical system, G be an amenable group and $\pi\colon A \to B(H)$ be a faithful representation of A. Then the induced representation $\operatorname{Ind}_\alpha \pi\colon A \times_\alpha G \to B(H \otimes L^2(G))$ is faithful.*

Proof. Let $l_{A\times G}$ denote the embedding of $A \times_\alpha G$ into the double crossed product $A \times_\alpha G \times_{\hat\alpha} G$, so $\ker(l_{A\times G})$ is an $\hat\alpha$-invariant ideal. Since G is amenable, the cosystem $(A \times_\alpha G, G, \hat\alpha)$ is normal, which means $\ker(l_{A\times G}) = 0$ and 0 is an $\hat\alpha$-invariant ideal of $A \times_\alpha G$. Thus by Corollary 2.9,

$$0 = \operatorname{Res}_{\hat\alpha} \operatorname{Ind}_{\hat\alpha}(0) = \operatorname{Ind}_\alpha \operatorname{Res}_\alpha(0) = \operatorname{Ind}_\alpha(\ker k_A) = \operatorname{Ind}_\alpha(0).$$

So if π is a faithful representation of A, then $\operatorname{Ind}_\alpha \pi$ is a faithful representation. □

Corollary 2.12 (Imai-Takai Duality)**.** *Let (A, G, α) be a dynamical system. Define*

$$\begin{aligned} j_A\colon A &\to M(A \otimes K(L^2(G))) \quad &&\text{by} \quad j_A(a) = \mathrm{id} \otimes M^{-1}(\alpha(a)),\\ j_G\colon G &\to M(A \otimes K(L^2(G))) \quad &&\text{by} \quad j_G(s) = 1 \otimes \lambda(s), \text{ and}\\ j_{C_0(G)}\colon C_0(G) &\to M(A \otimes K(L^2(G))) \quad &&\text{by} \quad j_{C_0(G)}(f) = 1 \otimes M(f). \end{aligned}$$

Then $\psi := (j_A \times j_G) \times j_{C_0(G)}\colon (A \times_\alpha G) \times_{\hat\alpha} G \to A \otimes K(L^2(G))$ is an isomorphism.

Proof. Since 0 is an invariant ideal of A, $\operatorname{Ind}_{\hat\alpha} \operatorname{Ind}_\alpha(0) = \ker\psi$ (Corollary 2.10), and $\operatorname{Ind}_{\hat\alpha}(K) = 0$ if $K \subseteq \ker(j_{A\times_\alpha G})$ (Corollary 2.4), it is enough to show that $\operatorname{Ind}_\alpha(0) \subseteq \ker(j_{A\times_\alpha G})$. Because $\operatorname{Ind}_\alpha$ preserves containment, $\operatorname{Ind}_\alpha(0)$ is contained in $\operatorname{Ind}_\alpha \operatorname{Res}_\alpha(\ker(j_{A\times_\alpha G}))$, and using the fact that $\ker(j_{A\times_\alpha G})$ is $\hat\alpha$-invariant, from Corollary 2.9 we deduce

$$\operatorname{Ind}_\alpha(0) \subseteq \operatorname{Ind}_\alpha \operatorname{Res}_\alpha(\ker(j_{A\times_\alpha G})) = \operatorname{Res}_{\hat\alpha} \operatorname{Ind}_{\hat\alpha}(\ker j_{A\times_\alpha G}) = \ker j_{A\times_\alpha G}. \qquad \square$$

3. Invariant ideals

The definition of δ-invariance used in this paper is different than that used in [8]. In that paper we needed to restrict a coaction to give a coaction on an ideal. It turns out that the condition required for this is different from that required to ensure there is a coaction on the quotient. However, there is a coaction on the quotient by a δ-invariant ideal by the following argument. If $I \in \mathcal{I}_\delta(A)$, that means $I = \operatorname{Res}_\delta \operatorname{Ind}_\delta(I)$, which implies I is in the image of $\operatorname{Res}_\delta$. Thus there exists a covariant pair (π, μ) such that $\ker\pi = I$. The covariance relation says $\pi \otimes \mathrm{id}(\delta(a)) = \mu \otimes \mathrm{id}(w_G)\ \pi(a) \otimes 1\ \mu \otimes \mathrm{id}(w_G^*)$, where $\pi \otimes \mathrm{id}$ means the map into $M(K(H) \otimes C^*(G))$. Since we are using minimal tensor products, $M(\pi(A) \otimes C^*(G))$ is mapped faithfully into $M(K(H) \otimes C^*(G))$, so the map $\pi \otimes \mathrm{id}$ into $M(K(H) \otimes C^*(G))$

has the same kernel as the map into $M(\pi(A)\otimes C^*(G))$. Since $\pi(A)\cong A/I$, we have $\ker(q\otimes \mathrm{id}\circ\delta)=\ker q$, where q is the quotient map. The coaction δ^I on the quotient A/I is then defined by $\delta^I(q(a))=q\otimes\mathrm{id}(\delta(a))$ [8, Proposition 2.2].

Proposition 3.1. *Let (A,G,δ) be a cosystem and I an ideal of A. Then*

(i) *the following are equivalent:*
  (a) $\mathrm{Res}_\delta(\mathrm{Ind}_\delta(I))=I$*,*
  (b) $I=\mathrm{Res}_\delta(\mathrm{Ext}_\delta(I))$*, and*
  (c) $I\in\mathcal{I}_\delta(A)$*;*
(ii) *if G is amenable, then* $\mathrm{Res}_\delta(\mathrm{Ind}_\delta(I))\subseteq I\subseteq \mathrm{Res}_\delta(\mathrm{Ext}_\delta(I))$*;*
(iii) $\mathrm{Ext}_\delta(I)=\mathrm{Ind}_\delta(I)$ *for* $I\in\mathcal{I}_\delta(A)$*;*
(iv) (a) $\mathcal{I}_\delta(A)=\mathrm{Im}\,\mathrm{Res}_\delta$ *and*
  (b) $\mathcal{I}_{\hat\delta}(A\times_\delta G)=\mathrm{Im}\,\mathrm{Ind}_\delta$*.*

Proof. (i) By definition, $I\in\mathcal{I}_\delta(A)$ if and only if $\mathrm{Res}_\delta\,\mathrm{Ind}_\delta(I)=I$, so (a) is equivalent to (c). Before we verify that (b) is equivalent to (c), we will show that $\mathcal{I}_\delta(A)=\mathrm{Im}\,\mathrm{Res}_\delta$.

If $I\in\mathcal{I}_\delta(A)$, then I is in the image of Res_δ. Conversely, suppose $I\in\mathrm{Im}\,\mathrm{Res}_\delta$. That means there is a covariant pair (π,μ) with $\ker\pi=I$, and the covariance relation says $\overline{\pi\otimes\mathrm{id}}(\delta(a))=[\mu\otimes\mathrm{id}(w_G)][\pi(a)\otimes 1][\mu\otimes\mathrm{id}(w_G^*)]$. Apply $\mathrm{id}\otimes\lambda$ to both sides to see that $\ker((\overline{\pi\otimes\lambda})\circ\delta)=\ker\pi$. Since

$$\mathrm{Ind}_\delta\,\pi=((\overline{\pi\otimes\lambda})\circ\delta)\times(1\otimes M),$$

$\mathrm{Res}_\delta\,\mathrm{Ind}_\delta(\ker\pi)=\ker((\overline{\pi\otimes\lambda})\circ\delta)$, and so $\mathrm{Res}_\delta\,\mathrm{Ind}_\delta(I)=I$. Thus $\mathcal{I}_\delta(A)=\mathrm{Im}\,\mathrm{Res}_\delta$.

Now suppose $\mathrm{Res}_\delta(\mathrm{Ext}_\delta(I))=I$. Then $I\in\mathrm{Im}\,\mathrm{Res}_\delta=\mathcal{I}_\delta(A)$. Conversely, suppose $I\in\mathcal{I}_\delta(A)$ so that $I\in\mathrm{Im}\,\mathrm{Res}_\delta$. Then $I=\mathrm{Res}_\delta(K)$ for some ideal K of $A\times_\delta G$, so $\mathrm{Res}_\delta(\mathrm{Ext}_\delta(I))=\mathrm{Res}_\delta\,\mathrm{Ext}_\delta\,\mathrm{Res}_\delta(K)=\mathrm{Res}_\delta(K)=I$, which shows that (c) is equivalent to (b).

(ii) We have $I\subseteq\mathrm{Res}_\delta\,\mathrm{Ext}_\delta(I)$ from [9, Lemma 1.1](ii). Now suppose G is amenable. Then λ is faithful, so that $\mathrm{id}\otimes\lambda\colon A\otimes C^*(G)\to M\Big(A\otimes K(L^2(G))\Big)$ is faithful. We want to show that $\mathrm{Res}_\delta\,\mathrm{Ind}_\delta(I)\subseteq I$, so let π be a representation of A such that $\ker\pi=I$. Suppose $a\in\mathrm{Res}_\delta\,\mathrm{Ind}_\delta(\ker\pi)$, that is, $\overline{\pi\otimes\lambda}(\delta(a))=0$. Then

$$(\overline{\mathrm{id}\otimes\lambda})\circ(\overline{\pi\otimes\mathrm{id}})(\delta(a))=0.$$

Since $\overline{\mathrm{id}\otimes\lambda}$ is faithful, $\overline{\pi\otimes\mathrm{id}}(\delta(a))=0$, and since $\mathrm{id}\otimes 1(\delta(a))=a$, it follows that $\pi(a)=0$.

(iii) Let $I\in\mathcal{I}_\delta(A)$ and $\pi\colon A\to B(H)$ be a representation of A with $\ker\pi=I$. Let $\tau\colon A/I\to B(H)$ be the associated faithful representation. Since I is δ-invariant, there is a coaction δ^I on the quotient A/I. By Corollary 2.4, the induced representation $\mathrm{Ind}_\delta\,\tau\colon(A/I)\times_{\delta^I}G\to B\Big(H\otimes L^2(G)\Big)$ is faithful. By [8, Theorem 2.3], there is an isomorphism $\psi\colon(A\times_\delta G)/\mathrm{Ext}_\delta(I)\to(A/I)\times_{\delta^I}G$, so there is a faithful representation of $(A\times_\delta G)/\mathrm{Ext}_\delta(I)$ given by $\eta=\mathrm{Ind}_\delta\,\tau\circ\psi$. The representation $\eta'\colon A\times_\delta G\to B\Big(H\otimes L^2(G)\Big)$ defined by $\eta'(a)=\eta(a+\mathrm{Ext}_\delta(I))$, has kernel $\mathrm{Ext}_\delta(I)$. To show that $\mathrm{Ind}_\delta(I)=\mathrm{Ext}_\delta(I)$, one can show that $\mathrm{Ind}_\delta\,\pi=\eta'$, by checking they agree on the generators.

(iv) We showed (a) in part (i), so it remains to show (b). Now suppose $K\in\mathcal{I}_{\hat\delta}(A\times_\delta G)$. By Corollary 2.3, $K=\mathrm{Res}_{\hat\delta}\,\mathrm{Ind}_{\hat\delta}(K)=\mathrm{Ind}_\delta\,\mathrm{Res}_\delta(K)$, so $\mathcal{I}_{\hat\delta}(A\times_\delta G)\subseteq$

$\operatorname{Im}\operatorname{Ind}_\delta$. For the reverse containment, we need to know that $\operatorname{Ind}_\delta(\ker\pi)$ is a $\hat{\delta}$-invariant ideal of $A\times_\delta G$. Routine calculations show that $(\operatorname{Ind}_\delta\pi, 1\otimes\rho)$ is covariant for $\hat{\delta}$; thus $\operatorname{Ind}_\delta(\ker\pi)\in\operatorname{Im}\operatorname{Res}_{\hat{\delta}}=\mathcal{I}_{\hat{\delta}}(A\times_\delta G)$, by Proposition 3.3(iv)(a). □

Corollary 3.2. *Let (A,G,δ) be a cosystem. Then*

(i) $\operatorname{Res}_\delta\colon\mathcal{I}_{\hat{\delta}}(A\times_\delta G)\to\mathcal{I}_\delta(A)$ *is a homeomorphism,*
(ii) $\operatorname{Res}_\delta\colon\mathcal{I}(A\times_\delta G)\to\mathcal{I}_\delta(A)$ *is a continuous, open surjection, and,*
(iii) *if G is amenable, then* $\operatorname{Res}_\delta\operatorname{Ind}_\delta\colon\mathcal{I}(A)\to\mathcal{I}_\delta(A)$ *is continuous, open and surjective.*

Proof. (i) Let $I\in\mathcal{I}_\delta(A)$. To show that $\operatorname{Res}|_{\mathcal{I}_{\hat{\delta}}(A\times_\delta G)}$ is onto, we must find an invariant ideal of $A\times_\delta G$ which maps to I. We claim $\operatorname{Ind}_\delta(I)$ will do. Firstly it is $\hat{\delta}$-invariant by Proposition 3.1(iii), and since I is invariant, $\operatorname{Res}_\delta\operatorname{Ind}_\delta(I)=I$ by definition.

Next we show $\operatorname{Res}|_{\mathcal{I}_{\hat{\delta}}(A\times_\delta G)}$ is injective. Suppose K and L are $\hat{\delta}$-invariant ideals of $A\times_\delta G$, and $\operatorname{Res}_\delta(K)=\operatorname{Res}_\delta(L)$. Since they are invariant, there exist ideals I and J of A such that $\operatorname{Ind}_\delta(I)=K$ and $\operatorname{Ind}_\delta(J)=L$. Thus $\operatorname{Res}_\delta\operatorname{Ind}_\delta(I)=\operatorname{Res}_\delta\operatorname{Ind}_\delta(J)$, and applying $\operatorname{Ind}_\delta$ to both sides gives $\operatorname{Ind}_\delta\operatorname{Res}_\delta\operatorname{Ind}_\delta(I)=\operatorname{Ind}_\delta\operatorname{Res}_\delta\operatorname{Ind}_\delta(J)$. From Proposition 3.1(iii) one can deduce $\operatorname{Ind}_\delta\operatorname{Res}_\delta\operatorname{Ind}_\delta=\operatorname{Ind}_\delta$, so $\operatorname{Ind}_\delta(I)=\operatorname{Ind}_\delta(J)$, which means $K=L$.

We know $\operatorname{Res}_\delta$ is continuous, so we show $\operatorname{Res}|_{\mathcal{I}_{\hat{\delta}}(A\times_\delta G)}$ is a homeomorphism by verifying that $\operatorname{Ind}_\delta$ is a continuous inverse. Well, $\operatorname{Ind}_\delta$ is continuous by Proposition 2.1, and $\operatorname{Res}_\delta\operatorname{Ind}_\delta(I)=I$ for invariant ideals by definition. Let K be a $\hat{\delta}$-invariant ideal of $A\times_\delta G$, so it is of the form $\operatorname{Ind}_\delta(I)=K$ for a δ-invariant ideal of A. Thus $\operatorname{Ind}_\delta\operatorname{Res}_\delta(K)=\operatorname{Ind}_\delta\operatorname{Res}_\delta(\operatorname{Ind}_\delta(I))=\operatorname{Ind}_\delta(I)=K$.

(ii) We need to show $\operatorname{Res}_\delta$ is open. Proposition 3.1(ii) says that $\operatorname{Ext}_\delta(I)=\operatorname{Ind}_\delta(I)$ for $I\in\mathcal{I}_\delta(A)$; thus $\operatorname{Ext}_\delta|_{\mathcal{I}_\delta(A)}$ is continuous. Now Res is open onto its range if and only if $\operatorname{Ext}|_{\operatorname{Im}\operatorname{Res}}$ (see §1) and since $\mathcal{I}_\delta(A)=\operatorname{Im}\operatorname{Res}_\delta$ (Proposition 3.1(iii)), this implies that $\operatorname{Res}_\delta$ is open onto its range.

(iii) Since $\operatorname{Res}_\delta\operatorname{Ind}_\delta$ is continuous, it remains to show $\operatorname{Res}_\delta\operatorname{Ind}_\delta$ is open. In Corollary 3.4(ii) we show that $\operatorname{Res}_{\hat{\delta}}\colon\mathcal{I}((A\times_\delta G)\times_{\hat{\delta}}G)\to\mathcal{I}_{\hat{\delta}}(A\times_\delta G)$ is open onto its range when G is amenable. That corollary also says that $\operatorname{Res}_{\hat{\delta}}$ restricted to $\mathcal{I}_{\hat{\hat{\delta}}}((A\times_\delta G)\times_{\hat{\delta}}G)$ is a homeomorphism with inverse $\operatorname{Ind}_{\hat{\delta}}$, so $\operatorname{Ind}_{\hat{\delta}}|_{\mathcal{I}_{\hat{\delta}}(A\times_\delta G)}$ is open onto its range. Thus $\operatorname{Ind}_{\hat{\delta}}|_{\operatorname{Im}\operatorname{Res}_{\hat{\delta}}}$ is open onto its range because $\mathcal{I}_{\hat{\delta}}(A\times_\delta G)=\operatorname{Im}\operatorname{Res}_{\hat{\delta}}$. We show that $\operatorname{Res}_\delta\operatorname{Ind}_\delta$ is open by showing that

$$\operatorname{Res}_\delta\operatorname{Ind}_\delta=(\operatorname{Res}_\phi\circ h)^{-1}\operatorname{Ind}_{\hat{\delta}}\operatorname{Res}_{\hat{\delta}}(\operatorname{Res}_\phi\circ h).$$

This makes sense because G is amenable, and Corollary 2.6 (Katayama duality), says that ϕ is an isomorphism, so that Res_ϕ is a homeomorphism. From Corollaries 2.5 and 2.1 we conclude

$$(\operatorname{Res}_\phi\circ h)\operatorname{Res}_\delta\operatorname{Ind}_\delta=\operatorname{Ind}_{\hat{\delta}}\operatorname{Ind}_\delta\operatorname{Res}_\delta\operatorname{Ind}_\delta=\operatorname{Ind}_{\hat{\delta}}\operatorname{Ind}_\delta=\operatorname{Ind}_{\hat{\delta}}\operatorname{Res}_{\hat{\delta}}(\operatorname{Res}_\phi\circ h).\qquad\square$$

The first three parts of the next proposition are in [9, Lemma 5.2(ii), (iii)], and the proof of the last part is the same as Proposition 3.1(iv). The proof of the corollary following mirrors Corollary 3.2.

Proposition 3.3. *Let (A,G,α) be a dynamical system and I an ideal of A. Then*

(i) *the following are equivalent:*
 (a) $\operatorname{Res}_\alpha(\operatorname{Ind}_\alpha(I))=I$,
 (b) $I=\operatorname{Res}_\alpha(\operatorname{Ext}_\alpha(I))$, *and*

(c) *$I \in \mathcal{I}_\alpha(A)$;*

(ii) $\operatorname{Res}_\alpha(\operatorname{Ind}_\alpha(I)) \subseteq I \subseteq \operatorname{Res}_\alpha(\operatorname{Ext}_\alpha(I))$;

(iii) $\operatorname{Ext}_\alpha(I) \subseteq \operatorname{Ind}_\alpha(I)$ *for* $I \in \mathcal{I}_\alpha(A)$, *with equality if* G *is amenable;*

(iv) (a) $\mathcal{I}_\alpha(A) = \operatorname{Im}\operatorname{Res}_\alpha$ *and*

(b) $\mathcal{I}_{\hat{\alpha}}(A \times_\alpha G) = \operatorname{Im}\operatorname{Ind}_\alpha$.

Corollary 3.4. *Let* (A, G, α) *be a dynamical system. Then*

(i) $\operatorname{Res}_\alpha \colon \mathcal{I}_{\hat{\alpha}}(A \times_\alpha G) \to \mathcal{I}_\alpha(A)$ *is a homeomorphism,*

(ii) *if* G *is amenable, then* $\operatorname{Res}_\alpha \colon \mathcal{I}(A \times_\alpha G) \to \mathcal{I}_\alpha(A)$ *is a continuous, open surjection, and,*

(iii) $\operatorname{Res}_\alpha \operatorname{Ind}_\alpha \colon \mathcal{I}(A) \to \mathcal{I}_\alpha(A)$ *is a continuous, open surjection.*

Remark 6. Gootman and Lazar proved that $\operatorname{Res}_\delta \operatorname{Ind}_\delta$ is open for spatial crossed products and G amenable [1, Proposition 4.6], and Green proved that $\operatorname{Res}_\alpha \operatorname{Ind}_\alpha$ is open [2, pp. 221-222]; our method of proof is completely new.

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