

**A NOTE ON FIXED POINT THEOREMS
FOR SEMI-CONTINUOUS CORRESPONDENCES ON $[0, 1]$**

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ABSTRACT. This paper presents a fixed point theorem for correspondences on $[0, 1]$. Some examples comparing it to related work and also some simple applications to game theory are included.

1. INTRODUCTION

A function that is either *continuous* or *nondecreasing* from $[0, 1]$ to itself has a fixed point. What are common points between these two results? This problem has been investigated independently by Milgrom and Roberts [4] and Guillaume [1]. Their findings can be summarized as:

Theorem 1.1. *If a real function $f : [0, 1] \rightarrow [0, 1]$ is upper semicontinuous on the right and lower semicontinuous on the left, then it has a fixed point.*

On the other hand, Strother [5] asserted that a continuous multivalued function on $[0, 1]$ has a fixed point. This is a surprising result in the sense that a fixed point theorem for a multivalued function usually requires the function to have convex values—for example, Kakutani’s theorem, or some theorems involving a continuous selection (for which Michael’s [3] result is often needed). Do we have an analogy to Theorem 1.1 for multivalued functions (which are often called correspondences in economics)? The answer is positive. Strother’s result was proved also by constructing a continuous selection. But our result is not. Moreover, we shall give an example to show that correspondences satisfying the hypothesis of our theorem may not allow a continuous selection. We also include some simple applications to game theory.

2. FIXED POINT THEOREMS

First we give some definitions. Let $T : \mathbb{R} \rightarrow 2^Y$ be a correspondence, where \mathbb{R} is the real line, Y is a metric space and 2^Y is the set of all nonempty subsets of Y . T is said to be upper semicontinuous on the right (RUS) if for any $\bar{x} \in \mathbb{R}$ and any sequence (x_n) such that $x_n \downarrow \bar{x}$, if a sequence (y_n) converging to \bar{y} is such that $y_n \in T(x_n) \forall n$, then $\bar{y} \in T(\bar{x})$; T is said to be lower semicontinuous on the

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left (LLS) if for any sequence $x_n \uparrow \bar{x}$ and every $\bar{y} \in T(\bar{x})$, there is a sequence (y_n) converging to \bar{y} such that for every n , one has $y_n \in T(x_n)$.

Now we are ready to prove an intermediate value theorem.

Proposition 2.1. *Let $T : [0, 1] \rightarrow 2^{\mathbb{R}}$ be an RUS and LLS correspondence. Suppose that $T([0, 1])$ is contained in some bounded set of \mathbb{R} and for any $y \in T(0)$ we have $y \geq 0$, while for any $y \in T(1)$ we have $y \leq 0$. Then there exists $x \in [0, 1]$ such that $0 \in T(x)$.*

Proof. Define $x_L = \inf\{x : \text{there exists some } y \in T(x) \text{ such that } y \leq 0\}$. Obviously x_L exists. If $x_L = 1$, we have $0 \in T(1)$. Indeed, otherwise, take any $y_L \in T(1)$. Then $y_L < 0$. Now take any sequence (x_n) in $[0, 1]$ such that $x_n \uparrow x_L$. Since T is LLS, we can find (y_n) such that $0 \leq y_n \in T(x_n)$ for each n and $y_n \rightarrow y_L$. This is a contradiction. In fact, $T(1) = \{0\}$ in this case.

Now suppose that $0 \leq x_L < 1$. Then there is a sequence (x_n) in $[0, 1]$ such that $x_n \downarrow x_L$ and there is a sequence (y_n) such that, for each n , $y_n \in T(x_n)$ and $y_n \leq 0$. Since the set $T([0, 1])$ is bounded, we can have a subsequence (y_{n_i}) of (y_n) converging to some $y_L = \lim y_{n_i} \leq 0$. Since T is RUS, we have $y_L \in T(x_L)$. If $x_L = 0$, since $y_L \in T(x_L) = T(0)$, we have $y_L \geq 0$. Thus $y_L = 0$, which implies $0 \in T(0)$ in this case. If $x_L > 0$, we can find a sequence (x_n) in $[0, 1]$ such that $x_n \uparrow x_L$. Since T is LLS, there exists (y_n) such that for each n , $y_n \in T(x_n)$ and $y_n \rightarrow y_L$. Since $y_n > 0$ for each n , we have $y_L \geq 0$. So $y_L = 0$, i.e. $0 \in T(x_L)$. \square

Theorem 2.2. *Let $T : [0, 1] \rightarrow 2^{[0,1]}$ be an RUS and LLS correspondence. Then T has a fixed point.*

Proof. Define $G : [0, 1] \rightarrow 2^{\mathbb{R}}$ by $G(x) = \{y - x : \text{for } y \in T(x)\}$; then G satisfies all conditions of Proposition 2.1. So there is \bar{x} such that $0 \in G(\bar{x})$, which means that $\bar{x} \in T(\bar{x})$. \square

Example 2.3. Let $F : [0, 1] \rightarrow 2^{[0,1]}$ be defined by

$$F(x) = \begin{cases} [1/3 - (x+1)/8, 2/3 + (x+1)/8], & x < 1/2; \\ [1/3, 2/3] \cup \{3/4\}, & x = 1/2; \\ [2/5, 3/5] \cup \{3/4\}, & x > 1/2. \end{cases}$$

It is easy to check that F is RUS and LLS, but it is neither upper semicontinuous nor lower semicontinuous. Moreover, it has non-convex values. Therefore the usual Kakutani fixed point theorem cannot be applied.

In [5], Strother proved the result we mentioned above by constructing a continuous selection of the continuous correspondence. We give an example to show that our conditions may not allow a continuous selection.

Example 2.4. Let $F : [0, 1] \rightarrow 2^{[0,1]}$ be defined by

$$F(x) = \begin{cases} [0, 1], & x \leq 1/2; \\ \{|\sin(1/(x-1/2))|, |\cos(1/(x-1/2))|\}, & x > 1/2. \end{cases}$$

Then F is RUS and LLS, but it is not possible that F has a continuous selection. For if it had, we denote it by f and suppose $f(.5) = a \in [0, 1]$. If $a = \sin(\pi/4) = \cos(\pi/4)$, we take $x_n = .5 + 1/(n\pi)$; then $F(x_n) = \{0, 1\}$, and $d(a, F(x_n)) = (1 - \sin(\pi/4))$. If $a \neq \sin(\pi/4)$, take $x_n = 1/2 + 1/(n\pi + \pi/4)$; then $d(a, F(x_n)) = |a - \sin(\pi/4)|$. So $F(x)$ has no continuous selection. This also demonstrates that F

is not continuous, for otherwise F would have a continuous selection by Strother's Theorem.

Again, since F is not convex-valued, the Kakutani fixed point theorem cannot be applied.

3. APPLICATIONS TO GAME THEORY

Consider a *two-person game* in which the two players are identified as Player 1 and Player 2. The strategy spaces for them are sets X_1 and X_2 , respectively. The best responses of Player 1 and Player 2 are defined by correspondences $T_1 : X_2 \rightarrow 2^{X_1}$ and $T_2 : X_1 \rightarrow 2^{X_2}$, respectively. Let $T : X_1 \times X_2 \rightarrow 2^{X_1} \times 2^{X_2}$ be defined by $T(x) = T_1(x_2) \times T_2(x_1)$ for $x = (x_1, x_2) \in X_1 \times X_2$. The set of *Nash equilibrium points* of this game is defined as the set of fixed points of T . Note that such a game has a non-empty set of equilibria if and only if the composite correspondence $T_1T_2 : X_1 \rightarrow 2^{X_1}$ or $T_2T_1 : X_2 \rightarrow 2^{X_2}$ has a fixed point.

Theorem 3.1. *For a two-person game on $[0, 1] \times Y$, where Y is a compact metric space, let $T_1(y)$ be the best response of Player 1 and $T_2(x)$ be the best response of Player 2. We suppose that T_1 is continuous with closed values and T_2 is RUS and LLS with closed values. Then there is a Nash equilibrium for this game.*

Proof. We define $T : [0, 1] \rightarrow 2^{[0,1]}$ by $T = T_1T_2$. If T has a fixed point \hat{x} , then there is $\hat{y} \in T_2(\hat{x})$ such that $\hat{x} \in T_1(\hat{y})$. So (\hat{x}, \hat{y}) will be a Nash equilibrium of the game.

Now we prove that T is RUS. Suppose that $x_n \downarrow x$ and $z_n \in T(x_n) \rightarrow z$. Then there is $y_n \in T_2(x_n)$ such that $z_n \in T_1(y_n)$. Since Y is compact, without loss of generality we suppose that (y_n) converges to y . Then $z \in T_1(y)$, since T_1 is continuous with closed values (hence it is closed). But $y \in T_2(x)$, since T_2 is RUS. Therefore $z \in T_1T_2(x) = T(x)$. That is, T is RUS.

Next we shall prove that T is LLS. Suppose that $x_n \uparrow x$ and $z \in T(x)$. First there is $y \in T_2(x)$ such that $z \in T_1(y)$. Since T_2 is LLS, there is $y_n \in T_2(x_n)$ such that $y_n \rightarrow y$. Since $y_n \rightarrow y$ and $z \in T_1(y)$, there is $z_n \in T_1(y_n)$ such that $z_n \rightarrow z$, since T_1 is lower semicontinuous. Now we have $z_n \in T_1T_2(x_n)$ such that $z_n \rightarrow z \in T(x)$. Thus T is LLS.

So by Theorem 2, there is a fixed point $\hat{x} \in T(\hat{x})$, i.e. there is a Nash equilibrium for the two person game. \square

Corollary 3.2. *For a two-person game on $[0, 1] \times Y$, where Y is a Banach space, let $T_1(y)$ be the best response of Player 1 and $T_2(x)$ be the best response of Player 2. We suppose that T_1 is lower semicontinuous with closed convex values and T_2 is RUS and LLS with closed values. Then there is a Nash equilibrium for this game.*

Proof. In fact, by Theorem 3.2'' in Michael [3] there is a continuous selection of T_1 . So the result follows. \square

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