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A VOLUME COMPARISON THEOREM FOR FINSLER MANIFOLDS

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Abstract. Let (M^n, F) be a symmetric Finsler manifold, endowed with the Busemann volume form, and let D be its unit disk bundle endowed with the canonical symplectic volume form. It is shown that $Vol(D) \leq C(n)Vol(M^n)$, where $C(n)$ is the volume of the unit disk in $\mathbb{R}^n$. Moreover, equality holds if and only if (M^n, F) is Riemannian.

1. Introduction

A *Finsler structure* on a manifold M^n is a function $F : TM \to \mathbb{R}$ which is homogenous of degree 1, strictly convex and smooth off the zero section. The function F is to be thought of as a "norm" in each tangent space T_pM, which is not necessarily Euclidean. The Finsler manifold (M, F) is said to be *symmetric* if $F(v) = F(-v)$.

There has been a revived interest in the study of Finsler manifolds recently, sparked by S.S. Chern and others, (e.g., [2], [3], [9]). There are also some older, beautiful geometric studies of Finsler manifolds; for example, [1], [5], [6], [7]. A basic problem is to find computable invariants that distinguish Finsler manifolds from Riemannian manifolds; the basic invariant is the Cartan 3-tensor A, which is, in essence, simply three derivatives of the energy function along the fibers. Riemannian manifolds are characterized by $A \equiv 0$. (see [2]).

In this note we will find a global integral invariant of Finsler metrics that attains its maximum exactly at the set of Riemannian metrics: the ratio of the symplectic volume of the unit disk bundle to the volume of the manifold. The philosophy behind the construction is that of Busemann: Finsler geometry should be thought of as the geometry of families of convex sets in $\mathbb{R}^n$, parametrized by a manifold. These geometries can be put together via the associated calculus of variations, i.e. the Hamiltonian geometry of (TM, ω, F), where ω is the pullback of the canonical symplectic form of T^*M via the Legendre transformation.

2. The Busemann volume form

Let (M, F) be a symmetric Finsler manifold. The *Busemann volume form* of (M, F) is defined as follows: let $e^1, \dots, e^n$ be a basis for T_xM, with dual basis

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$\eta_1, \ldots, \eta_n$. Given $x \in M$, let $D(x) \subset \mathbb{R}^n$ be given by

$$D(x) = \{y \in \mathbb{R}^n \,:\, F(y_i e^i) \leq 1\}$$

(summation convention will be used throughout). Denote by $V(x)$ the volume of $D(x)$, with respect to the standard Euclidean structure of $\mathbb{R}^n$. Then the Busemann volume form B_F is given by

$$B_F = \frac{C(n)}{V(x)} \eta_1 \wedge \ldots \wedge \eta_n,$$

where $C(n)$ is the volume of the unit disk in $\mathbb{R}^n$. It is easy to see that B does not depend on the choice of the (positively oriented) basis. The induced measure on M coincides with the Hausdorff measure given by the metric; thus B_F is in the metric sense the "right" volume form for a Finsler manifold (see [4], [9]). As usual, we define the volume of M by

$$vol(M) = \int_M B_F.$$

On the other hand, the punctured tangent bundle T_0M has a symplectic form ω, given by the pullback of the canonical symplectic form of T^*M by the Lagrange transformation $v \mapsto L_F(v)$. The top form $\frac{(-1)^n}{(2n)!}\omega^n$ is the canonical volume form on T_0M. In local coordinates $v = (x^i, y_i \frac{\partial}{\partial x^i})$ of TM, we can write

$$L(x, y) = \frac{1}{2}\frac{\partial F^2}{\partial y_i} dx^i, \qquad \frac{(-1)^n}{(2n)!}\omega^n = \det(g_{ij}) dx^1 \wedge \cdots \wedge dx^n \wedge dy_1 \cdots \wedge dy_n,$$

where $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y_i \partial y_j}$ is called the fundamental tensor of the Finsler metric F.

Let $C(n)$ and $c(n)$ be the volume of the unit disk and the unit sphere in $\mathbb{R}^n$, respectively. Then we have

Theorem 1. *Let D be the unit disk in T_0M, and let $Vol(D)$ be its volume with respect to the symplectic form. Then*

$$Vol(D) \leq C(n) vol(M),$$

with equality if and only if (M, F) is Riemannian.

Recall that $\omega = -d\alpha$ where α is the pullback by the Lagrange transformation of the canonical 1-form of T^*M. Then the unit tangent bundle $\{F = 1\}$ has an exact contact structure given by α, and corresponding volume form $\frac{1}{(n-1)!}\alpha \wedge (d\alpha)^{n-1}$. We have then the corresponding theorem for the unit tangent bundle:

Corollary. *Let S be the unit tangent bundle $\{F = 1\}$, and let $Vol(S)$ be its volume with respect to the contact structure. Then*

$$Vol(S) \leq c(n) vol(M),$$

with equality if and only if (M, F) is Riemannian.

3. Proof of Theorem 1

Cover M with a full measure coordinate patch giving us coordinates (x, y) on TM. Then

$$\omega^n = \det(g_{ij}) dx^1 \wedge \ldots \wedge dx^n \wedge dy_1 \wedge \ldots \wedge dy_n,$$

which we write in the more compact notation $\det(g_{ij}) dx \wedge dy$.

Integrating over the fiber, we have

$$\int_D \omega = \int_{x\in M}\int_{D_x}\omega = \int_{x\in M} dx\int_{D(x)} \det(g_{ij})(x,y)dy.$$

Let $D^*(x)$ be the dual of the convex set $D(x)$. Recall that given a convex set K in a vector space V, the *dual set* $K^* \subset V^*$ is defined by

$$K^* = \{\xi \in V^* \,:\, \xi(v) \le 1 \,\forall\, v \in K\}$$

(see the beautiful book [10] for more details on duality and convex geometry in general).

Note that the map

$$y_i\partial_{x^i} \mapsto \frac{1}{2}\frac{\partial F^2}{\partial y_i}(y)dx^i$$

is a diffeomorphism between $D(x)$ and $D^*(x)$, with Jacobian $\det(g_{ij})$. Therefore,

$$\int_{D(x)} \det(g_{ij})(x,y)dy = \int_{D^*(x)} dy = Vol(D^*(x)).$$

Integrating over M, we have

$$Vol(D) = \int_M Vol(D^*(x))dx = \frac{1}{C(n)}\int_M C(n)Vol(D^*(x))dx.$$

Let $\mu(x) = Vol(D(x))Vol(D^*(x))$. The volume product μ is an affine invariant of the convex body $D(x)$, and we have Santaló's inequality (see [8] and the references therein)

$$\mu(x) \le Vol(E)Vol(E^*) = C(n)^2,$$

where E is any ellipsoid. Moreover, equality occurs if and only if $D(x)$ is an ellipsoid.

Then we have

$$\begin{aligned} Vol(D) &= \int_M Vol(D^*(x))dx \\ &= \frac{1}{C(n)}\int_M \frac{C(n)\mu(x)}{Vol(D(x))}dx \\ &= \frac{1}{C(n)}\int_M \mu(x)B_F \\ &\le C(n)vol(M), \end{aligned}$$

which proves the inequality in Theorem 1. Equality occurs if and only if $\mu(x) \equiv C(n)^2$ which only happens if for each x, $F(x,y) \le 1$ is the interior of an ellipsoid symmetric with respect to the origin, which means that the metric is Riemannian.

The corollary follows from Theorem 1 and Stokes's Theorem.

Note that the proof of Theorem 1 and the non-symmetric version of Santaló's inequality also show the following for non-symmetric Finsler manifolds:

Let (M^n, F) be a non-symmetric Finsler manifold such that the centroid of the unit disk coincides with zero. Then $Vol(D) \le C(n)Vol(M)$, with equality if and only if the unit disk is an ellipsoid. Therefore such a Finsler metric is actually symmetric and Riemannian.

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