

A NEW CHARACTERISTIC OF MÖBIUS TRANSFORMATIONS BY USE OF APOLLONIUS QUADRILATERALS

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ABSTRACT. The purpose of this paper is to give a new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping. To this end a new concept of “Apollonius quadrilaterals” is used.

1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout the paper, unless otherwise stated, let $w = f(z)$ be a nonconstant meromorphic function of a complex variable z in $|z| < +\infty$. We consider the following Property A:

Property A. The function $w = f(z)$ transforms circles on the z -plane onto circles on the w -plane, including straight lines among circles.

The well-known principle of circle-transformation (see [1], [4], [5], [7, p. 160]) reads:

Theorem A. $w = f(z)$ satisfies Property A iff $w = f(z)$ is a Möbius transformation.

Before stating Property B we shall give the definition of Apollonius quadrilaterals on the complex plane.

Definition. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on the complex plane. If $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA}$ holds, then $ABCD$ is said to be an *Apollonius quadrilateral*. ($\overline{AB} = |z_1 - z_2|$ where z_1, z_2 are the complex numbers corresponding to the points A, B respectively.)

Example 1. Each of a square, a rhombus and a kite is an Apollonius quadrilateral.

Example 2. If from a point outside a circle a secant and two tangents are drawn, then the four points on the circle form an Apollonius quadrilateral.

Proof. The proof follows from the following theorem (see [3, p. 7]): If a chord of a circle is drawn from the point of contact of a tangent, the angle made by the chord with the tangent is equal to the angle subtended by the chord at a point on that part of the circumference which lies on the far side of the chord.

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We may now state Property B. In section 3 we shall prove that Property A implies the following Property B.

Property B. Suppose that $w = f(z)$ is analytic and univalent in a non-empty domain R on the z -plane. Let $ABCD$ be an arbitrary Apollonius quadrilateral contained in R . If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then $A'B'C'D'$ is also an Apollonius quadrilateral on the w -plane.

The purpose of this paper is to give a new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping and to give a new proof of the only if part of Theorem A. The results in this paper are contained in the following theorem and its corollary.

Main Theorem. *The function $w = f(z)$ satisfies Property B iff $w = f(z)$ is a Möbius transformation.*

Corollary. *This theorem gives a new proof of the only if part of Theorem A.*

2. LEMMAS

In sections 3 and 4 we shall apply the following three lemmas:

Lemma 1. *If the function $w = f(z)$ satisfies Property A, then $w = f(z)$ is univalent in $|z| < +\infty$.*

Proof. See [4].

Lemma 2 (The Theorem of Apollonius). *We include straight lines among circles.*

- (i) *The locus of a point, which moves so that the ratio of its distances from two fixed points is constant, is a circle with respect to which those points are inverse.*
- (ii) *If A, B are two fixed points inverse with respect to a given circle, then the ratio $\frac{PA}{PB}$ is constant for all positions of P on that circle.*

Proof. (i) See [6, p. 129].

(ii) See [6, pp. 129–130].

Lemma 3. *If the function $w = f(z)$ is analytic and univalent in a non-empty domain R , then $f'(z) \neq 0$ in R .*

Proof. See [9, p. 302].

3. PROOF THAT PROPERTY A IMPLIES PROPERTY B

Suppose that $w = f(z)$ is analytic in a non-empty domain R on the z -plane. Since, by hypothesis, $w = f(z)$ satisfies Property A, by Lemma 1 $w = f(z)$ is univalent in $|z| < +\infty$. Thus $w = f(z)$ is univalent in R . Let $ABCD$ be an arbitrary Apollonius quadrilateral contained in R . Then, by definition we obtain

$$\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA},$$

and so

$$(1) \quad \frac{\overline{AB}}{\overline{CB}} = \frac{\overline{AD}}{\overline{CD}}.$$

We denote (1) by N . By Lemma 2 (i) the locus of a point, which moves so that the ratio of its distances from two fixed points A, C is N ($= \text{const}$), is a circle with

respect to which A, C are inverse. We denote the above circle of Apollonius by K and let $K' = f(K)$. Since, by hypothesis, $w = f(z)$ satisfies Property A, K' is also a circle on the w -plane. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$, then by the fact that B, D are on K and by $K' = f(K)$ B', D' are on the circle K' . Since A, C are inverse with respect to the circle K , by the Reflection Principle [10, p. 155] of analytic functions it follows that $A' = f(A)$, $C' = f(C)$ are also inverse with respect to the circle $K' = f(K)$. Consequently, by Lemma 2 (ii) we obtain

$$\frac{\overline{A'B'}}{\overline{C'B'}} = \frac{\overline{A'D'}}{\overline{C'D'}}$$

which implies

$$\overline{A'B'} \cdot \overline{C'D'} = \overline{B'C'} \cdot \overline{D'A'}.$$

Therefore, the quadrilateral $A'B'C'D'$ on the w -plane is also an Apollonius quadrilateral, and so we get the desired result.

4. PROOF OF THE MAIN THEOREM

If $w = f(z)$ is a Möbius transformation, then, by the if part of Theorem A, $w = f(z)$ satisfies Property A. Thus, by the result in section 3, $w = f(z)$ satisfies Property B. Then, we shall prove the only if part of the main theorem, i.e., if $w = f(z)$ satisfies Property B, then $w = f(z)$ is a Möbius transformation. Since $w = f(z)$ is analytic and univalent in the domain R , by Lemma 3 it follows that

$$(2) \quad f'(z) \neq 0.$$

If x is an arbitrarily fixed point of R , then, by (2) we obtain

$$(3) \quad f'(x) \neq 0.$$

Let E be the point represented by x . Since $E \in R$, there exists a positive real number r such that the r closed neighbourhood of E is contained in R . We denote this closed neighbourhood by V . Throughout the rest of the proof let $ABCD$ denote an arbitrary square which is contained in V and whose centre is at E . Here the sense of A, B, C, D is counterclockwise. Since $ABCD$ is a square contained in V , we can represent A, B, C, D by complex numbers

$$x + y, x + iy, x - y, x - iy \quad (|y| \leq r),$$

respectively. Since $w = f(z)$ is univalent in R , A' ($= f(A)$), B' ($= f(B)$), C' ($= f(C)$), D' ($= f(D)$) are different points. Since, by hypothesis, $A'B'C'D'$ is an Apollonius quadrilateral, by definition we obtain

$$(4) \quad \overline{A'B'} \cdot \overline{C'D'} = \overline{B'C'} \cdot \overline{D'A'}.$$

Since A', B', C', D' are represented by

$$f(x + y), f(x + iy), f(x - y), f(x - iy),$$

respectively, we obtain

$$(5) \quad \overline{A'B'} = |f(x + y) - f(x + iy)|,$$

$$(6) \quad \overline{B'C'} = |f(x + iy) - f(x - y)|,$$

$$(7) \quad \overline{C'D'} = |f(x - y) - f(x - iy)|,$$

$$(8) \quad \overline{D'A'} = |f(x - iy) - f(x + y)|.$$

Substituting (5), (6), (7), (8) into (4) yields

$$\begin{aligned} & |(f(x + y) - f(x + iy))(f(x - y) - f(x - iy))| \\ & = |(f(x + iy) - f(x - y))(f(x - iy) - f(x + y))| \end{aligned}$$

and therefore

$$(9) \quad \frac{|(f(x + y) - f(x + iy))(f(x - y) - f(x - iy))|}{|(f(x + iy) - f(x - y))(f(x - iy) - f(x + y))|} = 1.$$

If we set

$$(10) \quad g(y) = \frac{(f(x + y) - f(x + iy))(f(x - y) - f(x - iy))}{(f(x + iy) - f(x - y))(f(x - iy) - f(x + y))},$$

then, by (9) we have

$$(11) \quad |g(y)| = 1.$$

Since the numerator and the denominator of $g(y)$ in (10) are analytic for all y satisfying $0 < |y| \leq r$ and since, by the fact that $w = f(z)$ is univalent in R , the denominator of $g(y)$ in (10) never vanishes in $0 < |y| \leq r$, $g(y)$ is analytic in $0 < |y| \leq r$. Next we will prove that $g(y)$ is also analytic at $y = 0$. As $y \rightarrow 0$, by L'Hopital's Rule (see [2]) and by (3) we have

$$(12) \quad \frac{f(x + y) - f(x + iy)}{f(x + iy) - f(x - y)} \rightarrow \frac{f'(x) - if'(x)}{if'(x) + f'(x)} = \frac{1 - i}{1 + i}$$

and

$$(13) \quad \frac{f(x - y) - f(x - iy)}{f(x - iy) - f(x + y)} \rightarrow \frac{-f'(x) + if'(x)}{-if'(x) - f'(x)} = \frac{1 - i}{1 + i}.$$

Hence, by (10), (12), (13), as $y \rightarrow 0$,

$$(14) \quad g(y) \rightarrow \left(\frac{1 - i}{1 + i}\right)^2 = -1.$$

If we define

$$(15) \quad g(0) = -1$$

by (14), by Riemann's Theorem on removable singularities, the function $g(y)$ is analytic at $y = 0$. Furthermore, by (15) the equality (11) still holds at $y = 0$. Thus $g(y)$ is analytic in $|y| \leq r$ and $|g(y)| = 1$ holds in $|y| \leq r$. Therefore, by the Maximum Modulus Principle [8, p. 141] of analytic functions we obtain

$$(16) \quad g(y) = K$$

in $|y| \leq r$, where K is a complex constant.

Setting $y = 0$ in (16) and using (15) it follows

$$(17) \quad K = -1.$$

By (10), (16), (17) we obtain

$$(18) \quad \begin{aligned} & (f(x + y) - f(x + iy))(f(x - y) - f(x - iy)) \\ & + (f(x + iy) - f(x - y))(f(x - iy) - f(x + y)) = 0 \end{aligned}$$

for all y satisfying $|y| \leq r$.

Using Leibnitz's Rule for differentiation, differentiating both sides of (18) four times with respect to y , setting $y = 0$ and simplifying the resulting equality yields

$$(19) \quad f'''(x)f'(x) - \frac{3}{2}f''(x)^2 = 0.$$

Since $x \in R$ (the domain) was arbitrarily fixed, we can replace x by a variable z and thus by (19) we have

$$f'''(z)f'(z) - \frac{3}{2}f''(z)^2 = 0$$

in R . By the Identity Theorem (see [7, p. 106]) the above equality holds in $|z| < +\infty$. Hence

$$\frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 = 0$$

holds for all z satisfying $f'(z) \neq 0$.

Thus, the Schwarzian derivative of f vanishes for all z satisfying $f'(z) \neq 0$. Therefore, by a well-known fact $f(z)$ is a Möbius transformation of z .

Proof of the Corollary. By hypothesis $w = f(z)$ satisfies Property A. Hence, by the result of section 3, $w = f(z)$ satisfies Property B. Consequently, by the main theorem $w = f(z)$ is a Möbius transformation.

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