SEMIGROUP REPRESENTATIONS, POSITIVE DEFINITE FUNCTIONS AND ABELIAN $C^*$-ALGEBRAS

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Abstract. It is shown that every $*$-representation of a commutative semi-group $S$ with involution via operators on a Hilbert space has an integral representation with respect to a unique, compactly supported, selfadjoint Radon spectral measure defined on the Borel sets of the character space of $S$. The main feature is that the proof, which is based on the theory of positive definite functions, makes no use whatsoever (directly or indirectly) of the theory of $C^*$-algebras or more general Banach algebra arguments. Accordingly, this integral representation theorem is used to give a new proof of the Gelfand-Naimark theorem for abelian $C^*$-algebras.

Let $H$ be a Hilbert space and $A \subseteq \mathcal{L}(H)$ an abelian $C^*$-algebra, that is, the identity operator $I$ belongs to $A$, the adjoint operator $T^* \in A$ whenever $T \in A$, and the (commutative) algebra $A$ is closed for the operator norm topology in the space $\mathcal{L}(H)$ of all continuous linear operators of $H$ into itself. The Gelfand-Naimark theorem can be formulated as follows; see [5, Theorem 12.22] for a very readable account of this result.

**Theorem 1.** Let $H$ be a Hilbert space and $A \subseteq \mathcal{L}(H)$ an abelian $C^*$-algebra. Then there exists a unique selfadjoint, Radon spectral measure $F$: $\mathcal{B}(\Delta) \to \mathcal{L}(H)$ such that

\begin{equation}
T = \int_{\Delta} \hat{T}(\delta) \, dF(\delta), \quad T \in A.
\end{equation}

Some explanation is in order. The space $\Delta$ in Theorem 1 is the structure space of $A$; it is a compact Hausdorff space and can be interpreted as the set of all non-zero, complex homomorphisms $\delta$: $A \to \mathbb{C}$ equipped with the topology of pointwise convergence on $A$. Given $T \in A$, the continuous function $\hat{T}: \Delta \to \mathbb{C}$ is defined by $\hat{T}(\delta) := \delta(T)$, for $\delta \in \Delta$; it is called the Gelfand transform of $T$. We denote by $\mathcal{B}(\Delta)$ the Borel $\sigma$-algebra of $\Delta$, i.e. the smallest $\sigma$-algebra containing all the open subsets of $\Delta$. To say a function $F$: $\mathcal{B}(\Delta) \to \mathcal{L}(H)$ is a selfadjoint spectral measure means that $F(\Delta) = I$, that each operator $F(A)$, for $A \in \mathcal{B}(\Delta)$, is selfadjoint, that $F$ is multiplicative (i.e. $F(A \cap B) = F(A)F(B)$ for all $A, B \in \mathcal{B}(\Delta)$), and that $F$ is countably additive for the weak (equivalently, the strong) operator topology in $\mathcal{L}(H)$, that is, $F_{x,y}$: $A \mapsto \langle F(A)x, y \rangle$, for $A \in \mathcal{B}(\Delta)$, is a $\sigma$-additive, $\mathbb{C}$-valued...
measure for each \( x, y \in H \). Since all the values \( F(A) \), for \( A \in \mathcal{B}(\Delta) \), are orthogonal projections, we see that \( F_{x,x}(A) = \|F(A)x\|^2 \) is actually non-negative, for each \( x \in H \). To say that \( F \) is Radon means that \( F_{x,x} \) is a Radon measure (i.e. inner regular), for each \( x \in H \); see [1, Ch. 2], for example. The selfadjointness of \( F \) implies that

\[
4F_{x,y} = F_{x+y,x+y} - F_{x-y,x-y} + iF_{x+iy,x+iy} - iF_{x-iy,x-iy}, \quad x, y \in H,
\]

and so each complex measure \( F_{x,y} \) is also Radon. Since \( \Delta \) is compact and \( \hat{T} \) is continuous, it is, in particular, a bounded Borel function and so the integral in (1) defines an element of \( \mathcal{L}(H) \) via the standard theory of integration with respect to spectral measures; see [5, Section 12.20].

Let \( S \) be a commutative semigroup with identity element (always denoted by \( e \)) and equipped with an involution \( s \mapsto s^- \) (i.e. \( (s^-)^- = s \) and \( (st)^- = s^-t^- \) for all \( s, t \in S \)). A character of \( S \) is any function \( \rho : S \to \mathbb{C} \) satisfying \( \rho(e) = 1 \) and \( \rho(st^-) = \rho(s)\rho(t) \) for all \( s, t \in S \). The set of all characters of \( S \) is denoted by \( S^* \); it is a completely regular space when equipped with the topology of pointwise convergence inherited from \( \mathbb{C}^S \). If \( H \) is a Hilbert space, then a map \( \mathcal{U} : S \to \mathcal{L}(H) \) is called a \(*\)-representation if \( \mathcal{U}(e) = I \) and \( \mathcal{U}(st^-) = \mathcal{U}(s)\mathcal{U}(t)^* \) for all \( s, t \in S \).

**Theorem 2.** Let \( S \) be a commutative semigroup with identity and an involution, and let \( \mathcal{U} : S \to \mathcal{L}(H) \) be a \(*\)-representation. Then there exists a unique selfadjoint, Radon spectral measure \( E : \mathcal{B}(S^*) \to \mathcal{L}(H) \) which has compact support, such that

\[
\mathcal{U}(s) = \int_{S^*} \hat{s}(\rho) \, dE(\rho), \quad s \in S.
\]

Again some explanation is needed. The support of any spectral measure \( E : \mathcal{B}(S^*) \to \mathcal{L}(H) \), denoted by \( \text{supp}(E) \), is defined to be the closed subset of \( S^* \) given by \( \bigcup_{x \in H} \text{supp}(E_{x,x}) \), where \( \text{supp}(E_{x,x}) \) is the usual support of a non-negative Radon measure [1, p. 22]. Finally, given \( s \in S \), the function \( \hat{s} : S^* \to \mathbb{C} \) is defined by \( \hat{s}(\rho) := \rho(s) \), for \( \rho \in S^* \). Since \( \hat{s} \) is continuous and \( \text{supp}(E) \) is compact, the spectral integral on the right-hand-side of (2) exists and is an element of \( \mathcal{L}(H) \).

A version of Theorem 2 is formulated in [2, Theorem 2.6] where it is indicated that the proof is a combination of an integral representation theorem for exponentially bounded, positive definite functions on semigroups (now-a-days referred to as the Berg-Maserick theorem) [2, Theorem 2.1], together with the method of proof given for an earlier version of Theorem 2 formulated for uniformly bounded representations \( U \) [3, Theorem 3.2]. An examination of that proof (i.e. when \( U \) is uniformly bounded) shows that an essential ingredient is the use of the theory of abelian \( C^* \)-algebras (cf. Section 2 of [3]) together with some more general Banach algebra arguments (cf. [3, p. 501]).

The aim of this note is to highlight the fact, contrary to the line of argument suggested above, that Theorems 1 and 2 are actually “independent” of one another. That is, we present a (new) proof of Theorem 2 which uses neither Gelfand-Naimark theory nor any Banach algebra arguments (either directly or indirectly). We then use Theorem 2 to establish Theorem 1, thereby giving a new proof of the Gelfand-Naimark theorem for abelian \( C^* \)-algebras. Conversely, via a quite different argument than that suggested by Berg and Maserick, we show that Theorem 2 also follows from Theorem 1.

In order to prove Theorem 2 we require a few preliminaries. A function \( \varphi : S \to \mathbb{C} \) is called positive definite if \( \sum_{j,k=1}^{n} c_j \overline{c_k} \varphi(s_j s_k^-) \geq 0 \) for all choices of \( n \in \mathbb{N} \),
Proposition 1. Let \( \alpha: S \to [0, \infty) \) be an absolute value on \( S \) and \( \varphi: S \to \mathbb{C} \) an \( \alpha \)-bounded, positive definite function. Then there exists \( \mu \in M_+(S^\alpha) \) such that
\[
\varphi(s) = \int_{S^\alpha} \hat{s}(\rho) \, d\mu(\rho), \quad s \in S,
\]
and \( \mu \) is unique within \( M_+(S^\alpha) \).

Remark 1. The function on the right-hand-side of (3), with domain \( S \), is denoted by \( \hat{\mu} \) and is called the \textit{generalized Laplace transform} of \( \mu \). It is important to note that the proof of Proposition 1 given in [1, Ch. 4, §2], which is based on the integral version of the Krein-Milman theorem, makes no use of any Banach algebra techniques what-so-ever.

Proof of Theorem 2. Define an absolute value \( \alpha: S \to [0, \infty) \) by \( \alpha(s) = \|U(s)\| \) for \( s \in S \). For \( x \in H \) fixed define \( \varphi_x: S \to \mathbb{C} \) by \( \varphi_x(s) = \langle U(s)x, x \rangle \), for \( s \in S \). Using the fact that \( U \) is a *-representation it follows that
\[
\sum_{j, k=1}^n c_j \gamma_k \varphi_x s_j, s_k \rangle = \left\| \sum_{j=1}^n c_j U(s_j)x \right\|^2 \geq 0
\]
for any finite sets \( \{c_1, \ldots, c_n\} \subseteq \mathbb{C} \) and \( \{s_1, \ldots, s_n\} \subseteq S \) and hence, \( \varphi_x \) is positive definite. Moreover,
\[
|\varphi_x(s)| \leq \|U(s)\| \cdot \|x\|^2 = \|x\|^2 \alpha(s), \quad s \in S,
\]
which shows that \( \varphi_x \) is \( \alpha \)-bounded. By Proposition 1 there is a unique Radon measure \( \mu_x \in M_+(S^\alpha) \) such that \( \varphi_x = \hat{\mu}_x \).

For \( x, y \in H \) define a complex Radon measure \( \mu_{x,y} \) by
\[
\mu_{x,y} = \frac{1}{4}(\mu_{x+y} - \mu_{x-y} + i\mu_{x+iy} - i\mu_{x-iy}),
\]
in which case \( \hat{\mu}_{x,y} = \frac{1}{2}(\mu_{x+y} - \hat{\mu}_{x-y} + i\hat{\mu}_{x+iy} - i\hat{\mu}_{x-iy}) \). It then follows from the definition of \( \hat{\mu}_z = \varphi_z \) for each \( z \in H \) and a direct calculation that
\[
\hat{\mu}_{x,y}(s) = \langle U(s)x, y \rangle, \quad s \in S.
\]

Fix \( B \in B(S^\ast) \) and define \( \Psi_B: H \times H \to \mathbb{C} \) by \( \Psi_B(x, y) = \mu_{x,y}(B) \) for each \( x, y \in H \). It is clear from (5) and the linearity and injectivity of the map \( \nu \mapsto \hat{\nu} \) on the space of compactly supported (complex) Radon measures on \( S^\ast \) [1, p. 96] that \( \Psi_B \) is a sesquilinear form on \( H \times H \). We proceed to show that \( \Psi_B \) is bounded.
Let \( \{x_1, \ldots, x_n\} \subseteq H \) and \( \{c_1, \ldots, c_n\} \subseteq \mathbb{C} \) be finite sets. Define a Radon measure \( \nu \) on \( B(S^*) \) by \( \nu = \sum_{j=1}^{n} c_j \tau_k \mu_{x_j, x_k} \). Then, for finite sets \( \{d_1, \ldots, d_m\} \subseteq \mathbb{C} \) and \( \{t_1, \ldots, t_m\} \subseteq \mathcal{S} \) we have

\[
\sum_{p,q=1}^{m} d_p \check{d}_q \langle t_p, t_q^\ast \rangle = \sum_{j,k} d_p \check{d}_q c_j \overline{c}_k \langle \mathcal{U}(t_p, t_q^\ast) x_j, x_k \rangle = \left\| \sum_{p,j} c_j d_p \mathcal{U}(t_p) x_j \right\|^2 \geq 0,
\]

which shows that \( \check{\nu} : \mathcal{S} \to \mathbb{C} \) is positive definite. By combining Theorem 2.5 on p. 93 and the remarks of \( \S 2.10 \) on p. 96 in [1] it follows that \( \nu \geq 0 \), i.e. \( \nu \in M_+(S^\alpha) \).

In particular,

\[
\sum_{j,k=1}^{n} c_j \overline{c}_k \Psi_B(x_j, x_k) = \sum_{j,k=1}^{n} c_j \overline{c}_k \mu_{x_j, x_k}(B) = \nu(B) \geq 0,
\]

which shows that \( \Psi_B \) is a positive definite kernel (in the sense of [1, p. 67], for example). By the Cauchy-Schwarz inequality for such kernels (cf. [1, p. 69], for example) we have \( \|\Psi_B(x, y)\|^2 \leq \Psi_B(x, x) \Psi_B(y, y) \), that is,

\[
|\mu_{x,y}(B)| \leq |\mu_{x,x}(B)|^{1/2} |\mu_{y,y}(B)|^{1/2} \leq \|x\| \cdot \|y\|, \quad x, y \in H,
\]

where the last inequality relies on the observation that

\[
\mu_{x,x}(B) \leq \mu_{x,x}(S^\alpha) = \int_{S^\alpha} \rho(e) d\mu_{x,x}(\rho) = \hat{\mu}_{x,x}(\rho) = \langle \mathcal{U}(e) x, x \rangle = \|x\|^2
\]

for all \( x \in H \). So, the sesquilinear form \( \Psi_B \) is indeed bounded and hence, there is \( E(B) \in \mathcal{L}(H) \) satisfying

\[
\langle E(B)x, y \rangle = \Psi_B(x, y) = \mu_{x,y}(B), \quad x, y \in H.
\]

To see that \( E(B)^* = E(B) \) we note that

\[
\langle E(B)^* x, y \rangle = \langle x, E(B)y \rangle = \langle E(B)y, x \rangle = \mu_{y,x}(B) = \mu_{x,y}(B) = \langle E(B)x, y \rangle
\]

for all \( x, y \in H \), where \( \mu_{y,x}(B) = \mu_{x,y}(B) \) follows again from the positive definiteness of the kernel \( \Psi_B \). Furthermore, \( E(S^*) = I \) since, by (5), we have

\[
\langle E(S^*)x, y \rangle = \mu_{x,x}(S^*) = \hat{\mu}_{x,x}(\rho) = \langle \mathcal{U}(e)x, y \rangle = \langle x, y \rangle, \quad x, y \in H.
\]

It is clear from (6) that \( E_{x,y} = \mu_{x,y} \) is \( \sigma \)-additive for all \( x, y \in H \). The identity (6) also shows that \( E : B(S^*) \to \mathcal{L}(H) \) is a Radon measure and that it is compactly supported since \( \text{supp}(E_{x,y}) = \text{supp}(\mu_{x,y}) \subseteq S^\alpha \) for all \( x, y \in H \).

To see that \( E \) is multiplicative we proceed as follows. Given a Borel set \( A \subseteq S^* \) and \( x, y \in H \) define measures \( \nu_A \) and \( \lambda_A \) by \( \nu_A(B) = \mu_{E(A)x,y}(B) \) and \( \lambda_A(B) = \langle E(A \cap B)x, y \rangle = \mu_{x,y}(A \cap B) \) for each \( B \in B(S^*) \). Using (5) we have

\[
\hat{\nu}_A(t) = \mu_{E(A)x,y}(t) = \langle \mathcal{U}(t) E(A)x, y \rangle = \langle E(A)x, \mathcal{U}(t)^* y \rangle = \mu_{x,\mathcal{U}(t)^* y}(A)
\]

and

\[
\hat{\lambda}_A(t) = \int_A \rho(t) d\mu_{x,y}(\rho).
\]

For a fixed \( t \in S \) it is clear from (7) and (8) that the mappings \( \tau_1 : A \mapsto \hat{\nu}_A(t) \) and \( \tau_2 : A \mapsto \hat{\lambda}_A(t) \) are Radon measures on \( S^* \), supported in \( S^\alpha \), and so we can also calculate \( \tau_1 \) and \( \tau_2 \). Indeed, for \( s \in S \), we have (by (5) and (7)) that

\[
\hat{\tau}_1(s) = \langle \mathcal{U}(s)x, \mathcal{U}(t)^* y \rangle = \langle \mathcal{U}(st)x, y \rangle = \int_{S^\alpha} \rho(s) \rho(t) d\mu_{x,y}(\rho).
\]
But, from (8) it is also clear that
\[ \hat{\tau}_2(s) = \int_{S^*} \rho(s) \, d\tau_2(\rho) = \int_{S^*} \rho(s) \rho(t) \, d\mu_{x,y}(\rho). \]

By uniqueness of generalized Laplace transforms [1, p. 96], it follows that \( \tau_1 = \tau_2 \) as compactly supported Radon measures (for any fixed \( t \in S \)). In particular, for \( A \) fixed we deduce that \( \hat{\nu}_A(t) = \hat{\lambda}_A(t) \). Since this is true for each \( t \in S \) it again follows that \( \nu_A = \lambda_A \) as measures and hence, for any Borel set \( B \subseteq S^* \) we have \( \lambda_A(B) = \nu_A(B) \), that is,
\[
\langle E(A \cap B)x, y \rangle = \lambda_A(B) = \nu_A(B) = \langle E(B)E(A)x, y \rangle, \quad x, y \in H.
\]

So, \( E \) is multiplicative.

Finally, \( \hat{s} : \varphi : S^* \to \mathbb{C} \) is continuous and hence bounded on \( S^\alpha \), for any \( s \in S \). By the theory of spectral integrals \( \int_{S^*} \hat{s}(\rho) \, dE(\rho) \) exists in \( \mathcal{L}(H) \). It is clear from (5) and (6) that \( U(s) = \int_{S^*} \hat{s}(\rho) \, dE(\rho) \), for \( s \in S \), and the proof of Theorem 2 is complete. \( \square \)

To deduce Theorem 1 from Theorem 2 we require a further result. A commutative, unital complex algebra \( A \) with an involution (see Definitions 10.1 and 11.14 in [5]) always induces a commutative semigroup with involution, namely let \( S = A \) and consider \( A \) just with respect to its given multiplication and involution. We then define \( S^* := \{ \rho \in S^* : \rho \text{ is linear} \} \) and, given any absolute value \( \alpha : S \to [0, \infty) \), define \( S^\alpha := S^\alpha \cap S^\otimes \). The following result can be found in [4, Corollary 1]; we stress again that its proof does not use any Banach algebra methods.

**Proposition 2.** Let \( S \) be the semigroup induced by a unital, commutative complex algebra with an involution. Let \( \alpha : S \to [0, \infty) \) be an absolute value. If \( \varphi : S \to \mathbb{C} \) is a positive definite, \( \alpha \)-bounded function which is linear on \( S \), then the unique Radon measure \( \mu \in M_+(S^\alpha) \) satisfying \( \hat{\mu} = \varphi \) has its support in \( S^\otimes \).

**Proof of Theorem 1.** Let \( A \subseteq \mathcal{L}(H) \) be an abelian \( C^* \)-algebra. Let \( S = A \), considered as a semigroup with respect to the multiplication in \( A \) and with the involution from \( A \). By Theorem 2 there exists a unique, selfadjoint Radon spectral measure \( E : B(S^*) \to \mathcal{L}(H) \) with compact support satisfying
\[
(9) \quad T = \int_{S^*} \hat{T}(\rho) \, dE(\rho), \quad T \in S;
\]
we have used the representation \( U(T) = T, \) for \( T \in S \). From the proof of Theorem 2, where the absolute value \( \alpha : S \to [0, \infty) \) used is \( \alpha(T) = ||U(T)|| = ||T||, \) for \( T \in S \), we have that \( \text{supp}(E) \subseteq S^\alpha \). Now, for \( x \in H \) fixed, the positive definite and \( \alpha \)-bounded function \( \varphi_x : S \to \mathbb{C} \) defined in the proof of Theorem 2 (i.e. \( \varphi_x(T) = \langle U(T)x, x \rangle = \langle Tx, x \rangle \), for \( T \in S \)) is clearly linear on \( S \). Then Proposition 2 implies that the unique measure \( \mu_x \in M_+(S^\alpha) \) satisfying \( \hat{\mu}_x = \varphi_x \) has its support in \( S^\otimes \).

Since \( E_{x,x} = \mu_x \) also supp(\( E \)) \subseteq S^\otimes. But, if \( \rho \in S^\otimes \), then
\[
|\rho(T)| \leq \alpha(T) = ||T||, \quad T \in A,
\]
which implies that \( ||\rho|| = 1 \). So, \( S^\otimes \) consists of all algebra homomorphisms \( \rho : S = A \to \mathbb{C} \) of norm 1. It is part of the definition that each semigroup character \( \rho \in S^\ast \) must satisfy \( \rho(s^{-1}) = \rho(s) \), which, in the present setting, becomes \( \rho(T^*) = \rho(T) \). But, elements of the structure space \( \Delta \) of \( A \) automatically satisfy this condition [5, Theorem 11.18]. Hence, \( S^\otimes \) is precisely \( \Delta \) and (9), which is actually an integral over \( S^\otimes = \Delta \), reduces to (1). \( \square \)
In conclusion we indicate how Theorem 2 also follows from Theorem 1. So, let $S$ and $U: S \to L(H)$ be as in Theorem 2. Define $A \subseteq L(H)$ to be the $C^*$-algebra generated by the $*$-closed, commutative family of operators $M = \{U(s): s \in S\}$. Of course, $A$ is the operator norm closure in $L(H)$ of the linear span of $M$. Denote the structure space of $A$ by $\Delta$. For each $\delta \in \Delta$ define $\hat{\delta}: \Delta \to \mathbb{C}$ by $\hat{\delta}(s) := \delta(\hat{U}(s)), s \in S$. From the homomorphism properties of $\delta$ and the fact that $U$ is a $*$-representation it is easily seen that $\hat{\delta} \in S^*$; the property $\hat{\delta}(s^*) = \overline{\delta(s)}$ again follows from the comments at the end of the previous paragraph. So, we can define $\Lambda: \Delta \to S^*$ by $\Lambda(\delta) := \delta$, for $\delta \in \Delta$, in which case $\hat{\delta} \circ \Lambda = (U(s))^\sim$ as functions on $\Delta$, for each $s \in S$. If $\Lambda(\delta_1) = \Lambda(\delta_2)$, then $\delta_1(U(s)) = \delta_2(U(s))$ for all $s \in S$ and hence, $\delta_1$ and $\delta_2$ agree on the linear span of $M$. Since the span of $M$ is dense in $A$ and both $\delta_1$ and $\delta_2$ are continuous on $A$, it follows that $\delta_1 = \delta_2$. This shows that $\Lambda$ is injective. Since $\Delta$ (resp. $S^*$) is equipped with the pointwise convergence topology on $A$ (resp. $S$), it is clear that $\Lambda$ is continuous. Then the compactness of $\Delta$ ensures that $\Lambda$ is a topological homeomorphism of $\Delta$ onto its range $K := \Lambda(\Delta) \subseteq S^*$. By Theorem 1 there is a unique selfadjoint Radon spectral measure $F: B(\Delta) \to L(H)$ satisfying

\[
T = \int_\Delta \hat{T}(\delta) \, dF(\delta), \quad T \in A.
\]

Define $E: B(S^*) \to L(H)$ by $E(A) = F(\Lambda^{-1}(A))$, for each $A \in B(S^*)$, in which case $E$ is a compactly supported (as supp($E$) $\subseteq K$), selfadjoint Radon spectral measure such that

\[
U(s) = \int_\Delta (U(s))^\sim(\delta) \, dF(\delta) = \int_{S^*} \hat{s}(\rho) \, dE(\rho), \quad s \in S;
\]

see [5, Theorem 13.28] for the last equality.

Suppose now that $G: B(S^*) \to L(H)$ is another compactly supported, selfadjoint Radon spectral measure satisfying

\[
U(s) = \int_{S^*} \hat{s}(\rho) \, dG(\rho), \quad s \in S.
\]

It follows that $\int_{S^*} \hat{s}(\rho) \, dG_{x,x}(\rho) = \int_{S^*} \hat{s}(\rho) \, dE_{x,x}(\rho)$ for all $s \in S$ and $x \in H$, that is, $\hat{G}_{x,x} = \hat{E}_{x,x}$ for all $x \in H$. Since both $G_{x,x}$ and $E_{x,x}$ are compactly supported elements of $M_a(S^*)$, it follows from the uniqueness of generalized Laplace transforms that $E_{x,x} = G_{x,x}$ for all $x \in H$. Then the selfadjointness of both $E$ and $G$ and the polarization identity imply that $E_{x,y} = G_{x,y}$ for all $x, y \in H$. In particular, it follows that $E(A) = G(A)$ for all $A \in B(S^*)$, that is, $E = G$. Hence, the measure $E$ satisfying (10) is unique and the proof is complete.

Remark 2. From the proof of Theorem 2 it is obvious that if all the measures $E_{x,x} = \mu_x$, for $x \in H$, concentrate on a subset $A \subseteq S^*$, then so does $E$. As a consequence we obtain Stone’s theorem for $S$ a locally compact abelian group (which is a semigroup with the special involution $s^- := s^{-1}$), and where $U$ is assumed to be weak operator continuous, i.e. $s \mapsto \{U(s)x, x\}$ is continuous for each $x \in H$. Then by Bochner’s theorem all the measures $\mu_x$ live on the dual group (by definition the set of all continuous characters), and so therefore does $E$.  

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