It is shown that every \( \ast \)-representation of a commutative semigroup \( S \) with involution via operators on a Hilbert space has an integral representation with respect to a unique, compactly supported, selfadjoint Radon spectral measure defined on the Borel sets of the character space of \( S \). The main feature is that the proof, which is based on the theory of positive definite functions, makes no use whatsoever (directly or indirectly) of the theory of \( C^\ast \)-algebras or more general Banach algebra arguments. Accordingly, this integral representation theorem is used to give a new proof of the Gelfand-Naimark theorem for abelian \( C^\ast \)-algebras.

Let \( H \) be a Hilbert space and \( \mathcal{A} \subseteq \mathcal{L}(H) \) an abelian \( C^\ast \)-algebra, that is, the identity operator \( I \) belongs to \( \mathcal{A} \), the adjoint operator \( T^\ast \in \mathcal{A} \) whenever \( T \in \mathcal{A} \), and the (commutative) algebra \( \mathcal{A} \) is closed for the operator norm topology in the space \( \mathcal{L}(H) \) of all continuous linear operators of \( H \) into itself. The Gelfand-Naimark theorem can be formulated as follows; see [5, Theorem 12.22] for a very readable account of this result.

**Theorem 1.** Let \( H \) be a Hilbert space and \( \mathcal{A} \subseteq \mathcal{L}(H) \) an abelian \( C^\ast \)-algebra. Then there exists a unique selfadjoint, Radon spectral measure \( F: \mathcal{B}(\Delta) \to \mathcal{L}(H) \) such that

\[
T = \int_{\Delta} \hat{T}(\delta) \, dF(\delta), \quad T \in \mathcal{A}.
\]

Some explanation is in order. The space \( \Delta \) in Theorem 1 is the structure space of \( \mathcal{A} \); it is a compact Hausdorff space and can be interpreted as the set of all non-zero, complex homomorphisms \( \delta: \mathcal{A} \to \mathbb{C} \) equipped with the topology of pointwise convergence on \( \mathcal{A} \). Given \( T \in \mathcal{A} \), the continuous function \( \tilde{T}: \Delta \to \mathbb{C} \) is defined by \( \tilde{T}(\delta) := \delta(T) \), for \( \delta \in \Delta \); it is called the Gelfand transform of \( T \). We denote by \( \mathcal{B}(\Delta) \) the Borel \( \sigma \)-algebra of \( \Delta \), i.e. the smallest \( \sigma \)-algebra containing all the open subsets of \( \Delta \). To say a function \( F: \mathcal{B}(\Delta) \to \mathcal{L}(H) \) is a selfadjoint spectral measure means that \( F(\Delta) = I \), that each operator \( F(A) \), for \( A \in \mathcal{B}(\Delta) \), is selfadjoint, that \( F \) is multiplicative (i.e. \( F(A \cap B) = F(A)F(B) \) for all \( A, B \in \mathcal{B}(\Delta) \)), and that \( F \) is countably additive for the weak (equivalently, the strong) operator topology in \( \mathcal{L}(H) \), that is, \( F_{x,y}: A \mapsto \langle F(A)x, y \rangle \), for \( A \in \mathcal{B}(\Delta) \), is a \( \sigma \)-additive, \( \mathbb{C} \)-valued
measure for each $x, y \in H$. Since all the values $F(A)$, for $A \in B(\Delta)$, are orthogonal projections, we see that $F_{x,x}(A) = \|F(A)x\|^2$ is actually non-negative, for each $x \in H$. To say that $F$ is Radon means that $F_{x,x}$ is a Radon measure (i.e. inner regular), for each $x \in H$; see [1, Ch. 2], for example. The selfadjointness of $F$ implies that

$$A F_{x,y} = F_{x+y,x+y} - F_{x-y,x-y} + i F_{x+iy,x+iy} - i F_{x-iy,x-iy}, \quad x, y \in H,$$

and so each complex measure $F_{x,y}$ is also Radon. Since $\Delta$ is compact and $\hat{T}$ is continuous, it is, in particular, a bounded Borel function and so the integral in (1) defines an element of $L(H)$ via the standard theory of integration with respect to spectral measures; see [5, Section 12.20].

Let $S$ be a commutative semigroup with identity element (always denoted by $e$) and equipped with an involution $s \mapsto s^*$ (i.e. $(s^*)^* = s$ and $(st)^* = s^*t^*$ for all $s,t \in S$). A character of $S$ is any function $\rho: S \to \mathbb{C}$ satisfying $\rho(e) = 1$ and $\rho(st^*) = \rho(s)\overline{\rho(t)}$ for all $s,t \in S$. The set of all characters of $S$ is denoted by $S^*$; it is a completely regular space when equipped with the topology of pointwise convergence inherited from $\mathbb{C}^S$. If $H$ is a Hilbert space, then a map $\mathcal{U}: S \to \mathcal{L}(H)$ is called a $*$-representation if $\mathcal{U}(e) = I$ and $\mathcal{U}(st^*) = \mathcal{U}(s)\mathcal{U}(t)^*$ for all $s,t \in S$.

**Theorem 2.** Let $S$ be a commutative semigroup with identity and an involution, and let $\mathcal{U}: S \to \mathcal{L}(H)$ be a $*$-representation. Then there exists a unique selfadjoint, Radon spectral measure $E: \mathcal{B}(S^*) \to \mathcal{L}(H)$ which has compact support, such that

$$\mathcal{U}(s) = \int_S \hat{s}(\rho) \ dE(\rho), \quad s \in S. \quad (2)$$

Again some explanation is needed. The support of any spectral measure $E: \mathcal{B}(S^*) \to \mathcal{L}(H)$, denoted by $\text{supp}(E)$, is defined to be the closed subset of $S^*$ given by $\bigcup_{x \in H} \text{supp}(E_{x,x})$, where $\text{supp}(E_{x,x})$ is the usual support of a non-negative Radon measure [1, p. 22]. Finally, given $s \in S$, the function $\hat{s}: S^* \to \mathbb{C}$ is defined by $\hat{s}(\rho) := \rho(s)$, for $\rho \in S^*$. Since $\hat{s}$ is continuous and $\text{supp}(E)$ is compact, the spectral integral on the right-hand-side of (2) exists and is an element of $\mathcal{L}(H)$.

A version of Theorem 2 is formulated in [2, Theorem 2.6] where it is indicated that the proof is a combination of an integral representation theorem for exponentially bounded, positive definite functions on semigroups (now-a-days referred to as the Berg-Maserick theorem) [2, Theorem 2.1], together with the method of proof given for an earlier version of Theorem 2 formulated for uniformly bounded representations $\mathcal{U}$ [3, Theorem 3.2]. An examination of that proof (i.e. when $\mathcal{U}$ is uniformly bounded) shows that an essential ingredient is the use of the theory of abelian $C^*$-algebras (cf. Section 2 of [3]) together with some more general Banach algebra arguments (cf. [3, p. 501]).

The aim of this note is to highlight the fact, contrary to the line of argument suggested above, that Theorems 1 and 2 are actually “independent” of one another. That is, we present a (new) proof of Theorem 2 which uses neither Gelfand-Naimark theory nor any Banach algebra arguments (either directly or indirectly). We then use Theorem 2 to establish Theorem 1, thereby giving a new proof of the Gelfand-Naimark theorem for abelian $C^*$-algebras. Conversely, via a quite different argument than that suggested by Berg and Maserick, we show that Theorem 2 also follows from Theorem 1.

In order to prove Theorem 2 we require a few preliminaries. A function $\varphi: S \to \mathbb{C}$ is called positive definite if $\sum_{j,k=1}^n c_j \overline{c_k} \varphi(s_j s_k^*) \geq 0$ for all choices of $n \in \mathbb{N},"
\{s_1,\ldots, s_n\} \subseteq S and \{c_1,\ldots, c_n\} \subseteq \mathbb{C}. In particular, every character \(\rho \in S^*\) is positive definite.

A function \(\alpha: \ S \to [0, \infty)\) satisfying \(\alpha(e) = 1\) is called an absolute value if \(\alpha(s^-) = \alpha(s)\) and \(\alpha(st) \leq \alpha(s) \alpha(t)\) for all \(s, t \in S\). A function \(f: \ S \to \mathbb{C}\) is called \(\alpha\)-bounded if there exists \(C > 0\) such that \(|f(s)| \leq C \alpha(s)\) for all \(s \in S\). If \(f\) happens to be positive definite and \(\alpha\)-bounded, then it is possible to choose \(C = \varphi(e)\). A character \(\rho \in S^*\) is \(\alpha\)-bounded iff \(|\rho| \leq \alpha\). Hence, the set \(S^\alpha\) of all \(\alpha\)-bounded characters is a compact subset of \(S^*\). For all of these notions and further properties we refer to [1, Ch. 4].

The space of all non-negative Radon measures on \(S^*\) is denoted by \(M_+(S^*)\). Given any absolute value \(\alpha: \ S \to [0, \infty)\) the subspace of \(M_+(S^*)\) consisting of all Radon measures supported in the compact subset \(S^\alpha \subseteq S^*\) is denoted by \(M_+(S^\alpha)\).

The following result is the Berg-Maserick theorem mentioned above.

**Proposition 1.** Let \(\alpha: \ S \to [0, \infty)\) be an absolute value on \(S\) and \(\varphi: \ S \to \mathbb{C}\) an \(\alpha\)-bounded, positive definite function. Then there exists \(\mu \in M_+(S^\alpha)\) such that

\[
\varphi(s) = \int_{S^*} \hat{s}(\rho) \, d\mu(\rho), \quad s \in S,
\]

and \(\mu\) is unique within \(M_+(S^\alpha)\).

**Remark 1.** The function on the right-hand-side of (3), with domain \(S\), is denoted by \(\mu\) and is called the *generalized Laplace transform* of \(\mu\). It is important to note that the proof of Proposition 1 given in [1, Ch. 4, §2], which is based on the integral version of the Krein-Milman theorem, makes no use of any Banach algebra techniques whatsoever.

**Proof of Theorem 2.** Define an absolute value \(\alpha: \ S \to [0, \infty)\) by \(\alpha(s) = ||U(s)||\) for \(s \in S\). For \(x \in H\) fixed define \(\varphi_x: \ S \to \mathbb{C}\) by \(\varphi_x(s) = \langle U(s)x, x \rangle\), for \(s \in S\). Using the fact that \(U\) is a *-representation it follows that

\[
\sum_{j,k=1}^n c_j \overline{c_k} \varphi_x(s_j s_k) = \left\| \sum_{j=1}^n c_j U(s_j)x \right\|^2 \geq 0
\]

for any finite sets \(\{c_1, \ldots, c_n\} \subseteq \mathbb{C}\) and \(\{s_1, \ldots, s_n\} \subseteq S\) and hence, \(\varphi_x\) is positive definite. Moreover,

\[
|\varphi_x(s)| \leq ||U(s)|| \cdot ||x||^2 = ||x||^2 \alpha(s), \quad s \in S,
\]

which shows that \(\varphi_x\) is \(\alpha\)-bounded. By Proposition 1 there is a unique Radon measure \(\mu_x \in M_+(S^\alpha)\) such that \(\varphi_x = \hat{\mu}_x\).

For \(x, y \in H\) define a complex Radon measure \(\mu_{x,y}\) by

\[
\mu_{x,y} = \frac{1}{4} (\mu_{x+y} - \mu_{x-y} + i\mu_{x+iy} - i\mu_{x-iy}),
\]

in which case \(\hat{\mu}_{x,y} = \frac{1}{2}(\hat{\mu}_{x+y} - \hat{\mu}_{x-y} + i\hat{\mu}_{x+iy} - i\hat{\mu}_{x-iy})\). It then follows from the definition of \(\hat{\mu}_x = \varphi_x\) for each \(z \in H\) and a direct calculation that

\[
\hat{\mu}_{x,y}(s) = \langle U(s)x, y \rangle, \quad s \in S.
\]

Fix \(B \in \mathcal{B}(S^*)\) and define \(\Psi_B: \ H \times H \to \mathbb{C}\) by \(\Psi_B(x, y) = \mu_{x,y}(B)\) for each \(x, y \in H\). It is clear from (5) and the linearity and injectivity of the map \(\nu \mapsto \hat{\nu}\) on the space of compactly supported (complex) Radon measures on \(S^*\) [1, p. 96] that \(\Psi_B\) is a sesquilinear form on \(H \times H\). We proceed to show that \(\Psi_B\) is bounded.
Let \( \{x_1, \ldots, x_n\} \subseteq H \) and \( \{c_1, \ldots, c_n\} \subseteq \mathbb{C} \) be finite sets. Define a Radon measure \( \nu \) on \( B(S^*) \) by \( \nu = \sum_{j,k=1}^n c_j \tau_k \mu_{x_j,x_k} \). Then, for finite sets \( \{d_1, \ldots, d_m\} \subseteq \mathbb{C} \) and \( \{t_1, \ldots, t_m\} \subseteq S \) we have

\[
\sum_{p,q=1}^m d_p \overline{d_q} \hat{\nu}(t_p t_q^\ast) = \sum_{j,k=1}^n \sum_{p,q} d_p \overline{d_q} c_j \tau_k \langle \mathcal{U}(t_p t_q^\ast)x_j, x_k \rangle = \left\| \sum_{p,j} c_j d_p \mathcal{U}(t_p)x_j \right\|^2 \geq 0,
\]

which shows that \( \hat{\nu} : S \rightarrow \mathbb{C} \) is positive definite. By combining Theorem 2.5 on p. 93 and the remarks of §2.10 on p. 96 in [1] it follows that \( \nu \geq 0 \), i.e. \( \nu \in M_+(S^\alpha) \).

In particular,

\[
\sum_{j,k=1}^n c_j \tau_k \Psi_B(x_j, x_k) = \sum_{j,k=1}^n c_j \tau_k \mu_{x_j,x_k}(B) = \nu(B) \geq 0,
\]

which shows that \( \Psi_B \) is a positive definite kernel (in the sense of [1, p. 67], for example). By the Cauchy-Schwarz inequality for such kernels (cf. [1, p. 69], for example) we have \( \|\Psi_B(x, y)\|^2 \leq \Psi_B(x, x) \Psi_B(y, y) \), that is,

\[
|\mu_{x,y}(B)| \leq \left( \mu_{x,x}(B) \right)^{1/2} \left( \mu_{y,y}(B) \right)^{1/2} \leq \|x\| \cdot \|y\|, \quad x, y \in H,
\]

where the last inequality relies on the observation that

\[
\mu_{x,x}(B) \leq \mu_{x,x}(S^\alpha) = \int_{S^*} \rho(x) d\mu_{x,x}(\rho) = \mu_{x,x}(e) = \langle \mathcal{U}(e)x, x \rangle = \|x\|^2
\]

for all \( x \in H \). So, the sesquilinear form \( \Psi_B \) is indeed bounded and hence, there is \( E(B) \in \mathcal{L}(H) \) satisfying

\[
(6) \quad \langle E(B)x, y \rangle = \Psi_B(x, y) = \mu_{x,y}(B), \quad x, y \in H.
\]

To see that \( E(B)^* = E(B) \) we note that

\[
\langle E(B)^*x, y \rangle = \langle x, E(B)y \rangle = \overline{\langle E(B)y, x \rangle} = \overline{\mu_{y,x}(B)} = \mu_{x,y}(B) = \langle E(B)x, y \rangle
\]

for all \( x, y \in H \), where \( \overline{\mu_{y,x}(B)} = \mu_{x,y}(B) \) follows again from the positive definiteness of the kernel \( \Psi_B \). Furthermore, \( E(S^*) = I \) since, by (5), we have

\[
\langle E(S^*)x, y \rangle = \mu_{x,x}(S^*) = \mu_{x,x}(e) = \langle \mathcal{U}(e)x, y \rangle = \langle x, y \rangle, \quad x, y \in H.
\]

It is clear from (6) that \( E \) is a positive definite measure and that it is supported since \( \text{supp}(E_{x,y}) = \text{supp}(\mu_{x,y}) \subseteq S^\alpha \) for all \( x, y \in H \).

To see that \( E \) is multiplicative we proceed as follows. Given a Borel set \( A \subseteq S^* \) and \( x, y \in H \) define measures \( \nu_{A} \) and \( \lambda_{A} \) by \( \nu_{A}(B) = \mu_{E(A)x,y}(B) \) and \( \lambda_{A}(B) = \langle E(A \cap B)x, y \rangle = \mu_{x,y}(A \cap B) \) for each \( B \in B(S^*) \). Using (5) we have

\[
(7) \quad \hat{\nu}_{A}(t) = \mu_{E(A)x,y}(t) = \langle \mathcal{U}(t)E(A)x, y \rangle = \langle E(A)x, \mathcal{U}(t)^*y \rangle = \mu_{x,\mathcal{U}(t)^*y}(A)
\]

and

\[
(8) \quad \hat{\lambda}_{A}(t) = \int_{A} \rho(t) d\mu_{x,y}(\rho).
\]

For a fixed \( t \in S \) it is clear from (7) and (8) that the mappings \( \gamma_1 : A \mapsto \hat{\nu}_{A}(t) \) and \( \gamma_2 : A \mapsto \hat{\lambda}_{A}(t) \) are Radon measures on \( S^* \), supported in \( S^\alpha \), and so we can also calculate \( \hat{\tau}_1 \) and \( \hat{\tau}_2 \). Indeed, for \( s \in S \), we have (by (5) and (7)) that

\[
\hat{\tau}_1(s) = \langle \mathcal{U}(s)x, \mathcal{U}(t)^*y \rangle = \langle \mathcal{U}(st)x, y \rangle = \int_{S^*} \rho(s) \rho(t) d\mu_{x,y}(\rho).
\]
So, from (8) it is also clear that
\[ \hat{\tau}_2(s) = \int_{S^*} \rho(s) \, d\tau_2(\rho) = \int_{S^*} \rho(s) \rho(t) \, d\mu_{x,y}(\rho). \]
By uniqueness of generalized Laplace transforms [1, p. 96], it follows that \( \tau_1 \equiv \tau_2 \) as compactly supported Radon measures (for any fixed \( t \in S \)). In particular, for \( A \) fixed we deduce that \( \hat{\nu}_A(t) = \hat{\lambda}_A(t) \). Since this is true for each \( t \in S \) it again follows that \( \nu_A = \lambda_A \) as measures and hence, for any Borel set \( B \subseteq S^* \) we have
\[ \lambda_A(B) = \nu_A(B), \]
that is,
\[ \langle E(A \cap B)x, y \rangle = \lambda_A(B) = \nu_A(B) = \langle E(B)E(A)x, y \rangle, \quad x, y \in H. \]
So, \( E \) is multiplicative.

Finally, \( \hat{s} \colon S^* \to \mathbb{C} \) is continuous and hence bounded on \( S^\alpha \), for any \( s \in S \). By the theory of spectral integrals \( \int_{S^*} \hat{s}(\rho) \, dE(\rho) \) exists in \( \mathcal{L}(H) \). It is clear from (5) and (6) that \( \mathcal{U}(s) = \int_{S^*} \hat{s}(\rho) \, dE(\rho) \), for \( s \in S \), and the proof of Theorem 2 is complete.

To deduce Theorem 1 from Theorem 2 we require a further result. A commutative, unital complex algebra \( \mathcal{A} \) with an involution (see Definitions 10.1 and 11.14 in [5]) always induces a commutative semigroup with involution, namely let \( S = \mathcal{A} \) and consider \( \mathcal{A} \) just with respect to its given multiplication and involution. We then define \( S^\alpha := \{ \rho \in S^* : \rho \text{ is linear} \} \), and, given any absolute value \( \alpha : S \to [0, \infty) \), define \( S^\alpha := S^\alpha \cap S^\alpha \). The following result can be found in [4, Corollary 1]; we stress again that its proof does not use any Banach algebra methods.

**Proposition 2.** Let \( S \) be the semigroup induced by a unital, commutative complex algebra with an involution. Let \( \alpha : S \to [0, \infty) \) be an absolute value. If \( \varphi : S \to \mathbb{C} \) is a positive definite, \( \alpha \)-bounded function which is linear on \( S \), then the unique Radon measure \( \mu \in M_{\infty}(S^\alpha) \) satisfying \( \hat{\mu} = \varphi \) has its support in \( S^\alpha \).

**Proof of Theorem 1.** Let \( A \subseteq \mathcal{L}(H) \) be an abelian \( C^* \)-algebra. Let \( S = \mathcal{A} \), considered as a semigroup with respect to the multiplication in \( \mathcal{A} \) and with the involution from \( \mathcal{A} \). By Theorem 2 there exists a unique, selfadjoint Radon spectral measure \( E : B(S^*) \to \mathcal{L}(H) \) with compact support satisfying
\[ T = \int_{S^*} \hat{T}(\rho) \, dE(\rho), \quad T \in S; \]
we have used the representation \( \mathcal{U}(T) = T \), for \( T \in S \). From the proof of Theorem 2, where the absolute value \( \alpha : S \to [0, \infty) \) used is \( \alpha(T) = \| \mathcal{U}(T) \| = \| T \| \), for \( T \in S \), we have that \( \text{supp}(E) \subseteq S^\alpha \). Now, for \( x \in H \) fixed, the positive definite and \( \alpha \)-bounded function \( \varphi_x : S \to \mathbb{C} \) defined in the proof of Theorem 2 (i.e. \( \varphi_x(T) = \langle \mathcal{U}(T)x, x \rangle = \langle Tx, x \rangle \), for \( T \in S \)) is clearly linear on \( S \). Then Proposition 2 implies that the unique measure \( \mu_x \in M_{\infty}(S^\alpha) \) satisfying \( \hat{\mu}_x = \varphi_x \) has its support in \( S^\alpha \). Since \( E_{x,x} = \mu_x \) also supp(\( E \)) \( \subseteq S^\alpha \). But, if \( \rho \in S^\alpha \), then
\[ |\rho(T)| \leq \alpha(T) = \| T \|, \quad T \in \mathcal{A}, \]
which implies that \( \| \rho \| = 1 \). So, \( S^\alpha \) consists of all algebra homomorphisms \( \rho : S(= \mathcal{A}) \to \mathbb{C} \) of norm 1. It is part of the definition that each semigroup character \( \rho \in S^* \) must satisfy \( \rho(s^-) = \rho(s) \) which, in the present setting, becomes \( \rho(T^*) = \rho(T) \). But, elements of the structure space \( \Delta \) of \( \mathcal{A} \) automatically satisfy this condition [5, Theorem 11.18]. Hence, \( S^\alpha \) is precisely \( \Delta \) and (9), which is actually an integral over \( S^\alpha = \Delta \), reduces to (1).
In conclusion we indicate how Theorem 2 also follows from Theorem 1. So, let $S$ and $\mathcal{U}: S \to \mathcal{L}(H)$ be as in Theorem 2. Define $\mathcal{A} \subseteq \mathcal{L}(H)$ to be the $C^*$-algebra generated by the $\ast$-closed, commutative family of operators $\mathcal{M} = \{\mathcal{U}(s): s \in S\}$. Of course, $\mathcal{A}$ is a selfadjoint Radon spectral measure such that $E \to \Lambda: \Delta \to \mathbb{C}$ follows from the comments at the end of the previous paragraph. So, we can define the measure $E$ satisfying (10) is unique and the proof is complete.

Remark 2. From the proof of Theorem 2 it is obvious that if all the measures $E_{x,x} = \mu_x$, for $x \in H$, concentrate on a subset $A \subseteq S^*$, then so does $E$. As a consequence we obtain Stone’s theorem for $S$ a locally compact abelian group (which is a semigroup with the special involution $s^{-1} := s^{-1}$), and where $\mathcal{U}$ is assumed to be weak operator continuous, i.e. $s \mapsto \langle \mathcal{U}(s)x, x \rangle$ is continuous for each $x \in H$. Then by Bochner’s theorem all the measures $\mu_x$ live on the dual group (by definition the set of all continuous characters), and so therefore does $E$. 

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