

## SEMIGROUP REPRESENTATIONS, POSITIVE DEFINITE FUNCTIONS AND ABELIAN $C^*$ -ALGEBRAS

P. RESSEL AND W. J. RICKER

(Communicated by Palle E. T. Jorgensen)

ABSTRACT. It is shown that every  $*$ -representation of a commutative semigroup  $S$  with involution via operators on a Hilbert space has an integral representation with respect to a unique, compactly supported, selfadjoint Radon spectral measure defined on the Borel sets of the character space of  $S$ . The main feature is that the proof, which is based on the theory of positive definite functions, makes no use what-so-ever (directly or indirectly) of the theory of  $C^*$ -algebras or more general Banach algebra arguments. Accordingly, this integral representation theorem is used to give a new proof of the Gelfand-Naimark theorem for abelian  $C^*$ -algebras.

Let  $H$  be a Hilbert space and  $\mathcal{A} \subseteq \mathcal{L}(H)$  an *abelian  $C^*$ -algebra*, that is, the identity operator  $I$  belongs to  $\mathcal{A}$ , the adjoint operator  $T^* \in \mathcal{A}$  whenever  $T \in \mathcal{A}$ , and the (commutative) algebra  $\mathcal{A}$  is closed for the operator norm topology in the space  $\mathcal{L}(H)$  of all continuous linear operators of  $H$  into itself. The Gelfand-Naimark theorem can be formulated as follows; see [5, Theorem 12.22] for a very readable account of this result.

**Theorem 1.** *Let  $H$  be a Hilbert space and  $\mathcal{A} \subseteq \mathcal{L}(H)$  an abelian  $C^*$ -algebra. Then there exists a unique selfadjoint, Radon spectral measure  $F: \mathcal{B}(\Delta) \rightarrow \mathcal{L}(H)$  such that*

$$(1) \quad T = \int_{\Delta} \widehat{T}(\delta) dF(\delta), \quad T \in \mathcal{A}.$$

Some explanation is in order. The space  $\Delta$  in Theorem 1 is the *structure space* of  $\mathcal{A}$ ; it is a compact Hausdorff space and can be interpreted as the set of all non-zero, complex homomorphisms  $\delta: \mathcal{A} \rightarrow \mathbb{C}$  equipped with the topology of pointwise convergence on  $\mathcal{A}$ . Given  $T \in \mathcal{A}$ , the continuous function  $\widehat{T}: \Delta \rightarrow \mathbb{C}$  is defined by  $\widehat{T}(\delta) := \delta(T)$ , for  $\delta \in \Delta$ ; it is called the *Gelfand transform* of  $T$ . We denote by  $\mathcal{B}(\Delta)$  the *Borel  $\sigma$ -algebra* of  $\Delta$ , i.e. the smallest  $\sigma$ -algebra containing all the open subsets of  $\Delta$ . To say a function  $F: \mathcal{B}(\Delta) \rightarrow \mathcal{L}(H)$  is a *selfadjoint spectral measure* means that  $F(\Delta) = I$ , that each operator  $F(A)$ , for  $A \in \mathcal{B}(\Delta)$ , is selfadjoint, that  $F$  is multiplicative (i.e.  $F(A \cap B) = F(A)F(B)$  for all  $A, B \in \mathcal{B}(\Delta)$ ), and that  $F$  is countably additive for the weak (equivalently, the strong) operator topology in  $\mathcal{L}(H)$ , that is,  $F_{x,y}: A \mapsto \langle F(A)x, y \rangle$ , for  $A \in \mathcal{B}(\Delta)$ , is a  $\sigma$ -additive,  $\mathbb{C}$ -valued

---

Received by the editors March 3, 1997.

1991 *Mathematics Subject Classification.* Primary 47A67, 47B15, 47D03, 47D25.

The support of the Deutscher Akademischer Austauschdienst (DAAD) is gratefully acknowledged by the second author.

measure for each  $x, y \in H$ . Since all the values  $F(A)$ , for  $A \in \mathcal{B}(\Delta)$ , are orthogonal projections, we see that  $F_{x,x}(A) = \|F(A)x\|^2$  is actually non-negative, for each  $x \in H$ . To say that  $F$  is *Radon* means that  $F_{x,x}$  is a Radon measure (i.e. inner regular), for each  $x \in H$ ; see [1, Ch. 2], for example. The selfadjointness of  $F$  implies that

$$4F_{x,y} = F_{x+y,x+y} - F_{x-y,x-y} + iF_{x+iy,x+iy} - iF_{x-iy,x-iy}, \quad x, y \in H,$$

and so each complex measure  $F_{x,y}$  is also Radon. Since  $\Delta$  is compact and  $\widehat{T}$  is continuous, it is, in particular, a bounded Borel function and so the integral in (1) defines an element of  $\mathcal{L}(H)$  via the standard theory of integration with respect to spectral measures; see [5, Section 12.20].

Let  $S$  be a *commutative semigroup* with identity element (always denoted by  $e$ ) and equipped with an *involution*  $s \mapsto s^-$  (i.e.  $(s^-)^- = s$  and  $(st)^- = s^-t^-$  for all  $s, t \in S$ ). A *character* of  $S$  is any function  $\rho: S \rightarrow \mathbb{C}$  satisfying  $\rho(e) = 1$  and  $\rho(st^-) = \rho(s)\overline{\rho(t)}$  for all  $s, t \in S$ . The set of all characters of  $S$  is denoted by  $S^*$ ; it is a completely regular space when equipped with the topology of pointwise convergence inherited from  $\mathbb{C}^S$ . If  $H$  is a Hilbert space, then a map  $\mathcal{U}: S \rightarrow \mathcal{L}(H)$  is called a *\*-representation* if  $\mathcal{U}(e) = I$  and  $\mathcal{U}(st^-) = \mathcal{U}(s)\mathcal{U}(t)^*$  for all  $s, t \in S$ .

**Theorem 2.** *Let  $S$  be a commutative semigroup with identity and an involution, and let  $\mathcal{U}: S \rightarrow \mathcal{L}(H)$  be a \*-representation. Then there exists a unique selfadjoint, Radon spectral measure  $E: \mathcal{B}(S^*) \rightarrow \mathcal{L}(H)$  which has compact support, such that*

$$(2) \quad \mathcal{U}(s) = \int_{S^*} \hat{s}(\rho) dE(\rho), \quad s \in S.$$

Again some explanation is needed. The *support* of any spectral measure  $E: \mathcal{B}(S^*) \rightarrow \mathcal{L}(H)$ , denoted by  $\text{supp}(E)$ , is defined to be the closed subset of  $S^*$  given by  $\bigcup_{x \in H} \text{supp}(E_{x,x})$ , where  $\text{supp}(E_{x,x})$  is the usual support of a non-negative Radon measure [1, p. 22]. Finally, given  $s \in S$ , the function  $\hat{s}: S^* \rightarrow \mathbb{C}$  is defined by  $\hat{s}(\rho) := \rho(s)$ , for  $\rho \in S^*$ . Since  $\hat{s}$  is continuous and  $\text{supp}(E)$  is compact, the spectral integral on the right-hand-side of (2) exists and is an element of  $\mathcal{L}(H)$ .

A version of Theorem 2 is formulated in [2, Theorem 2.6] where it is indicated that the proof is a combination of an integral representation theorem for exponentially bounded, positive definite functions on semigroups (now-a-days referred to as the Berg-Maserick theorem) [2, Theorem 2.1], together with the method of proof given for an earlier version of Theorem 2 formulated for *uniformly bounded* representations  $\mathcal{U}$  [3, Theorem 3.2]. An examination of that proof (i.e. when  $\mathcal{U}$  is uniformly bounded) shows that an essential ingredient is the use of the theory of abelian  $C^*$ -algebras (cf. Section 2 of [3]) together with some more general Banach algebra arguments (cf. [3, p. 501]).

The aim of this note is to highlight the fact, contrary to the line of argument suggested above, that Theorems 1 and 2 are actually “independent” of one another. That is, we present a (new) proof of Theorem 2 which uses neither Gelfand-Naimark theory nor any Banach algebra arguments (either directly or indirectly). We then use Theorem 2 to establish Theorem 1, thereby giving a new proof of the Gelfand-Naimark theorem for abelian  $C^*$ -algebras. Conversely, via a quite different argument than that suggested by Berg and Maserick, we show that Theorem 2 also follows from Theorem 1.

In order to prove Theorem 2 we require a few preliminaries. A function  $\varphi: S \rightarrow \mathbb{C}$  is called *positive definite* if  $\sum_{j,k=1}^n c_j \bar{c}_k \varphi(s_j s_k^-) \geq 0$  for all choices of  $n \in \mathbb{N}$ ,

$\{s_1, \dots, s_n\} \subseteq S$  and  $\{c_1, \dots, c_n\} \subseteq \mathbb{C}$ . In particular, every character  $\rho \in S^*$  is positive definite.

A function  $\alpha: S \rightarrow [0, \infty)$  satisfying  $\alpha(e) = 1$  is called an *absolute value* if  $\alpha(s^-) = \alpha(s)$  and  $\alpha(st) \leq \alpha(s)\alpha(t)$  for all  $s, t \in S$ . A function  $f: S \rightarrow \mathbb{C}$  is called  $\alpha$ -*bounded* if there exists  $C > 0$  such that  $|f(s)| \leq C\alpha(s)$  for all  $s \in S$ . If  $f$  happens to be positive definite and  $\alpha$ -bounded, then it is possible to choose  $C = \varphi(e)$ . A character  $\rho \in S^*$  is  $\alpha$ -bounded iff  $|\rho| \leq \alpha$ . Hence, the set  $S^\alpha$  of all  $\alpha$ -bounded characters is a compact subset of  $S^*$ . For all of these notions and further properties we refer to [1, Ch. 4].

The space of all non-negative Radon measures on  $S^*$  is denoted by  $M_+(S^*)$ . Given any absolute value  $\alpha: S \rightarrow [0, \infty)$  the subspace of  $M_+(S^*)$  consisting of all Radon measures supported in the compact subset  $S^\alpha \subseteq S^*$  is denoted by  $M_+(S^\alpha)$ . The following result is the Berg-Maserick theorem mentioned above.

**Proposition 1.** *Let  $\alpha: S \rightarrow [0, \infty)$  be an absolute value on  $S$  and  $\varphi: S \rightarrow \mathbb{C}$  an  $\alpha$ -bounded, positive definite function. Then there exists  $\mu \in M_+(S^\alpha)$  such that*

$$(3) \quad \varphi(s) = \int_{S^*} \hat{s}(\rho) d\mu(\rho), \quad s \in S,$$

and  $\mu$  is unique within  $M_+(S^*)$ .

*Remark 1.* The function on the right-hand-side of (3), with domain  $S$ , is denoted by  $\hat{\mu}$  and is called the *generalized Laplace transform* of  $\mu$ . It is important to note that the proof of Proposition 1 given in [1, Ch. 4, §2], which is based on the integral version of the Krein-Milman theorem, makes no use of any Banach algebra techniques what-so-ever.

*Proof of Theorem 2.* Define an absolute value  $\alpha: S \rightarrow [0, \infty)$  by  $\alpha(s) = \|\mathcal{U}(s)\|$  for  $s \in S$ . For  $x \in H$  fixed define  $\varphi_x: S \rightarrow \mathbb{C}$  by  $\varphi_x(s) = \langle \mathcal{U}(s)x, x \rangle$ , for  $s \in S$ . Using the fact that  $\mathcal{U}$  is a  $*$ -representation it follows that

$$\sum_{j,k=1}^n c_j \bar{c}_k \varphi_x(s_j s_k^-) = \left\| \sum_{j=1}^n c_j \mathcal{U}(s_j)x \right\|^2 \geq 0$$

for any finite sets  $\{c_1, \dots, c_n\} \subseteq \mathbb{C}$  and  $\{s_1, \dots, s_n\} \subseteq S$  and hence,  $\varphi_x$  is positive definite. Moreover,

$$|\varphi_x(s)| \leq \|\mathcal{U}(s)\| \cdot \|x\|^2 = \|x\|^2 \alpha(s), \quad s \in S,$$

which shows that  $\varphi_x$  is  $\alpha$ -bounded. By Proposition 1 there is a unique Radon measure  $\mu_x \in M_+(S^\alpha)$  such that  $\varphi_x = \hat{\mu}_x$ .

For  $x, y \in H$  define a complex Radon measure  $\mu_{x,y}$  by

$$(4) \quad \mu_{x,y} = \frac{1}{4}(\mu_{x+y} - \mu_{x-y} + i\mu_{x+iy} - i\mu_{x-iy}),$$

in which case  $\hat{\mu}_{x,y} = \frac{1}{4}(\hat{\mu}_{x+y} - \hat{\mu}_{x-y} + i\hat{\mu}_{x+iy} - i\hat{\mu}_{x-iy})$ . It then follows from the definition of  $\hat{\mu}_z = \varphi_z$  for each  $z \in H$  and a direct calculation that

$$(5) \quad \hat{\mu}_{x,y}(s) = \langle \mathcal{U}(s)x, y \rangle, \quad s \in S.$$

Fix  $B \in \mathcal{B}(S^*)$  and define  $\Psi_B: H \times H \rightarrow \mathbb{C}$  by  $\Psi_B(x, y) = \mu_{x,y}(B)$  for each  $x, y \in H$ . It is clear from (5) and the linearity and injectivity of the map  $\nu \mapsto \hat{\nu}$  on the space of compactly supported (complex) Radon measures on  $S^*$  [1, p. 96] that  $\Psi_B$  is a sesquilinear form on  $H \times H$ . We proceed to show that  $\Psi_B$  is bounded.

Let  $\{x_1, \dots, x_n\} \subseteq H$  and  $\{c_1, \dots, c_n\} \subseteq \mathbb{C}$  be finite sets. Define a Radon measure  $\nu$  on  $\mathcal{B}(S^*)$  by  $\nu = \sum_{j,k=1}^n c_j \bar{c}_k \mu_{x_j, x_k}$ . Then, for finite sets  $\{d_1, \dots, d_m\} \subseteq \mathbb{C}$  and  $\{t_1, \dots, t_m\} \subseteq S$  we have

$$\sum_{p,q=1}^m d_p \bar{d}_q \hat{\nu}(t_p t_q^-) = \sum_{j,k} \sum_{p,q} d_p \bar{d}_q c_j \bar{c}_k \langle \mathcal{U}(t_p t_q^-) x_j, x_k \rangle = \left\| \sum_{p,j} c_j d_p \mathcal{U}(t_p) x_j \right\|^2 \geq 0,$$

which shows that  $\hat{\nu}: S \rightarrow \mathbb{C}$  is positive definite. By combining Theorem 2.5 on p. 93 and the remarks of §2.10 on p. 96 in [1] it follows that  $\nu \geq 0$ , i.e.  $\nu \in M_+(S^\alpha)$ . In particular,

$$\sum_{j,k=1}^n c_j \bar{c}_k \Psi_B(x_j, x_k) = \sum_{j,k=1}^n c_j \bar{c}_k \mu_{x_j, x_k}(B) = \nu(B) \geq 0,$$

which shows that  $\Psi_B$  is a positive definite kernel (in the sense of [1, p. 67], for example). By the Cauchy-Schwarz inequality for such kernels (cf. [1, p. 69], for example) we have  $|\Psi_B(x, y)|^2 \leq \Psi_B(x, x) \Psi_B(y, y)$ , that is,

$$|\mu_{x,y}(B)| \leq [\mu_{x,x}(B)]^{1/2} [\mu_{y,y}(B)]^{1/2} \leq \|x\| \cdot \|y\|, \quad x, y \in H,$$

where the last inequality relies on the observation that

$$\mu_{x,x}(B) \leq \mu_{x,x}(S^\alpha) = \int_{S^*} \rho(e) d\mu_{x,x}(\rho) = \hat{\mu}_{x,x}(e) = \langle \mathcal{U}(e)x, x \rangle = \|x\|^2$$

for all  $x \in H$ . So, the sesquilinear form  $\Psi_B$  is indeed bounded and hence, there is  $E(B) \in \mathcal{L}(H)$  satisfying

$$(6) \quad \langle E(B)x, y \rangle = \Psi_B(x, y) = \mu_{x,y}(B), \quad x, y \in H.$$

To see that  $E(B)^* = E(B)$  we note that

$$\langle E(B)^* x, y \rangle = \langle x, E(B)y \rangle = \overline{\langle E(B)y, x \rangle} = \overline{\mu_{y,x}(B)} = \mu_{x,y}(B) = \langle E(B)x, y \rangle$$

for all  $x, y \in H$ , where  $\overline{\mu_{y,x}(B)} = \mu_{x,y}(B)$  follows again from the positive definiteness of the kernel  $\Psi_B$ . Furthermore,  $E(S^*) = I$  since, by (5), we have

$$\langle E(S^*)x, y \rangle = \mu_{x,y}(S^*) = \hat{\mu}_{x,y}(e) = \langle \mathcal{U}(e)x, y \rangle = \langle x, y \rangle, \quad x, y \in H.$$

It is clear from (6) that  $E_{x,y} = \mu_{x,y}$  is  $\sigma$ -additive for all  $x, y \in H$ . The identity (6) also shows that  $E: \mathcal{B}(S^*) \rightarrow \mathcal{L}(H)$  is a Radon measure and that it is compactly supported since  $\text{supp}(E_{x,y}) = \text{supp}(\mu_{x,y}) \subseteq S^\alpha$  for all  $x, y \in H$ .

To see that  $E$  is multiplicative we proceed as follows. Given a Borel set  $A \subseteq S^*$  and  $x, y \in H$  define measures  $\nu_A$  and  $\lambda_A$  by  $\nu_A(B) = \mu_{E(A)x, y}(B)$  and  $\lambda_A(B) = \langle E(A \cap B)x, y \rangle = \mu_{x,y}(A \cap B)$  for each  $B \in \mathcal{B}(S^*)$ . Using (5) we have

$$(7) \quad \hat{\nu}_A(t) = \hat{\mu}_{E(A)x, y}(t) = \langle \mathcal{U}(t)E(A)x, y \rangle = \langle E(A)x, \mathcal{U}(t)^* y \rangle = \mu_{x, \mathcal{U}(t)^* y}(A)$$

and

$$(8) \quad \hat{\lambda}_A(t) = \int_A \rho(t) d\mu_{x,y}(\rho).$$

For a fixed  $t \in S$  it is clear from (7) and (8) that the mappings  $\tau_1: A \mapsto \hat{\nu}_A(t)$  and  $\tau_2: A \mapsto \hat{\lambda}_A(t)$  are Radon measures on  $S^*$ , supported in  $S^\alpha$ , and so we can also calculate  $\hat{\tau}_1$  and  $\hat{\tau}_2$ . Indeed, for  $s \in S$ , we have (by (5) and (7)) that

$$\hat{\tau}_1(s) = \langle \mathcal{U}(s)x, \mathcal{U}(t)^* y \rangle = \langle \mathcal{U}(st)x, y \rangle = \int_{S^*} \rho(s)\rho(t) d\mu_{x,y}(\rho).$$

But, from (8) it is also clear that

$$\hat{\tau}_2(s) = \int_{S^*} \rho(s) d\tau_2(\rho) = \int_{S^*} \rho(s)\rho(t) d\mu_{x,y}(\rho).$$

By uniqueness of generalized Laplace transforms [1, p. 96], it follows that  $\tau_1 \equiv \tau_2$  as compactly supported Radon measures (for any fixed  $t \in S$ ). In particular, for  $A$  fixed we deduce that  $\hat{\nu}_A(t) = \hat{\lambda}_A(t)$ . Since this is true for each  $t \in S$  it again follows that  $\nu_A = \lambda_A$  as measures and hence, for any Borel set  $B \subseteq S^*$  we have  $\lambda_A(B) = \nu_A(B)$ , that is,

$$\langle E(A \cap B)x, y \rangle = \lambda_A(B) = \nu_A(B) = \langle E(B)E(A)x, y \rangle, \quad x, y \in H.$$

So,  $E$  is multiplicative.

Finally,  $\hat{s}: S^* \rightarrow \mathbb{C}$  is continuous and hence bounded on  $S^\alpha$ , for any  $s \in S$ . By the theory of spectral integrals  $\int_{S^*} \hat{s}(\rho) dE(\rho)$  exists in  $\mathcal{L}(H)$ . It is clear from (5) and (6) that  $\mathcal{U}(s) = \int_{S^*} \hat{s}(\rho) dE(\rho)$ , for  $s \in S$ , and the proof of Theorem 2 is complete.  $\square$

To deduce Theorem 1 from Theorem 2 we require a further result. A commutative, unital complex algebra  $\mathcal{A}$  with an involution (see Definitions 10.1 and 11.14 in [5]) always induces a commutative semigroup with involution, namely let  $S = \mathcal{A}$  and consider  $\mathcal{A}$  just with respect to its given multiplication and involution. We then define  $S^\otimes := \{\rho \in S^* : \rho \text{ is linear}\}$  and, given any absolute value  $\alpha: S \rightarrow [0, \infty)$ , define  $S^\otimes := S^\alpha \cap S^\otimes$ . The following result can be found in [4, Corollary 1]; we stress again that its proof does not use any Banach algebra methods.

**Proposition 2.** *Let  $S$  be the semigroup induced by a unital, commutative complex algebra with an involution. Let  $\alpha: S \rightarrow [0, \infty)$  be an absolute value. If  $\varphi: S \rightarrow \mathbb{C}$  is a positive definite,  $\alpha$ -bounded function which is linear on  $S$ , then the unique Radon measure  $\mu \in M_+(S^\alpha)$  satisfying  $\hat{\mu} = \varphi$  has its support in  $S^\otimes$ .*

*Proof of Theorem 1.* Let  $A \subseteq \mathcal{L}(H)$  be an abelian  $C^*$ -algebra. Let  $S = \mathcal{A}$ , considered as a semigroup with respect to the multiplication in  $\mathcal{A}$  and with the involution from  $\mathcal{A}$ . By Theorem 2 there exists a unique, selfadjoint Radon spectral measure  $E: \mathcal{B}(S^*) \rightarrow \mathcal{L}(H)$  with compact support satisfying

$$(9) \quad T = \int_{S^*} \hat{T}(\rho) dE(\rho), \quad T \in S;$$

we have used the representation  $\mathcal{U}(T) = T$ , for  $T \in S$ . From the proof of Theorem 2, where the absolute value  $\alpha: S \rightarrow [0, \infty)$  used is  $\alpha(T) = \|\mathcal{U}(T)\| = \|T\|$ , for  $T \in S$ , we have that  $\text{supp}(E) \subseteq S^\alpha$ . Now, for  $x \in H$  fixed, the positive definite and  $\alpha$ -bounded function  $\varphi_x: S \rightarrow \mathbb{C}$  defined in the proof of Theorem 2 (i.e.  $\varphi_x(T) = \langle \mathcal{U}(T)x, x \rangle = \langle Tx, x \rangle$ , for  $T \in S$ ) is clearly linear on  $S$ . Then Proposition 2 implies that the unique measure  $\mu_x \in M_+(S^\alpha)$  satisfying  $\hat{\mu}_x = \varphi_x$  has its support in  $S^\otimes$ . Since  $E_{x,x} = \mu_x$  also  $\text{supp}(E) \subseteq S^\otimes$ . But, if  $\rho \in S^\otimes$ , then

$$|\rho(T)| \leq \alpha(T) = \|T\|, \quad T \in \mathcal{A},$$

which implies that  $\|\rho\| = 1$ . So,  $S^\otimes$  consists of all algebra homomorphisms  $\rho: S(= \mathcal{A}) \rightarrow \mathbb{C}$  of norm 1. It is part of the definition that each semigroup character  $\rho \in S^*$  must satisfy  $\rho(s^-) = \overline{\rho(s)}$  which, in the present setting, becomes  $\rho(T^*) = \overline{\rho(T)}$ . But, elements of the structure space  $\Delta$  of  $\mathcal{A}$  automatically satisfy this condition [5, Theorem 11.18]. Hence,  $S^\otimes$  is precisely  $\Delta$  and (9), which is actually an integral over  $S^\otimes = \Delta$ , reduces to (1).  $\square$

In conclusion we indicate how Theorem 2 also follows from Theorem 1. So, let  $S$  and  $\mathcal{U}: S \rightarrow \mathcal{L}(H)$  be as in Theorem 2. Define  $\mathcal{A} \subseteq \mathcal{L}(H)$  to be the  $C^*$ -algebra generated by the  $*$ -closed, commutative family of operators  $\mathcal{M} = \{\mathcal{U}(s): s \in S\}$ . Of course,  $\mathcal{A}$  is the operator norm closure in  $\mathcal{L}(H)$  of the linear span of  $\mathcal{M}$ . Denote the structure space of  $\mathcal{A}$  by  $\Delta$ . For each  $\delta \in \Delta$  define  $\tilde{\delta}: S \rightarrow \mathbb{C}$  by  $\tilde{\delta}(s) := \delta(\mathcal{U}(s))$ ,  $s \in S$ . From the homomorphism properties of  $\delta$  and the fact that  $\mathcal{U}$  is a  $*$ -representation it is easily seen that  $\tilde{\delta} \in S^*$ ; the property  $\tilde{\delta}(s^-) = \overline{\tilde{\delta}(s)}$  again follows from the comments at the end of the previous paragraph. So, we can define  $\Lambda: \Delta \rightarrow S^*$  by  $\Lambda(\delta) := \tilde{\delta}$ , for  $\delta \in \Delta$ , in which case  $\hat{s} \circ \Lambda = (\mathcal{U}(s))^\wedge$  as functions on  $\Delta$ , for each  $s \in S$ . If  $\Lambda(\delta_1) = \Lambda(\delta_2)$ , then  $\delta_1(\mathcal{U}(s)) = \delta_2(\mathcal{U}(s))$  for all  $s \in S$  and hence,  $\delta_1$  and  $\delta_2$  agree on the linear span of  $\mathcal{M}$ . Since the span of  $\mathcal{M}$  is dense in  $\mathcal{A}$  and both  $\delta_1$  and  $\delta_2$  are continuous on  $\mathcal{A}$ , it follows that  $\delta_1 = \delta_2$ . This shows that  $\Lambda$  is injective. Since  $\Delta$  (resp.  $S^*$ ) is equipped with the pointwise convergence topology on  $\mathcal{A}$  (resp.  $S$ ), it is clear that  $\Lambda$  is continuous. Then the compactness of  $\Delta$  ensures that  $\Lambda$  is a topological homeomorphism of  $\Delta$  onto its range  $K := \Lambda(\Delta) \subseteq S^*$ . By Theorem 1 there is a unique selfadjoint Radon spectral measure  $F: \mathcal{B}(\Delta) \rightarrow \mathcal{L}(H)$  satisfying

$$T = \int_{\Delta} \hat{T}(\delta) dF(\delta), \quad T \in \mathcal{A}.$$

Define  $E: \mathcal{B}(S^*) \rightarrow \mathcal{L}(H)$  by  $E(A) = F(\Lambda^{-1}(A))$ , for each  $A \in \mathcal{B}(S^*)$ , in which case  $E$  is a compactly supported (as  $\text{supp}(E) \subseteq K$ ), selfadjoint Radon spectral measure such that

$$(10) \quad \mathcal{U}(s) = \int_{\Delta} (\mathcal{U}(s))^\wedge(\delta) dF(\delta) = \int_{S^*} \hat{s}(\rho) dE(\rho), \quad s \in S;$$

see [5, Theorem 13.28] for the last equality.

Suppose now that  $G: \mathcal{B}(S^*) \rightarrow \mathcal{L}(H)$  is another compactly supported, selfadjoint Radon spectral measure satisfying

$$\mathcal{U}(s) = \int_{S^*} \hat{s}(\rho) dG(\rho), \quad s \in S.$$

It follows that  $\int_{S^*} \hat{s}(\rho) dG_{x,x}(\rho) = \int_{S^*} \hat{s}(\rho) dE_{x,x}(\rho)$  for all  $s \in S$  and  $x \in H$ , that is,  $\hat{G}_{x,x} = \hat{E}_{x,x}$  for all  $x \in H$ . Since both  $G_{x,x}$  and  $E_{x,x}$  are compactly supported elements of  $M_+(S^*)$ , it follows from the uniqueness of generalized Laplace transforms that  $E_{x,x} = G_{x,x}$  for all  $x \in H$ . Then the selfadjointness of both  $E$  and  $G$  and the polarization identity imply that  $E_{x,y} = G_{x,y}$  for all  $x, y \in H$ . In particular, it follows that  $E(A) = G(A)$  for all  $A \in \mathcal{B}(S^*)$ , that is,  $E = G$ . Hence, the measure  $E$  satisfying (10) is unique and the proof is complete.

*Remark 2.* From the proof of Theorem 2 it is obvious that if all the measures  $E_{x,x} = \mu_x$ , for  $x \in H$ , concentrate on a subset  $A \subseteq S^*$ , then so does  $E$ . As a consequence we obtain Stone’s theorem for  $S$  a locally compact abelian group (which is a semigroup with the special involution  $s^- := s^{-1}$ ), and where  $\mathcal{U}$  is assumed to be weak operator continuous, i.e.  $s \mapsto \langle \mathcal{U}(s)x, x \rangle$  is continuous for each  $x \in H$ . Then by Bochner’s theorem all the measures  $\mu_x$  live on the dual group (by definition the set of all *continuous* characters), and so therefore does  $E$ .

## REFERENCES

1. C. Berg, J. P. R. Christensen and P. Ressel, *Harmonic analysis on semigroups, Theory of positive definite and related functions*, Graduate Texts in Mathematics Vol. 100, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1984. MR **86b**:43001
2. C. Berg and P. H. Maserick, *Exponentially bounded positive definite functions*, Illinois J. Math. **28** (1984), 162–179. MR **85i**:43012
3. P. H. Maserick, *Spectral theory of operator-valued transformations*, J. Math. Anal. Appl. **41** (1973), 497–507. MR **49**:7828
4. P. Ressel, *Integral representations on convex semigroups*, Math. Scand. **61** (1987), 93–111. MR **89e**:43012
5. W. Rudin, *Functional analysis*, McGraw Hill Book Co., New York-San Francisco-St. Louis, 1973. MR **51**:1315

MATH.-GEOGR. FAKULTÄT, KATHOLISCHE UNIVERSITÄT EICHSTÄTT, D-85071 EICHSTÄTT, GERMANY

SCHOOL OF MATHEMATICS, UNIVERSITY OF NEW SOUTH WALES, SYDNEY, NEW SOUTH WALES, 2052 AUSTRALIA