POLYNOMIAL APPROXIMATION WITH VARYING WEIGHTS
ON COMPACT SETS OF THE COMPLEX PLANE

IGOR E. PRITSKER

(Communicated by Theodore W. Gamelin)

Abstract. For a compact set $E$ with connected complement, let $A(E)$ be the uniform algebra of functions continuous on $E$ and analytic interior to $E$. We describe $A(E, W)$, the set of uniform limits on $E$ of sequences of the weighted polynomials $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, as $n \to \infty$, where $W \in A(E)$ is a nonvanishing weight on $E$. If $E$ has empty interior, then $A(E, W)$ is completely characterized by a zero set $Z_W \subset E$. However, if $E$ is a closure of Jordan domain, the description of $A(E, W)$ also involves an inner function.

In both cases, we exhibit the role of the support of a certain extremal measure, which is the solution of a weighted logarithmic energy problem, played in the descriptions of $A(E, W)$.

1. Introduction

Let $E$ be a compact set in the complex plane $\mathbb{C}$ with the connected complement $\mathbb{C} \setminus E$. We denote the uniform algebra of continuous on $E$ and analytic in the interior of $E$ functions by $A(E)$ (see, e.g., [5, p. 25]). Clearly, the corresponding uniform norm for any $f \in A(E)$ is defined by

$$\|f\|_E := \max_{z \in E} |f(z)|.$$

Consider a weight function $W \in A(E)$ such that $W(z) \neq 0$ for any $z \in E$, and define the weighted polynomials $W^n(z)P_n(z)$, where $P_n(z)$ is an algebraic polynomial in $z$ with complex coefficients, with $\deg P_n \leq n$. Note that the power of weight varies with the degree of polynomial. We are interested in a description of the function set $A(E, W)$, consisting of the uniform limits on $E$ of sequences of the weighted polynomials $\{W^n(z)P_n(z)\}_{n=0}^{\infty}$, as $n \to \infty$. It is well known that if $W(z) \equiv 1$ on $E$ then $A(E, 1) = A(E)$ by Mergelyan’s theorem [5, p. 48]. In general, we have that $A(E, W) \subset A(E)$.

Our problem originated in the work of Lorentz [12] on incomplete polynomials on the real line. Surveys of results in this area, dealing with weighted approximation on the real line, can be found in [22] and [18, Ch. VI]. The most recent developments are in [8]–[10].
The questions of density of the weighted polynomials in the set of analytic functions in a domain have been considered in [4], [15] and [16]. In particular, [16] contains a necessary and sufficient condition such that any analytic in a bounded open set function is uniformly approximable by the weighted polynomials $W^n(z)P_n(z)$ on compact subsets. However, the description of $A(E,W)$ seems to be much more complicated, in that no general necessary and sufficient condition is known (in terms of the weight $W(z)$), even for the real interval case, i.e., for $E = [a, b] \subset \mathbb{R}$.

We shall approach the above mentioned problems on $A(E,W)$, using ideas of the theories of uniform algebras and of weighted potentials.

The second section of this paper deals with certain general inclusions of $A(E,W)$ and some weighted potential background. We state the results on $A(E,W)$ for $E$ having empty interior in Section 3. The corresponding results for $E$ being the closure of the unit disk $D$ or, more generally, of a Jordan domain $G$, are developed in Section 4. Section 5 contains all proofs. We conclude with remarks and open problems in Section 6.

2. Inclusion of $A(E,W)$ as a Closed Ideal and Weighted Potentials

We start with

Proposition 2.1. $A(E,W)$, endowed with norm (1.1), is a closed function algebra (not necessarily containing constants and separating points).

We have already remarked that $A(E,W) \subset A(E)$. To make this inclusion more precise, let us introduce the algebra $[W(z),zW(z)]$ generated by the two functions $W(z)$ and $zW(z)$, which is the uniform closure of all polynomials in $W(z)$ and $zW(z)$ (with constant terms included) on $E$. Clearly, $[W(z),zW(z)] \subset A(E)$. Furthermore, since any weighted polynomial $W^n(z)P_n(z)$ is an element of $[W(z),zW(z)]$, then $A(E,W) \subset [W(z),zW(z)]$. Thus, we arrive at the following

Proposition 2.2. $A(E,W) \subset [W(z),zW(z)] \subset A(E)$.

The next fact is rather simple but important.

Proposition 2.3. $A(E,W)$ is a closed ideal of $[W(z),zW(z)]$.

It turns out that in many cases $[W(z),zW(z)] = A(E)$, so that $A(E,W)$ becomes a closed ideal of $A(E)$ by Proposition 2.3. This situation is the most interesting for us, because then we can employ characterizations of the closed ideals of $A(E)$ for some types of the compact $E$. This is done in Sections 3 and 4.

Proposition 2.4. $[W(z),zW(z)] = A(E)$ iff $1/W(z) \in [W(z),zW(z)]$.

Unfortunately, we do not know any effectively verifiable necessary and sufficient condition on the weight $W(z)$, so that the equality $[W(z),zW(z)] = A(E)$ is valid. Nevertheless, a number of sufficient conditions can be given, guaranteeing that the two algebras $[W(z),zW(z)]$ and $A(E)$ coincide.

Proposition 2.5. Each of the following conditions implies that $[W(z),zW(z)] = A(E)$:

(a) The point $\zeta = 0$ belongs to the unbounded component of $\overline{E \setminus W(E)}$;
(b) $E$ is the closure of a Jordan domain or a Jordan arc, and $W(z)$ is one-to-one on $E$;
(c) $E$ is a Jordan arc and $W(z)$ is of bounded variation on $E$;
(d) $E$ is a Jordan arc and $W(z)$ is locally one-to-one on $E$;
(e) $E = \overline{G}$, where $G$ is a Jordan domain bounded by an analytic curve, and $W'(z) \in A(\overline{G})$.

Other sufficient conditions, implying the conclusion of Proposition 2.5, can be found in [1], [6], [21, §30] and [20]. Obviously, if the interior of $E$ is empty, then $A(E) = C(E)$ by definition, where $C(E)$ is the algebra of all continuous functions on $E$.

A very important tool for analyzing the behavior of the weighted polynomials $W^n P_n$ is the theory of logarithmic potentials with external fields (cf. [18]). Assuming that $E$ has positive logarithmic capacity (cf. [23, p. 55]), then
\begin{equation}
    w(z) := \begin{cases} |W(z)|, & z \in E, \\
    0, & z \notin E, \end{cases}
\end{equation}
is an admissible weight for the weighted logarithmic energy problem on $E$ considered in Section 1.1 of [18]. This enables us to use certain results of [18], which we summarize below for the convenience of the reader. Recall that the logarithmic potential of a compactly supported Borel measure $\mu$ is given by (cf. [23, p. 53])
\begin{equation}
    U^\mu(z) := \int \frac{1}{|z-t|} d\mu(t).
\end{equation}

**Proposition 2.6.** There exists a positive unit Borel measure $\mu_w$, with support $S_w := \text{supp } \mu_w \subset \partial E$, such that for any polynomial $P_n(z), \deg P_n \leq n$, we have
\begin{equation}
    |W^n(z)P_n(z)| \leq \|W^n P_n\|_{S_w} \exp(n(F_w - U^\mu_w(z) + \log |W(z)|)), \quad z \in E,
\end{equation}
where $F_w$ is a constant.

Furthermore, the inequality
\begin{equation}
    U^\mu_w(z) - \log |W(z)| \geq F_w
\end{equation}
holds quasi-everywhere on $E$, and
\begin{equation}
    U^\mu_w(z) - \log |W(z)| \leq F_w, \quad \text{for any } z \in S_w.
\end{equation}

By saying quasi-everywhere (q.e.), we mean that a property holds everywhere, with the exception of a set of zero logarithmic capacity. The measure $\mu_w$ is the solution of a weighted energy problem, corresponding to the weight $w(z)$ of (2.1) (see Section 1.1 of [18]).

It follows from (2.3) and (2.4) that the norm of a weighted polynomial $W^n P_n$ essentially “lives” on $S_w$, i.e.,
\begin{equation}
    |W^n(z)P_n(z)| \leq \|W^n P_n\|_{S_w}
\end{equation}
for quasi-every $z \in E$. We can replace, in fact, the words “quasi-everywhere on $E$” in (2.4) and (2.6) by “everywhere on $E$”, by requiring a certain regularity for the set $E$. In particular, the following is valid (see Corollary III.2.6 of [18]).

**Proposition 2.7.** Suppose that for every point $z_0 \in E$, the set $\{z : |z - z_0| < \delta, \ z \in E\}$ has positive capacity for any $\delta > 0$. Then
\begin{equation}
    \|W^n P_n\|_E = \|W^n P_n\|_{S_w}
\end{equation}
for any polynomial $P_n, \deg P_n \leq n$.

Consequently, the Shilov boundary (cf. [21, pp. 36–39]) of $A(E,W)$ is contained in $S_w$. 
One may consult Section III.2 of [18] on further details, regarding the supremum norm of weighted polynomials.

3. Weighted approximation on sets with empty interior

Let E be a compact set with connected complement and empty interior. Obviously, \( A(E) = C(E) \) in this case. We characterize \( A(E;W) \) in terms of a certain zero set.

**Theorem 3.1.** Suppose that E has a connected complement and an empty interior, and that \( W \in C(E) \) is a nonvanishing weight on E. Assume that \( [W(z), zW(z)] = C(E) \).

Then, there exists a closed set \( Z_W \subset E \) such that

\[
 f \in A(E;W) \text{ if and only if } f \in C(E) \text{ and } f|_{Z_W} \equiv 0.
\]

It is clear that \( A(E;W) = C(E) \) if and only if the set \( Z_W \) is empty. This is true, for example, for \( W(z) \equiv 1 \) on E.

Theorem 3.1 generalizes a recent result of Kuijlaars (see Theorem 3 of [10]), related to polynomial approximation with varying weights on the real line. However, it has a new part even in the latter case, allowing us to consider the complex valued weights \( W(z) \) on subsets of the real line.

A description of the set \( Z_W \) in terms of the weight \( W(z) \) is unknown in general. We can only show that \( Z_W \) must contain the complement of \( S_w \) (see Proposition 2.6) in E.

**Theorem 3.2.** Let E be an arbitrary compact set with the connected complement \( \mathbb{C}\setminus E \) and let \( W \in A(E) \) be a nonvanishing weight on E. Suppose that for every point \( z_0 \in E \), the set \( \{ z : |z - z_0| < \delta, z \in E \} \) has positive logarithmic capacity for any \( \delta > 0 \). Assume further that \( \mathbb{C}\setminus S_w \) is connected and \( [W(z), zW(z)] = C(S_w) \) on \( S_w \).

If \( f \in A(E;W) \), then \( f(z) = 0 \) for any \( z \in E\setminus S_w \). In particular, if E has empty interior, then \( E\setminus S_w \subset Z_W \).

The proof of Theorem 3.2 is based on an idea of Kuijlaars (see Theorem 2 and its proof in [10]).

If E is a compact subset of the real line and the weight \( W(z) \) is real valued, then condition (a) of Proposition 2.5 is clearly satisfied, so that \( [W(z), zW(z)] = C(E) \). Therefore, the conclusion of Theorem 3.1 is valid, and coincides with that of Theorem 3 of [10]. Furthermore, if for any point in E, the intersection of its arbitrary neighborhood with E has positive logarithmic capacity, then \( E\setminus S_w \subset Z_W \).

Since \( [W(z), zW(z)] = C(S_w) \) on \( S_w \) by Proposition 2.5(a), Theorem 3.2 essentially reduces to Theorem 2 of [10] in this case, which in turn contains an earlier result of Theorem 4.1 of [22].

We mention two examples here just for illustrative purposes. A number of additional examples, with their complete discussions, is in [22], [18, Ch. VI], and [8]-[9].

**Example 3.3** (Incomplete Polynomials). Let \( E = [0, 1] \) and \( W(x) = x^{\theta/(1-\theta)} \), where \( 0 < \theta < 1 \). It is known that \( S_w = [\theta^2, 1] \) and that \( A([0, 1], x^{\theta/(1-\theta)}) \) consists of all continuous functions on \([0, 1]\), vanishing on \([0, \theta^2]\) (see [19] and [18, Sect. VI.1]). Thus, the set \( Z_W \) of Theorem 3.1 is just \([0, \theta^2]\) = \( E\setminus S_w \) in this case.

However, it is not always true that \( Z_W = E\setminus S_w \), as the next example shows.
Example 3.4 (Exponential Weights). Let \( E = [-2, 2] \) and \( W(x) = \exp(-\gamma_\alpha |x|^\alpha) \), where \( \alpha > 0 \) and
\[
\gamma_\alpha = \frac{\Gamma(\alpha/2) \Gamma(1/2)}{2\Gamma(\alpha/2 + 1/2)}.
\]
It is known that \( S_w = [-1, 1] \) and that
\[
A([-2, 2], \exp(-\gamma_\alpha |x|^\alpha)) = \begin{cases} \{ f \in C([-2, 2]) : f|_{[-2, -1] \cup [1, 2]} \equiv 0 \} & \text{for } \alpha \geq 1, \\ \{ f \in C([-2, 2]) : f|_{[-2, -1] \cup [1, 2] \cup \{0\}} \equiv 0 \} & \text{for } 0 < \alpha < 1, \end{cases}
\]
where the range \( \alpha > 1 \) was studied in [13], and the case of the remaining interval \( 0 < \alpha \leq 1 \) was covered by the results of [14]. In our context, (3.1) means that \( Z_W = [-2, -1] \cup [1, 2] = E \setminus S_w \) for \( \alpha \geq 1 \) and that \( Z_W = [-2, -1] \cup [1, 2] \cup \{0\} = E \setminus S_w \cup \{0\} \) for \( 0 < \alpha < 1 \).

4. Weighted Approximation on the Unit Disk and on Jordan Domains

The first result of this section is a consequence of the well-known description of closed ideals of \( A(D) \), where \( D \) is the unit disk, due to Beurling (unpublished) and Rudin [17] (see also [7, pp. 82-87] for a discussion). Recall that \( g \) is an inner function if it is analytic in \( D \), with \( \|g\|_{H^\infty} \leq 1 \), and \( |g(e^{i\theta})| = 1 \) almost everywhere on the unit circle (cf. [7, p. 62]). By the factorization theorem, every inner function can be uniquely expressed in the form
\[
g(z) = B(z)S(z), \quad z \in D,
\]
where \( B(z) \) is a Blaschke product and \( S(z) \) is a singular function, i.e.,
\[
S(z) := \exp \left( -\int \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\nu_s(\theta) \right), \quad z \in D,
\]
with \( \nu_s \) being a positive measure on the unit circle, singular with respect to \( d\theta \) (see [7, pp. 63-67]).

**Theorem 4.1.** Let a nonvanishing weight \( W \in A(D) \) be such that \([W(z), zW(z)] = A(D)\). Assume that \( A(D, W) \) contains a not identically zero function.

Then there exist a closed set \( H_W \subset \partial D \) of Lebesgue measure zero and an inner function \( g_W \) satisfying
(i) every accumulation point of the zeros of its Blaschke product is in \( H_W \);
(ii) the measure \( \nu_s \) of its singular function is supported on \( H_W \);
such that
\[
f \in A(D, W) \text{ if and only if } f = g_W h, \text{ where } h \in A(D) \text{ and } h|_{H_W} \equiv 0.
\]

The case of a Jordan domain \( G \) can be reduced to that of the unit disk, using a canonical conformal mapping \( \phi : G \to D \) and its inverse \( \psi := \phi^{-1} \). We state the corresponding result for completeness.

**Theorem 4.2.** Let a nonvanishing weight \( W \in A(G) \), where \( G \) is a Jordan domain, be such that \([W(z), zW(z)] = A(G)\). Assume that \( A(G, W) \) contains a not identically zero function.
Then there exist a closed set \( H_W \subset \partial D \) and an inner function \( g_W \), as in Theorem 4.4, such that
\[
f \in A(\overline{G}, W) \text{ if and only if } f = (g_W \circ \phi)h,
\]
where \( h \in A(\overline{G}) \) and \( h|_{\partial\psi(H_W)} \equiv 0 \).

It follows from Theorems 4.1 and 4.2 that \( A(D, W) = A(D) \) or \( A(G, W) = A(G) \) if and only if \( g_W \equiv 1 \) and \( H_W \) is empty.

**Example 4.3** (Exponential weight on the Szegő domain). Let \( W(z) = e^{-z} \) and \( G \) be the Szegő domain
\[
\{ z : |ze^{1-z}| < 1 \text{ and } |z| < 1 \},
\]
which is bounded by a piecewise analytic curve (the Szegő curve) with the only corner point \( z = 1 \) (see [15] for more information and the graph of \( G \)). We remark that the function \( \phi(z) = ze^{1-z} \) maps \( G \) conformally onto \( D \). It follows from Proposition 3.1 of [15] that \( f_0(z) := (z-1)e^{-z} \) belongs to \( A(\overline{G}, e^{-z}) \). Thus, either \( H_W = \{1\} \) or \( H_W \) is empty. Clearly, \( g_W \equiv 1 \) and we obtain from Theorem 4.2 that if \( f \in A(\overline{G}) \) and \( f(1) = 0 \), then \( f \in A(\overline{G}, e^{-z}) \). This implies the result of Theorem 3.2 of [15], in particular. It was also conjectured in [15] that \( H_W = \{1\} \) (in the present notation), but this problem remains open.

Our next goal is to exhibit the role of the set \( S_w \) (see Proposition 2.6) in the case of weighted approximation on Jordan domains. Since \( W \in A(\overline{G}) \) is analytic in \( G \), then \( S_w \subset \partial G \) by Theorem IV.1.10(a) of [18] and (2.1). The following result shows that \( S_w = \partial G \) is necessary for nontrivial weighted approximation on \( G \).

**Theorem 4.4.** Let \( G \) be a Jordan domain and let \( W \in A(\overline{G}) \) be a nonvanishing weight. Assume that \( S_w \) is a proper subset of \( \partial G \) and that \( |W(z)|, zW(z) = C(S_w) \) on \( S_w \). Then \( A(\overline{G}, W) \) contains the identically zero function only.

We construct below a specific example of the weight \( W(z) \), satisfying the conditions of Theorem 4.4.

**Example 4.5.** Let \( G \) be a Jordan domain, as before, and let \( \gamma \) be a proper closed subarc of \( \partial G \). Define the conformal mapping \( \Phi : \mathbb{C}\backslash\gamma \to \{ w : |w| > 1 \} \), normalized by \( \Phi(\infty) = \infty \) and \( \lim_{z \to \infty} \Phi(z)/z > 0 \). We extend \( \Phi \) to \( \gamma \), using boundary limits from inside of \( G \), so that \( \Phi \in A(\overline{G}) \) is one-to-one on \( \overline{G} \). Thus, we can consider approximation by weighted polynomials on \( \overline{G} \), with \( W(z) = 1/\Phi(z), z \in \overline{G} \).

Denote the Green function of the domain \( \mathbb{C}\backslash\gamma \), with pole at \( \infty \), by \( g(z, \infty) \) [23, p. 14], and the classical equilibrium distribution of \( \gamma \) (in the sense of logarithmic potential theory) by \( \mu_\gamma \) [23, p. 55]. Then,
\[
g(z, \infty) = \log \frac{1}{\text{cap}(\gamma)} - U^{\mu_\gamma}(z), \quad z \in \mathbb{C},
\]
where \( \text{cap}(\gamma) \) is the logarithmic capacity of \( \gamma \) (see [18, Sect. I.4]). Furthermore, since \( g(z, \infty) = \log |\Phi(z)|, z \in \mathbb{C} \) (cf. [23, p. 18]), we obtain
\[
U^{\mu_\gamma}(z) - \log |1/\Phi(z)| = \log \frac{1}{\text{cap}(\gamma)}, \quad z \in \mathbb{C}.
\]
It follows from (4.4) and Theorem I.3.3 of [18] that \( \mu_\gamma \) is the solution of a weighted energy problem (cf. [18, Sec. I.1]) for the weight \( w(z) = 1/|\Phi(z)| \) on \( \overline{G} \). Hence, we have shown that \( S_w = \text{supp } \mu_\gamma = \gamma \).
Observe that the weighted polynomials $W^n(z)P_n(z)$, with $W(z) = 1/\Phi(z)$, $z \in \mathbb{C}$, can approximate an arbitrary analytic function in $G$ on compact subsets of $G$ by Theorem 1.1 of [16] and (4.4). On the other hand, Proposition 2.5(b) and Theorem 4.4 indicates that they can only approximate the identically zero function uniformly on $\mathbb{C}$.

5. Proofs

Proof of Proposition 2.1. We have to show that $A(E, W)$ is closed under addition, multiplication by constants and by functions of $A(E, W)$, and under uniform limits. Suppose that $W^n P_n \to f \in A(E, W)$ and $W^n Q_n \to g \in A(E, W)$ uniformly on $E$, as $n \to \infty$. Then $W^n (P_n + Q_n) \to (f + g)$, as $n \to \infty$, so that $(f + g) \in A(E, W)$. If $\alpha \in \mathbb{C}$ then $W^n \alpha P_n \to \alpha f$, as $n \to \infty$, and $\alpha f \in A(E, W)$. Observe that

$$||fg - W^{2n} P_n Q_n||_E \leq ||f - W^n Q_n||_E + ||W^n Q_n - W^{2n} P_n Q_n||_E \leq \limsup_{n \to \infty} ||f - W^n P_n||_E \to 0,$$

as $n \to \infty$, i.e., $fg \in A(E, W)$. Applying the standard diagonalization argument, we see that $A(E, W)$ is closed in norm (1.1).

Proof of Proposition 2.3. Assume that $f \in A(E, W)$ and $W^n P_n \to f$ uniformly on $E$, as $n \to \infty$. Then, for any pair of nonnegative integers $k$ and $\ell$ such that $k \geq \ell$, we have

$$||f(z) W^k(z) z^\ell - W^{n+k}(z) z^\ell P_n(z)||_E \leq \limsup_{n \to \infty} ||f - W^n P_n||_E \to 0,$$

which gives that $f(z) W^k(z) z^\ell \in A(E, W)$. Since $A(E, W)$ is closed under addition and multiplication by constants (by Proposition 2.1), then the product of $f$ and any polynomial in $W(z)$ and $zW(z)$ belongs to $A(E, W)$. Thus, if $g \in [W(z), zW(z)]$ then $fg \in A(E, W)$ follows immediately, because $A(E, W)$ is closed in the uniform norm on $E$ (cf. Proposition 2.1). The proof is now complete in view of Propositions 2.1 and 2.2.

Proof of Proposition 2.4. Obviously, if $[W(z), zW(z)] = A(E)$ then $1/W(z) \in A(E) = [W(z), zW(z)]$. Assume that $1/W(z) \in [W(z), zW(z)]$. It follows that $z \in [W(z), zW(z)]$ and, consequently, every polynomial in $z$ is in $[W(z), zW(z)]$. Since $[W(z), zW(z)]$ is uniformly closed on $E$ by definition, then $A(E) \subset [W(z), zW(z)]$ by Mergelyan’s theorem [5, p. 48]. Thus, Proposition 2.2 implies at once that $A(E) = [W(z), zW(z)]$.

Proof of Proposition 2.5. First, we remark that $W(z)$ and $zW(z)$ together separate different points of any set $E$.

(a) Observe that $W(E)$, the image of $E$ in $\zeta$-plane under the mapping $\zeta = W(z)$, is compact. By assumption, function $1/\zeta$ is analytic on the polynomial convex hull of $W(E)$ and can be uniformly approximated there by polynomials in $\zeta$ (by Mergelyan’s theorem). Returning to $z$-plane, we obtain that $1/W(z)$ is uniformly approximable on $E$ by polynomials in $W(z)$. It follows that $[W(z), zW(z)] = A(E)$ by Proposition 2.4.

(b) The mapping $\zeta = W(z)$ can be continued as a homeomorphism between $z$-plane and $\zeta$-plane (cf. [11, p. 535]). Since $W(z)$ doesn’t vanish on $E$, then $\zeta = 0$ belongs to the domain $\mathbb{C} \setminus W(E) = W(\mathbb{C} \setminus E)$, which contains $\zeta = \infty$. Hence, (b) follows from (a).
(c) If \( E = [0, 1] \) then (c) is a direct consequence of Theorem 2 of [2]. For \( E \) being a Jordan arc, we consider a homeomorphic parametrization of \( E \) by \( \tau: [0, 1] \to E \). Since \( W \circ \tau(x) \) is of bounded variation on \([0, 1]\), we have, as before, that \( [W \circ \tau(x), \tau(x)(W \circ \tau(x))] = C[0, 1] \). Clearly, \( \tau \) induces an isometric isomorphism between \( C([0, 1]) \) and \( C(E) \). Thus, the result follows after returning to \( E \) with the help of \( \tau^{-1} \).

(d) is implied by Theorem 1 of [1] for \( E = [0, 1] \). The case of a Jordan arc can be reduced to that of the interval as in the proof of (c).

(e) First, assume that \( E = \overline{D} \). Then (e) follows at once from [24, p. 135] (see also [3]). It is well known that the conformal mapping \( \phi \) is described by its zero set (see [21, p. 32]).

Proof of Proposition 2.6. Since \( W(z) \) is a continuous nonvanishing function on \( E \) and \( w(z) \) of (2.1) is so too, then the existence of \( \mu_w \) and inequalities (2.4)-(2.5) follow from Theorem I.1.3 of [18]. Moreover, \( W(z) \) is analytic in the interior of \( E \), which implies that \( S_w \subset \partial E \) by Theorem IV.1.10(a) of [18] and (2.1). The inequality (2.3) is a direct consequence of Theorem III.2.1 of [18].

Proof of Theorem 3.1. We have that \( \{W(z), zW(z)\} = C(E) \) by the assumption of the theorem. Thus, \( A(E, W) \) is a closed ideal of \( C(E) \) (cf. Proposition 2.3), which is known to be described by its zero set (see [21, p. 32]).

Proof of Theorem 3.2. We essentially follow the proof of Theorem 2 of [10]. Suppose that there exist \( f_0 \in A(E, W) \) and \( z_0 \in E \setminus S_w \) such that \( f_0(z_0) \neq 0 \) and \( W^n P_n \to f_0 \) uniformly on \( E \), as \( n \to \infty \).

It is clear that \( f_0|_{S_w} \in A(S_w, W) \). Recall that \( S_w \subset \partial E \) by Proposition 2.6, i.e., \( S_w \) has empty interior. Applying Theorem 3.1, with \( E \) replaced by \( S_w \), we obtain that \( A(S_w, W) \) is described by the zero set \( Z_W^* \subset S_w \). Observe that multiplying \( A(S_w, W) \) by \( (z - z_0)W(z) \), we obtain a closed ideal of \( [W(z), zW(z)] = C(S_w) \) (cf. Proposition 2.3), which consists of all functions, uniformly approximable on \( S_w \) by the weighted polynomials \( W^n(z)Q_n(z) \) such that \( Q_n(z_0) = 0 \), as \( n \to \infty \). On the other hand, the zero set of the ideal \( (z - z_0)W(z)A(S_w, W) \) coincides with that of \( A(S_w, W) \). It follows that \( (z - z_0)W(z)A(S_w, W) = A(S_w, W) \) (see [21, p. 32]) and that \( f_0|_{S_w} \in (z - z_0)W(z)A(S_w, W) \).

Thus, there exists a sequence of the weighted polynomials \( \{W^n Q_n\}_{n=0}^\infty \), with \( Q_n(z_0) = 0 \), uniformly convergent to \( f_0 \) on \( S_w \), as \( n \to \infty \). Since \( W^n(z)(P_n(z) - Q_n(z)) \) converges to zero uniformly on \( S_w \) and converges to \( f_0(z_0) \neq 0 \) for \( z = z_0 \in E \setminus S_w \), as \( n \to \infty \), then we obtain a direct contradiction with (2.7) for some sufficiently large \( n \).

Consequently, if \( f \in A(E, W) \), then \( f(z) = 0 \) for any \( z \in E \setminus S_w \). Furthermore, the same is true for any \( z \in E \setminus S_w \) by the continuity of \( f(z) \).

Proof of Theorem 4.1. Since \( [W(z), zW(z)] = A(\overline{D}) \) by the assumption of the theorem, then \( A(\overline{D}, W) \) is a closed ideal of \( A(\overline{D}) \) by Proposition 2.3. The result now follows from the description of nontrivial closed ideals of the disk algebra (see [17] and [7, pp. 82-87]).

Proof of Theorem 4.2. First, we obtain that \( A(\overline{G}, W) \) is a closed ideal of \( A(\overline{G}) \) (cf. Proposition 2.3). It is well known that the conformal mapping \( \phi \) extends to a
homeomorphism between $\overline{G}$ and $D$, defining an isometric isomorphism between the algebras $A(\overline{G})$ and $A(D)$, and their closed ideals. Thus, we apply the result of [17] to the isomorphic image of $A(\overline{G}, W)$ in $A(D)$, as in the proof of Theorem 4.1, and return to $A(\overline{G}, W)$ with the help of the inverse conformal mapping $\psi$. 

Proof of Theorem 4.4. Since $G$ is a Jordan domain, then the set $\{z : |z - z_0| < \delta, \ z \in \overline{G}\}$ has positive logarithmic capacity for any $z_0 \in \overline{G}$ and $\delta > 0$. It is clear that $S_w$ is contained in some Jordan arc, as a proper closed subset of $\partial G$, so that $\overline{G}\setminus S_w$ is connected. Observe that all conditions of Theorem 3.2 are satisfied in this case, which yields that any function $f \in A(\overline{G}, W)$ must vanish on $(\overline{G}\setminus S_w) = \overline{G}$. 

6. Further remarks

One of the main assumptions of the theorems in Sections 3 and 4 is that $[W(z), zW(z)] = A(E)$. This equality is valid for a wide classes of sets $E$ and weights $W \in A(E)$, as described in Proposition 2.5. On the other hand, it is clear that our results may be extended further. The main ingredients of such extentions are:

(i) a proof of the equality $[W(z), zW(z)] = A(E)$;
(ii) a description of the closed ideals of $A(E)$.

Nevertheless, we believe that a problem of much greater interest and difficulty is to uncover a more explicit relation between the zero set $Z_W$ (or $H_W$ and the inner function $g_W$) and the weight $W(z)$. A considerable progress has been achieved on this problem for real valued weights on the real line (see [22], [18, Ch. VI] and [8]-[9]), but a general description of the set $Z_W$ through $W(z)$ is unknown even in the latter case.

References

D. S. Lubinsky and V. Totik, Weighted polynomial approximation with Freud weights, Constr. Approx. 10 (1994), 301–315. MR 95i:41007


Institute for Computational Mathematics, Department of Mathematics and Computer Science, Kent State University, Kent, Ohio 44242-0001

E-mail address: pritsker@mcs.kent.edu

Current address: Department of Mathematics, Case Western Reserve University, 10900 Euclid Avenue, Cleveland, Ohio 44106-7058

E-mail address: iep@po.cwru.edu