

THE FARY-MILNOR THEOREM IN HADAMARD MANIFOLDS

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ABSTRACT. The Fary-Milnor theorem is generalized: Let γ be a simple closed curve in a complete simply connected Riemannian 3-manifold of nonpositive sectional curvature. If γ has total curvature less than or equal to 4π , then γ is the boundary of an embedded disk. The example of a trefoil knot which moves back and forth arbitrarily close to a geodesic segment shows that the bound 4π is sharp in any such space. The original theorem was for closed curves in Euclidean 3-space and the proof by integral geometry did not apply to spaces of variable curvature. Now, instead, a combinatorial proof has been devised.

1. INTRODUCTION

The Fary-Milnor Theorem states that the total curvature of a knot in E^3 is greater than 4π [F], [M]. Fary proved Borsuk's conjecture that the total curvature was greater than or equal to 4π ; independently, Milnor showed that it was strictly greater. The original proofs were by beautiful integral-geometric arguments. We know of only one non-integral-geometric proof to have been proposed in the intervening decades, namely that by Brickell and Hsiung [BHs]. One advantage of their proof is that it also works in hyperbolic space H^3 , although not in spaces of variable curvature. The question of whether the Fary-Milnor Theorem can be extended to general nonpositively curved manifolds recently arose in several quarters. If Hadamard manifolds are in fact the right setting for this theorem, it seems that another essentially different proof is needed. Integral-geometric methods do not extend to this setting, and there seems to be no transformation of a Hadamard manifold to Euclidean or hyperbolic space that is nonincreasing on the total curvature of a given arbitrary curve. In this paper, we give a new, combinatorial proof of the theorem that exhibits it as a fundamental property of Hadamard 3-manifolds.

Main Theorem. *A simple closed curve of total curvature at most 4π in a Hadamard 3-manifold X is isotopic to a geodesic triangle and hence bounds an imbedded disk. Thus the total curvature of a knot in X is greater than 4π .*

The example of a trefoil knot which moves back and forth arbitrarily close to a geodesic segment shows that the bound 4π is sharp in any such space.

Independently, Carsten Schmitz has given a quite different proof that the total curvature of a knot in X is greater than or equal to 4π [S]. Schmitz' proof makes use of an idea of Pannwitz [P] and has interesting historical connections.

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The rest of the paper is organized as follows. Section 2 gives the basic properties of total curvature in Hadamard manifolds (complete, connected, simply connected, nonpositively curved Riemannian manifolds). In fact these properties hold in CAT(0) spaces, and are presented in that setting to clarify their essential features. Section 3 contains the proof of the Main Theorem for polygonal curves. Section 4 completes the proof by showing that a simple curve of finite total curvature in a Hadamard manifold is isotopic to an inscribed polygon.

2. TOTAL CURVATURE

The material in this section owes a strong debt to Chapter 5 of the book [AR] by Alexandrov and Reshetnyak, on irregular curves in Euclidean space. Our setting is a CAT(0) space X , of which a Hadamard manifold is an example. (Readers interested only in the latter can skip the next paragraph; those wishing more information on CAT(K) spaces can consult [BN], [Ba], [BHa].)

By definition, a CAT(0) space X is a metric space such that any two points are joined by a unique distance-realizing geodesic, and for any distance-realizing geodesic triangle $\triangle ABC$, the distance between points on $\triangle ABC$ is no greater than the distance between corresponding points on the Euclidean triangle $\triangle A'B'C'$ with the same sidelengths. It follows that the angles of $\triangle ABC$ are no greater than the corresponding Euclidean angles; hence the excess of $\triangle ABC$, namely the difference between the sum of its angles and π , is nonpositive. Here two rectifiable arcs starting at A have an *angle* $\alpha \in [0, \pi]$ if for all points B, C , one on each arc and neither equal to A , the corresponding Euclidean angle $\angle B'A'C'$ converges to α as B, C converge to A . An arc from A has a *direction* if the angle with itself exists, and hence is 0; two such arcs have the *same direction* if their angle is 0. The directions at A form a metric space under angles. (The fundamental theory of these spaces is due to Alexandrov [A1], [A2].)

A *curve* in X is a continuous map of a closed interval or oriented circle. A *chain* V on a curve γ is a set of points corresponding to finitely many parameter values in order. If γ consists of geodesics joining adjacent pairs in V , then γ and V form a *polygon with vertex chain* V . A polygon σ is *inscribed* in a curve γ if its chain of vertices has an oriented reparametrization as a chain on γ .

The *total curvature* $\kappa(\sigma)$ of a polygon σ is

$$\sum (\pi - \alpha_j),$$

where the α_j are the angles of σ (i.e., between left and right directions) at the interior vertices.

Lemma 2.1. *If a polygon σ is inscribed in a polygon γ , then $\kappa(\sigma) \leq \kappa(\gamma)$.*

Proof. It suffices to show that adjoining one vertex to a polygon does not decrease its total curvature. Let $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ be the vertices of σ and adjoin A_i to obtain γ . Denote by α_j (respectively $\tilde{\alpha}_j$) the angle at A_j of γ (respectively σ), writing π if A_j is not interior. Let β_{i-1} be the angle at A_{i-1} between the outgoing edges of σ and γ , and β_{i+1} be the angle at A_{i+1} between their incoming edges.

Then

$$\begin{aligned} \kappa(\gamma) - \kappa(\sigma) &= (\tilde{\alpha}_{i-1} - \alpha_{i-1}) + (\pi - \alpha_i) + (\tilde{\alpha}_{i+1} - \alpha_{i+1}) \\ (1) \qquad \qquad &\geq -\beta_{i-1} + (\pi - \alpha_i) - \beta_{i+1} \\ (2) \qquad \qquad &\geq 0. \end{aligned}$$

The inequality (2) states that the excess of $\triangle A_{i-1}A_iA_{i+1}$ is nonpositive, and (1) follows from the triangle inequality for angles. \square

For any curve γ , the total curvature $\kappa(\gamma)$ may now be defined as the supremum of $\kappa(\sigma)$ over all polygons σ inscribed in γ . For Hadamard manifolds and smooth curves, $\kappa(\gamma)$ coincides with the Riemannian total curvature.

Lemma 2.2. *A curve γ of finite total curvature has left and right directions at every point.*

Proof. Let B and C approach A along γ with B between A and C . Then the total curvature of the arc between A and C approaches 0. Therefore $\angle ABC$ approaches π , as does its Euclidean comparison angle $\angle A'B'C'$. Thus $\angle B'A'C'$ approaches 0, so the corresponding direction exists. \square

Although we do not need it here, we remark that rectifiability of γ also follows from finite total curvature by straightforward comparison to the Euclidean case; for that case, see [AR, p. 44].

Lemma 2.3. *If a polygon σ is inscribed in a curve γ and $\kappa(\sigma) = \kappa(\gamma) < \infty$, then each edge of σ determines, together with the arc of γ that it replaces, a closed curve bounding a convex subset of X that is isometric to a planar convex set. Furthermore, at each interior vertex A of σ , the angle of γ at A is the sum of the angle of σ at A , the angle between the incoming edges of γ and σ , and the angle between their outgoing edges.*

Proof. First suppose γ is a polygon. It suffices to consider the case in which σ has vertices $A_1, \dots, A_{i-1}, A_{i+k}, \dots, A_n$ for $i > 2$, and γ is obtained by adjoining k vertices A_i, \dots, A_{i+k-1} between A_{i-1} and A_{i+k} .

Suppose $k = 1$, as in the proof of Lemma 2.1. Since equality holds in (2), the sum of the angles of $\triangle A_{i-1}A_iA_{i+1}$ is π . By a theorem of Alexandrov [A1], [A2], $\triangle A_{i-1}A_iA_{i+1}$ bounds a convex subset that is isometric to a planar triangular region. By (1), $\alpha_{i-1} = \tilde{\alpha}_{i-1} + \beta_{i-1}$ and $\alpha_{i+1} = \tilde{\alpha}_{i+1} + \beta_{i+1}$, whence the statement holds for $k = 1$.

Now consider arbitrary k , and let $\tilde{\sigma}$ be the polygon that is inscribed in γ by omitting the vertex A_i . We may assume by induction that the theorem applies to σ inscribed in $\tilde{\sigma}$ and to $\tilde{\sigma}$ inscribed in γ . Therefore the closed polygon with vertex chain $A_{i-1}, A_i, \dots, A_{i+k}$ bounds a body S obtained by gluing along $A_{i-1}A_{i+1}$ the planar convex sets bounded by $\triangle A_{i-1}A_{i+1}A_i$ and the closed polygon with vertex chain $A_{i-1}, A_{i+1}, \dots, A_{i+k}$. Since the angle sums at A_{i-1} and A_{i+1} are at most π , then S is isometric in its intrinsic metric to a planar convex set. Since $\angle A_{i-2}A_{i-1}A_i = \angle A_{i-2}A_{i-1}A_{i+k} + \angle A_{i+k}A_{i-1}A_{i+1} + \angle A_{i+1}A_{i-1}A_i$, it follows from the triangle inequality for angles that $\angle A_{i+k}A_{i-1}A_i = \angle A_{i+k}A_{i-1}A_{i+1} + \angle A_{i+1}A_{i-1}A_i$, and so $\angle A_{i+k}A_{i-1}A_i$ is the same in the intrinsic metric of S as in X . Similarly, any two points B, C in S subtend at A_{i-1} the same intrinsic angle in S as in X . In the intrinsic metric of S , the triangle with these vertices is Euclidean. Therefore by

angle comparisons, its edge BC in S can be no longer than the X -distance between B and C , and hence is a geodesic in X . Thus S is isometric in the metric of X to a planar convex set, and the theorem holds for polygons. The general theorem follows by an easy limit argument. \square

As an immediate corollary, we have Fenchel's theorem for $\text{CAT}(0)$ spaces. For Hadamard manifolds, this was proved by Szenthe [Sz]; see also [T] and [BHs].

Corollary 2.4. *A closed curve γ in a $\text{CAT}(0)$ space X has total curvature at least 2π . If equality holds, then γ either is a geodesic bigon or bounds a convex subset of X that is isometric to a planar convex set.*

Proof. Since any closed curve has an inscribed bigon and the total curvature of a bigon is 2π , this follows from the definition of total curvature and Lemma 2.3. \square

3. REDUCTION BY TRIANGULAR ISOTOPIES

In the remaining two sections, we work only in a Hadamard manifold X . The Main Theorem as stated here fails for general $\text{CAT}(0)$ spaces; see the end of the paper for related open questions. We note that the constructions below are independent of dimension. However, in a Hadamard manifold of dimension greater than 3, any simple closed curve of finite total curvature bounds an imbedded disk, as follows easily by inverse exponentiation from the same statement for Euclidean space, and Theorem 4.1 below.

Theorem 3.1 (Fary-Milnor for polygons). *If γ is a simple closed polygon in X with total curvature at most 4π , then γ is isotopic to a geodesic triangle, and hence bounds an imbedded disk.*

Our primary tool is *triangular isotopy*:

Lemma 3.2. *Let A, B, C be consecutive vertices of a simple polygon such that all of the segments $AQ, Q \in BC$, do not intersect the rest of the polygon. Then there is an isotopy of X which moves the subarc ABC to AC and fixes the rest of the polygon.*

The proof of this lemma is given in Section 4. Now we show how to use triangular isotopies to prove the Fary-Milnor theorem for polygons.

Outline of proof of Theorem 3.1 (Main Construction). Suppose the total curvature of γ is less than 4π . The key fact motivating the following construction is that no chain of inscriptions starting from γ can lead to a doubly covered bigon, since the latter has total curvature 4π .

We proceed by induction on the number $n \geq 3$ of nontrivial vertices, where the case $n = 3$ is clear. Let A_1, A_2, \dots, A_n be the vertex chain of γ . The first vertex is never changed, but is also labeled B_1 . Triangular isotopies will be used to replace an edge $B_k A_{k+1}$ and a segment $A_{k+1} B_{k+1}$ by the segment $B_k B_{k+1}$ (see Figure 1). Since the new polygon is inscribed in the old one, the total curvature is not increased by the replacement. Note that when a vertex A_{k+1} is replaced due to the effect of a triangular isotopy the new vertex will be called B_{k+1} . However, we shall continue to call the polygons that we get by reduction by the same name, γ (note that they are all inscribed in the original one).

If some triangular isotopy could cut off an entire triangle $\triangle B_k A_{k+1} A_{k+2}$, then the number of vertices would be reduced by 1, and the inductive step would be

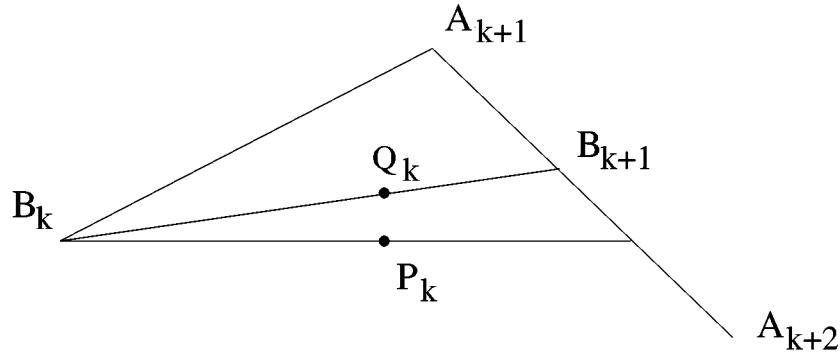


FIGURE 1

completed. Otherwise, as a point Q moves from A_{k+1} to A_{k+2} and $B_k Q$ sweeps across that triangle, $B_k Q$ must intersect the part of γ running in the cyclic order from A_{k+2} to B_k , excluding B_k . We denote the point of intersection by P_k ; if there is more than one such point of intersection, we choose the first one as we move cyclically forward on γ from A_{k+1} ; or if the whole segment $A_{k+2} A_{k+3}$ consists of such intersection points, we take P_k to be the midpoint of that segment. We say that P_k blocks the cutting off of an entire triangle from γ .

Given a blocking point P_k , we stop the triangular isotopy so that the sweep segment $B_k Q$ falls just short of P_k , at a segment $B_k B_{k+1}$ on which the nearest point to P_k is denoted Q_k . The pair P_k, Q_k forms a *virtual double point* of γ . By specifying what we mean by “falls just short of”, we shall see that for the pair P_{k+1}, Q_{k+1} forming the next virtual double point, P_{k+1} must lie in a subarc of γ that runs in the cyclic order from slightly before Q_k through B_{k+1} to Q_{k+1} and P_k , and slightly beyond P_k . Otherwise, the closed 4-gon with vertex chain $Q_k, Q_{k+1}, P_k, P_{k+1}$ would be inscribed and could be taken too close to a doubly covered bigon.

Thus our attention will be confined to decreasing nested subarcs of the original polygon, containing a decreasing number of its original vertices. Therefore, unless a sweep fails to be blocked and the induction step is completed, the blocking point P_k must eventually occur in $A_{k+2} A_{k+3}$. When that happens we have a terminal procedure which always cuts off an entire triangle to complete the inductive step. In this way, Theorem 3.1 is shown to hold when γ has total curvature less than 4π . If the total curvature is 4π , a rigidity argument (Lemma 3.3) will complete the proof. This ends our outline of the proof. \square

The rest of the section is devoted to filling in the argument. Suppose the total curvature of the original polygon is $4\pi - 2c$ where $c > 0$. Our description of “falls just short of” will be specifications of the radius ϵ_k for an open ball $B(P_k, \epsilon_k)$, and of how much smaller than that $d(P_k, Q_k)$ should be. The requirement on ϵ_k is that $B(P_k, \epsilon_k)$ only intersect γ in the edge or pair of edges (if P_k is a vertex) on which P_k lies before we start the current isotopy. Moreover, if $P_i, i > k$, falls in $B(P_k, \epsilon_k)$, then $B(P_i, \epsilon_i)$ is taken to lie in $B(P_k, \epsilon_k)$. We choose $d(P_k, Q_k) < \epsilon_k$ so small that $\angle P_k P Q_k < c$ holds for every P outside $B(P_k, \epsilon_k)$.

We claim that the next blocking point P_{k+1} cannot occur in the connected component of $\gamma \setminus B(P_k, \epsilon_k)$ not containing B_{k+1} . Equivalently, P_{k+1} must lie in the

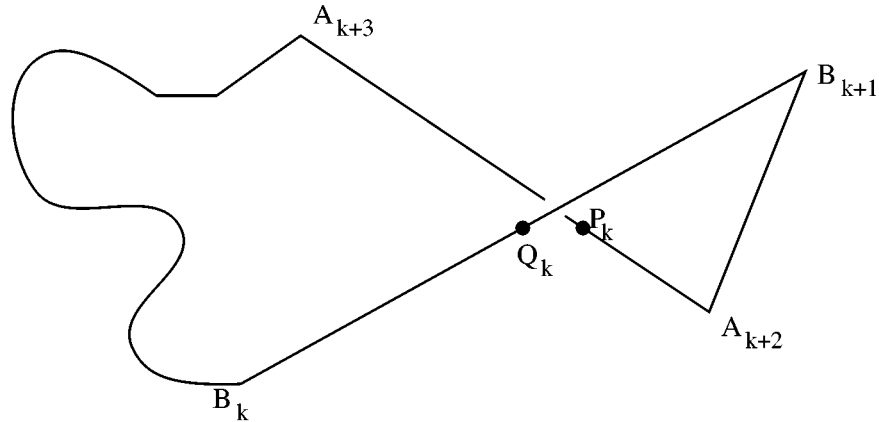


FIGURE 2

subarc of γ that begins as it enters $B(P_k, \epsilon_k)$, runs in the cyclic order through $Q_k, B_{k+1}, Q_{k+1}, P_k$, and ends as it leaves $B(P_k, \epsilon_k)$. Suppose P_{k+1} were in the complementary subarc. Then the closed polygon with vertex chain $Q_k, Q_{k+1}, P_k, P_{k+1}$ would be inscribed in γ ; and by careful readjustment of specifications we could argue that this polygon had total curvature greater than $4\pi - 2c$. However, we avoid these technicalities by examining the result of carrying the sweep procedure based at B_{k+1} through to an actual double point, allowing for the moment Q_{k+1} to coincide with P_{k+1} . The resulting nonsimple polygon γ would still be inscribed in the original one, and would have an inscribed closed polygon σ with vertex chain $Q_k, P_{k+1}, P_k, P_{k+1}$. Since the angles of σ would all be less than c , and would vanish at P_k and Q_k , the total curvature of σ would be greater than $4\pi - 2c$, a contradiction.

The proof follows directly from the preceding claim. Indeed, successive sweeps are confined to decreasing nested subarcs of the original polygon, since the blocking point P_k for the sweep from B_k lies in the subarc of that polygon running in order from A_{k+1} to the point where that polygon leaves $B(P_1, \epsilon_1)$. Now suppose we are at the terminal step, when $P_k \in A_{k+2}A_{k+3}$. Then we finish with 3 more triangular sweeps (see Figure 2):

- move A_{k+2} to $P_k = B_{k+2}$ with B_{k+1} as pivot;
- move B_{k+1} to Q_k with P_k as pivot;
- move P_k to A_{k+3} with Q_k as pivot.

The first sweep is not blocked, since any blocking point clearly would fall in the component of $\gamma \setminus B(P_k, \epsilon_k)$ not containing B_{k+1} , and we have just seen that this is impossible. The second sweep is not blocked for the same reason. Now in addition take Q_k to lie in a convex neighborhood N of $P_k A_{k+3}$ that excludes the arc of γ running from near A_{k+3} on $A_{k+3}A_{k+4}$ through B_k and up to $B(P_k, \epsilon_k)$. (Here indices are taken mod n .) The third sweep takes place entirely in N , and need only be prevented from intersecting $A_{k+3}A_{k+4}$ by taking N sufficiently small. This completes the proof in the case of total curvature less than 4π . The following lemma completes the proof of Theorem 3.1.

Lemma 3.3. *In a Hadamard manifold, a simple closed polygon with total curvature 4π is isotopic to a geodesic triangle.*

Proof. Consider any simple closed polygon γ that is not isotopic to a triangle. Carry out the successive sweep procedure of the preceding proof (here we do not rule out the possibility of cutting off triangles). Since γ is not isotopic to a triangle, there must be a closed 4-gon that lies as close as we please to a doubly covered bigon, and is inscribed in a closed polygon that is inscribed in γ and has no more vertices than γ . Taking limits, we conclude that for any knot γ , some doubly covered bigon σ is inscribed in a closed polygon $\tilde{\sigma}$ inscribed in γ .

Now suppose γ has total curvature 4π . Then so does $\tilde{\sigma}$, and Lemma 2.3 applies to σ inscribed in $\tilde{\sigma}$. Thus each edge of the doubly covered bigon σ determines, together with the arc of $\tilde{\sigma}$ that it replaces, a closed curve bounding a convex subset S_i ($i = 1, \dots, 4$) of X that is isometric to a planar convex set. Consider an isometric copy σ' of σ taking values in a line L in the Euclidean plane Π , and take the union of copies S'_i ($i = 1, \dots, 4$) in Π of the S_i , based on σ' and lying on alternating sides of L . Then the union S' of the S'_i is star-shaped about any point in σ' . We claim that S' and the union S of the S_i in X are isometric in their intrinsic metrics. Indeed, at each vertex of σ , the corresponding angle of $\tilde{\sigma}$ is the sum of the angles between the image segment of σ and the left and right tangents of $\tilde{\sigma}$, by Lemma 2.3. It follows, as in the proof of that lemma, that in $S_i \cup S_{i+1}$ (indices mod 4), any point O in the relative interior of the image segment of σ has a neighborhood that is a totally geodesic surface in X . It is easy to conclude that S' and $\exp_O^{-1} S$ are isometric planar sets. Thus \exp_O determines a one-one local isometry of S' onto S , where the relative interior of S is a locally totally geodesic surface in X . The preimage of $\tilde{\sigma}$ is a one-sidedly locally convex curve in the plane of total curvature 4π .

Now we may similarly apply Lemma 2.3 to $\tilde{\sigma}$ inscribed in γ . Thus we construct the union of four convex sets in Π , each of which is based on a subsegment of L containing σ' , and which lie on alternating sides of L . Isometric to this set is a planar set in $T_O X$, on which the exponential map is a one-one local isometry, and whose image has locally totally geodesic relative interior and contains γ . The preimage of γ is a one-sidedly locally convex planar curve of total curvature 4π , and is simple because γ is simple. This is a contradiction, since any simple, one-sidedly locally convex planar curve is convex and hence has total curvature 2π . \square

4. ISOTOPIC POLYHEDRAL APPROXIMATIONS

In this section we complete the proof of the Main Theorem by showing that isotopic polyhedral approximations exist; this was also the preliminary step in Milnor's proof.

Theorem 4.1. *A simple curve γ of finite total curvature in a Hadamard manifold is isotopic to an inscribed geodesic polygon.*

Proof. This proof is patterned after Milnor's proof, except that we find it more convenient to use solid cones and "spindles" where he uses Euclidean cylinders. A *spindle* is just a pair of cones with a common base, i.e., a suspension of a disk, which we call its *central disk*. The base of a cone or the central disk of a spindle is always taken to be a *bisecting disk*, namely, the image under the exponential map at a point of γ of a disk in the tangent space bisecting the left and right directions. Except at a cusp a bisecting disk locally separates the curve into two branches issuing from its center. (A *cusp* is a point at which the two one-sided directions coincide, and so accounts for a total curvature π .) The *axis* of a cone is the geodesic from the vertex to the center of the base; of a spindle, the union of two such geodesics.

The desired isotopy will be a chain of finitely many local isotopies, each fixing the boundary of the cone or spindle on which it is defined. First we may dispose of the cusps by using cone isotopies to move a curve in a neighborhood of a cusp into a pair of segments joined at the former cusp. Then we cover the resulting curve by open spindles. Taking a partition of the curve for which adjacent points are closer than the Lebesgue number of that covering, and which includes the spindle centers, we may use spindle isotopies to move each subarc of the partition to the spindle axis while leaving the rest of the curve fixed. It only remains to describe these cone and spindle isotopies.

The *generators* of a cone (including the two cones in a spindle) are the geodesic segments from the vertex to the points of the base, with constant speed parameter $t \in [0, 1]$. The sets on which t is constant form a fibration of the cone by disks. Extend a polar coordinate parametrization of the base to the fibering disks, taking it to be constant along generators. For two points in the interior of a disk, there is a disk isotopy that moves one to the other, fixes the boundary and depends continuously on the two points. This continuous dependence passes to the cone or spindle, allowing an isotopy that moves one curve transverse to the fibers into another.

To construct a suitable cone at a cusp P , consider bisecting disks on one branch of γ near P . Fix the radius of the disks to be $a/2$, where $B(P, a)$ is a ball which intersects γ only in the two branches from P . By finiteness of total curvature, the angles between a geodesic secant PA and the tangents of the corresponding arc of γ approach 0 as A approaches P along γ . Hence the angle between PA and the bisecting disk at A is bounded away from 0. Therefore the solid angle at P subtended by the disk will eventually include the tangent at P and all the chords to both branches. We fix one such disk as the base of a cone with vertex P . Since the angles between the outward tangents of the branches and the outward ends of the chords may be assumed acute, t is monotonic along each branch.

The isotopy for a cone at a cusp is carried out in three steps. First the branch running from the vertex to the center of the base is deformed to the axis. In the second step, the second branch is moved radially so that its polar radius r is constantly r_0 , the polar radius of its intersection with the base. The radial isotopies must be chosen to fix the center. The third step is specified in terms of the polar angle θ . Along the branch this polar angle has a continuously determined value starting from the base point, where $t = 1$, up to the vertex, where $t = 0$; we denote this value for a given t by $\varphi(t)$. The complication special to cusps is that φ could be unbounded as the branch approaches the vertex by spiraling wildly around the other branch. Letting the isotopy parameter be $s \in [0, 1]$ we define the isotopy F as follows:

- (1) The parameters t and r do not change as s varies.
- (2) For $r \leq r_0$ the angle parameter is given by $\theta \circ F(P, s) = \theta(P) - s(\varphi(t) - \varphi(1))$.
- (3) Let r_1 be the outer radius of the base. Then for $r_0 \leq r \leq r_1$ we let

$$\theta \circ F(P, s) = \theta(P) - \frac{s(r_1 - r)(\varphi(t) - \varphi(1))}{r_1 - r_0}.$$

To construct a suitable spindle at a noncusp point P , we move along γ from each side toward the bisecting disk at P . Again the size of the disk is chosen to prevent the spindle from intersecting γ except in the branches from the center. By finiteness of total curvature, for a point A on γ close enough to P , the angle formed

with the disk by the geodesic AP is bounded away from 0. Therefore the solid angle subtended by the disk will eventually include the tangent at A and all the chords to the intervening arc. We fix two such points as the vertices of a spindle. The angles the chords make with each arc are again acute, so on each arc t is monotonic. Note that any subspindle obtained by suspending the same disk over two points, one in each branch of γ within the spindle, is also a suitable spindle.

Now take a partition as described above. By deleting partition points, we may arrange that no corresponding subarc lies in the interior of half a spindle. By taking subspindles, every subarc either joins the vertices of a spindle or joins the spindle center and a vertex. Now either a spindle or a cone isotopy will carry the subarc to the axis. \square

The Triangular Isotopy Lemma (Lemma 3.2) is a variant of the preceding theorem:

Proof of Lemma 3.2. For each $Q \in BC - C$ there is a bisecting disk through Q for the curve AQC which supports a spindle disjoint from the rest of the polygon, with vertices A and some $P \in QC$; P is chosen after choosing the disk, near enough to Q so that AP intersects the disk. There is also some such spindle with vertices A and C and containing AC with bisecting disk through $Q_0 \in BC$. These spindles provide an open covering of BQ_0 so that a finite number of them, together with the last one based at Q_0 , support a chain of isotopies moving ABC to AC leaving the rest of the polygon fixed. In each isotopy, the axis of the spindle is moved to the segment connecting the vertices. \square

5. REMARKS

It is easy to construct a $CAT(0)$ space X in which a curve of total curvature 4π need not bound an imbedded disk. Indeed, let X be the result of gluing four Euclidean halfplanes along their boundaries, and L be the image of the boundaries. Let γ be an imbedded closed curve in X that crosses L four times, has angle π at each crossing, and forms with each of its four chords in L a convex set in one of the four sheets of X .

We do not have a counterexample to the statement that a curve of total curvature less than 4π in a $CAT(0)$ space bounds an imbedded disk. Even for $CAT(0)$ spaces that are manifolds, however, serious modifications of our approach would be needed since isotopies cannot be generated in the way we described. Note that the union of geodesics from a vertex of a triangle to its opposite side need not be a disk: the possibility of bifurcating geodesics necessitates much greater care.

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