THE FUNCTION \((b^x - a^x)/x\): INEQUALITIES AND PROPERTIES

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Abstract. In the article, some properties and inequalities of the function 
\(\int_a^b t^{x-1} dt\) are given by analytic method and the mathematical induction.

1. Introduction

Let
\[
g(x) = \left(b^x - a^x\right)/x, \quad b > a > 0, \quad x \neq 0; \quad g(0) = \ln b - \ln a.
\]

Define a function \(U_n(x; t)\) such that
\[
U_0(x; t) = t^x, \quad x U_n(x; t)/\partial x - (n + 1)U_n(x; t) = U_{n+1}(x; t)
\]
for \(n \in \mathbb{N}, t \in [a, b]\).

It is easy to see that
\[
g^{(n)}(x) = \int_a^b (\ln t)^n t^{x-1} dt, \quad b > a > 0, \quad n \in \mathbb{N}.
\]

In this article, by analytic method and mathematical induction, the functions 
\(g(x)\) and \(U_n(x; t)\) are researched, and some properties and inequalities of them are 
given, that is,

**Proposition 1.** Let \(g = g(x) = \int_a^b t^{x-1} dt\). Then, for \(k, i, j \in \mathbb{N}\)
\[
g^{(2(i+k)+1)}g^{(2(j+k)+1)} < g^{(2k)}g^{(2(i+j+k+1))}.
\]
The ratio \(g^{(2(j+k)+1)}(x)/g^{(2k)}(x)\) is increasing in \(x\).

**Proposition 2.** The function \(g(x)\) satisfies
\[
g^{(n)}(x) = \left(U_n(x; b) - U_n(x; a)\right)/x^{n+1},
\]
\[
\partial U_n(x, t)/\partial t = x^{n+1}(\ln t)^n t^{x-1}.
\]
Proposition 3. The function \( g(x) \) is absolutely and regularly monotonic on \((-\infty, +\infty)\) for \( a > 1 \), or on \((0, +\infty)\) for \( b > a^{-1} > 1 \), completely and regularly monotonic on \((-\infty, +\infty)\) for \( 0 < a < b < 1 \), or on \((-\infty, 0)\) for \( 1 < b < a^{-1} \). Furthermore, \( g(x) \) is absolutely convex on \((-\infty, +\infty)\).

Proposition 4. The function \( g(x + \gamma)/g(x) \) is increasing (or decreasing) in \( x \) for \( \gamma > 0 \) (or \( \gamma < 0 \)). \( [g(x + t)/g(x)]^{1/\lambda} \), \( t \neq 0 \), is increasing with \( t \).

Proposition 5. For \( \gamma \geq 1, x \geq 1, 0 < a < b \), the following inequality holds:

\[
\frac{b^{x+\gamma} - a^{x+\gamma}}{b^x - a^x} \geq \frac{x + \gamma}{x} \left(\frac{a + b}{2}\right)^\gamma.
\]

But (7) may not hold for \( 0 < \gamma < 1 \) or \( 0 < x < 1 \). If \( \gamma > 0, x > 0 \), then

\[
\frac{b^{x+\gamma} - a^{x+\gamma}}{b^x - a^x} \geq \frac{x + \gamma}{x} (ab)^{\gamma/2}.
\]

Note that Proposition 5 refines and extends the inequalities in [2], and verifies the conjecture by the first author in [2].

Using the method of this article, we can generalize the extended means to a two-parameter family of nonhomogeneous mean values; see [3].

2. Definitions and Lemmas

The following definitions and lemmas are necessary.

Definition 1. A function \( f(t) \) is said to be absolutely monotonic on \((a, b)\) if it has derivatives of all orders and \( f^{(k)}(t) \geq 0 \), \( t \in (a, b) \), \( k \in \mathbb{N} \).

Definition 2. A function \( f(t) \) is said to be completely monotonic on \((a, b)\) if it has derivatives of all orders and \((-1)^k f^{(k)}(t) \geq 0 \), \( t \in (a, b) \), \( k \in \mathbb{N} \).

Definition 3. A function \( f(t) \) is said to be absolutely convex on \((a, b)\) if it has derivatives of all orders and \( f^{(2k)}(t) \geq 0 \), \( t \in (a, b) \), \( k \in \mathbb{N} \).

Definition 4. A function \( f(t) \) is said to be regularly monotonic if it and its derivatives of all orders have constant sign (\( + \) or \( - \); not all the same) on \((a, b)\).

Lemma 1. Let \( f, h : [a, b] \to \mathbb{R} \) be integrable functions, both increasing or both decreasing. Furthermore, let \( p : [a, b] \to \mathbb{R}^+ \) be an integrable function. Then

\[
\int_a^b p(t)f(t)dt \int_a^b p(t)h(t)dt \leq \int_a^b p(t)dt \int_a^b p(t)f(t)h(t)dt.
\]

If one of the functions of \( f \) or \( h \) is nonincreasing and the other nondecreasing, then the inequality in (9) is reversed.

Lemma 2. Let \( f : [a, b] \to \mathbb{R} \) be a convex function; then

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}.
\]

Inequalities (10) and (9) are called the Hermite-Hadamard and the Tchebycheff integral inequality, respectively.

Lemma 3. For \( x, y > 0, x \neq y \),

\[
\frac{x + y}{2} > \frac{x - y}{\ln x - \ln y} > \sqrt{xy}.
\]
Inequality (11) is called the logarithmic mean inequality. For proofs of these lemmas, see [1, 4].

3. Proofs of the propositions

3.1. Inequality (4) is a special case of Lemma 1 applied to the functions \( p(t) = (\ln t)^{2k} t^{x-1}, f(t) = (\ln t)^{2i+1} \) and \( h(t) = (\ln t)^{2j+1} \). For \( i, j, k \in \mathbb{N}, x \in \mathbb{R} \) and \( t \in [a, b] \).

Inequality (4) and direct calculation produces

\[
\left( \frac{g^{(2(j+k+1))}}{g^{(2k)}} \right)' = \frac{g^{(2(j+k+1))}g^{(m)}g^{(2(j+k+1))}}{(g^{(2k)})^2} > 0.
\]

Therefore, the desired Proposition 1 follows.

3.2. Using (2) and direct computation yield

\[
g^{(n+1)}(x) = \frac{d}{dx} \left( \frac{U_n(x; b) - U_n(x; a)}{x^{n+1}} \right) = \left[ x \tau \frac{U_n(x; b)}{x} - (n + 1) \frac{U_n(x, b)}{x} \right] - \left[ x \tau \frac{U_n(x, a)}{x} - (n + 1) \frac{U_n(x, a)}{x} \right] = \frac{U_{n+1}(x; b) - U_{n+1}(x; a)}{x^{n+2}}.
\]

By mathematical induction on \( n \), (5) is valid.

Differentiating (2) with respect to \( t \) gives

\[
\frac{\partial U_n(x, t)}{\partial t} = x \frac{\partial U_n}{\partial x} \frac{\partial x}{\partial t} - (n + 1) \tau \frac{U_n}{\partial x} - (n + 1) \tau \frac{U_n}{\partial t}.
\]

Therefore, from mathematical induction on \( n \), we obtain (6).

This completes the proof of Proposition 2.

3.3. First, we consider the case of \( x > 0 \). It is clear that \( g^{(2k)}(x) \geq 0 \), \( g(x) \) is an absolutely convex function on \( (0, +\infty) \).

When \( a > 1 \), \( \partial U_n / \partial t > 0 \), thus \( U_n(x, t) \) increases with respect to \( t \), and \( g^{(n)}(x) > 0 \) follows from (5). Therefore \( g(x) \) is absolutely monotonic on \( (0, +\infty) \).

When \( 0 < a < b < 1, \partial U_{2k+1}(x; t) > 0 \); then \( U_{2k+1}(x, t) \) decreases in \( t \), thus \( g^{(2k+1)}(x) < 0 \), that is, \( (-1)^k g^{(k)}(x) > 0 \), \( g(x) \) is completely monotonic on \( (0, +\infty) \).

Direct computation results in

\[
\lim_{x \to 0^+} g^{(2k+1)}(x) = \frac{\ln b}{k + 1} - \frac{\ln a}{k + 1} > 0
\]

for \( b > a^{-1} \geq 1 \). Since \( g^{(2k+1)}(x) \) is increasing, \( g^{(2k+1)}(x) > 0 \) for \( x > 0 \). Therefore, \( g(x) \) is absolutely monotonic on \( (0, +\infty) \) for \( b > a^{-1} \geq 1 \).

Second, we consider the case of \( x < 0 \). For \( n = 2k + 1, k \in \mathbb{N} \), when \( a > 1, \partial U_n / \partial t > 0 \), \( U_n(x, t) \) increases in \( t \), from (5) it follows that \( g^{(2k+1)}(x) > 0 \). When \( 0 < a < b < 1, \partial U_n / \partial t < 0 \), \( U_n(x, t) \) is decreasing with \( t \), \( g^{(2k+1)}(x) < 0 \).

For \( n = 2k, \partial U_n / \partial t \leq 0, U_n(x, t) \) is decreasing in \( t \), \( g^{(2k)}(x) \geq 0 \), thus \( g(x) \) is absolutely convex on \( (-\infty, 0) \).

Therefore, \( g(x) \) is an absolutely monotonic function on \( (-\infty, 0) \) for \( a > 1 \). \( g(x) \) is completely monotonic on \( (-\infty, 0) \) for \( 0 < a < b < 1 \).
Since $g^{(2k+1)}(x), x < 0$, is increasing, inequality (12) is reversed for $1 < b < a^{-1}$, thus $g^{(2k+1)}(x) < 0$ for $1 < b < a^{-1}$ and $x < 0$, $g(x)$ is a completely monotonic function for $1 < b < a^{-1}$ on $(-\infty, 0)$.

From the definitions in section 2, Proposition 3 is valid.

3.4. Let $f(x) = g(x+\gamma)/g(x), \gamma \neq 0$. From Proposition 1, it is clear that $g'(x)/g(x)$ is increasing, thus

(13) \[ g'(x+\gamma)/g(x+\gamma) \geq g'(x)/g(x) \]

holds for $\gamma > 0$, and is reversed for $\gamma < 0$. Straightforward computation leads to

(14) \[ f'(x) = [g'(x+\gamma)g(x) - g(x+\gamma)g'(x)]/g^2(x). \]

Combining (13) and (14) produces that $f'(x) > 0$ for $\gamma > 0$ and $f'(x) < 0$ for $\gamma < 0$, therefore $f(x)$ increases (or decreases) for $\gamma > 0$ (or $< 0$).

Assume

\[ p(t, \theta) = [g(t+\theta)/g(t)]^{1/\theta}, \theta \neq 0; \]

\[ p(t, 0) = \exp(g'(t)/g(t)), t \in \mathbb{R}. \]

It is clear that $p(t, 0)$ increases with $t$. Computing straightforwardly gives

\[ \ln p(t, \theta) = \frac{g'(t+\theta)}{g(t+\theta)} - \frac{1}{\theta} \ln \frac{g(t+\theta)}{g(t)} \]

By the mean value theorem, it is easy to see that

\[ \ln \frac{g(t+\theta)}{g(t)} = \frac{g'(t+\xi)}{g(t+\xi)} \theta < \frac{g'(t+\theta)}{g(t+\theta)} \theta, \]

where $\xi$ is between 0 and $\theta$, $\theta \neq 0$. Therefore $\ln p(t, \theta)\theta > 0$, and $p(t, \theta)$ is increasing in $\theta, \theta \neq 0$. The proof of Proposition 4 is completed.

3.5. Since $f(x)$ is increasing for $\gamma > 0$

(15) \[ \frac{x(b^\gamma - a^x)}{(x+\gamma)(b^\gamma - a^x)} \geq \frac{b^{1+\gamma} - a^{1+\gamma}}{(1+\gamma)(b-a)}, \quad x \geq 1, \gamma > 0, 0 < a < b. \]

Since $t^\gamma (\gamma \geq 1)$ is convex, from (10) we have

(16) \[ \left(\frac{a + b}{2}\right)^\gamma \leq \frac{1}{b-a} \int_a^b t^\gamma dt = \frac{b^{1+\gamma} - a^{1+\gamma}}{(b-a)(1+\gamma)}, \gamma \geq 1. \]

Combining (15) and (16) yields (7).

Since $t^\gamma (0 < \gamma < 1)$ is concave, then (16) is reversed without equality. Hence (7) may not hold for $0 < \gamma < 1$.

Since $f(x)$ increases for $\gamma > 0, x > 0$, then we have

(17) \[ \frac{x(b^\gamma - a^x)}{(x+\gamma)(b^\gamma - a^x)} \geq \frac{b^\gamma - a^\gamma}{\gamma(\ln b - \ln a)}, \quad x \geq 0, \gamma > 0, \]

which, combined with the logarithmic mean inequality (11), yields (8), but (7) may not hold. Therefore, Proposition 5 is verified.

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