PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 126, Number 11, November 1998, Pages 3211–3221
S 0002-9939(98)04443-8

ALGEBRAIC LATTICES AND NONASSOCIATIVE STRUCTURES

ANTONIO FERNÁNDEZ LÓPEZ AND EULALIA GARCÍA RUS

(Communicated by Lance W. Small)

Abstract. Uniform elements in algebraic lattices are studied and their relationship with some nonassociative extensions of Goldie's Second Theorem is shown.

Introduction

An axiomatization of the properties of the lattice of the ideals of a nonassociative algebra (associative pair or Jordan system) leads us to the natural notion of algebraic lattice. Then we study the structure of the uniform elements of a semiprime algebraic lattice and, by specializing to the lattice of the ideals of a semiprime algebraic system (algebra, triple or pair), we obtain a quite general structure theorem characterizing those algebraic systems which are essential subdirect products of prime ones. This theorem is a key tool in some recent (nonassociative, local, and pair) versions of Goldie's Second Theorem.

The reader is referred to [1] for basic notions on lattices. For nonassociative algebras we adopt as general reference the books [13] and [21], and for Jordan systems [14], [15] and [18].

1. Algebraic lattices

Let $(L, \leq)$ be a complete lattice with greatest element denoted by 1 and least element denoted by 0. We will say that $(L, \leq)$ is an *algebraic lattice* if it is endowed with a binary operation $x \star y$ satisfying the following conditions:

(1.1) If $x \leq y$, then $z \star x \leq z \star y$ and $x \star z \leq y \star z$ for all $z \in L$.
(1.2) $x \star y \leq x \wedge y$ for all $x, y \in L$.
(1.3) $x \star (\bigvee_\alpha y_\alpha) \leq \bigvee_\alpha (x \star y_\alpha)$.

It follows from (1.1) and transitivity that

(1.4) if $x \leq y$ and $z \leq v$, then $x \star z \leq y \star v$.

We also note that by (1.1), (1.3) is equivalent to

(1.5) $x \star (\bigvee_\alpha y_\alpha) = \bigvee_\alpha (x \star y_\alpha)$.

Moreover, it follows from (1.2) that

Received by the editors November 20, 1996 and, in revised form, April 8, 1997.
1991 *Mathematics Subject Classification.* Primary 06B05, 16P60, 17C10.
Key words and phrases. Algebraic lattice, uniform element, essential subdirect product.
This work was supported by DGICYT Grant PB093-0990 and by the "Plan Andaluz de Investigación y Desarrollo Tecnológico".

(1.6) $x \star 0 = 0 = 0 \star x$.

An algebraic lattice will be said to be *semiprime* (*prime*) if $x^2 := x \star x = 0$ implies $x = 0$ ($x \star y = 0$ implies $x = 0$ or $y = 0$).

Examples. (1) Let A be a nonassociative algebra over an arbitrary ring of scalars, and X, Y non-empty subsets of A. Denote by $[X]$ the ideal of A generated by X, and by XY the linear span of all products xy, $x \in X$ and $y \in Y$. Now let B and C be ideals of A. Define the product of B by C as the ideal $B \star C := [BC]$. It is easy to verify that the lattice $\mathcal{L}(A)$ of all ideals of A endowed with the $\star$-product is an algebraic lattice. Moreover, since $B \star C = 0$ if and only if $BC = 0$, we have that A is semiprime (prime) if and only if the algebraic lattice $(\mathcal{L}(A), \star)$ is semiprime (prime). Note that if A is actually an associative (even alternative) algebra then $B \star C$ is merely BC.

(2) Let (J, U) be a quadratic Jordan algebra. By [14, Prop. 4.1.10], if B and C are ideals of J, then $U_B C$ is also an ideal. Now it is easy to see that the lattice $\mathcal{L}(J)$ of all ideals of J endowed with the product defined by $B \star C := U_B C$ is an algebraic lattice, which is semiprime (prime) if and only if J is semiprime (prime).

(3) Recall that an associative pair (of the first kind) is a pair (A^+, A^-) of Φ-modules (Φ is an arbitrary unital commutative ring of scalars) equipped with trilinear maps $(x, y, z) \to xyz$ from $A^\sigma \times A^{-\sigma} \times A^\sigma$ to A^σ satisfying the identities

$$uv(xyz) = u(vxy)z = (uvx)yz.$$

Every associative algebra A gives rise to an associative pair $\mathcal{A} = (A, A)$ under the triple product abc, where juxtaposition denotes the product of A. A more interesting example is given by $\mathcal{B} = (hom(M, N), hom(N, M))$, where M and N are left modules over a Φ-algebra A, under the triple product abc where juxtaposition now denotes the mapping composition. Let $\mathcal{A} = (A^+, A^-)$ be an associative pair. Given $\mathcal{B} = (B^+, B^-)$ and $\mathcal{C} = (C^+, C^-)$ ideals of $\mathcal{A}$, we have that

$$\mathcal{B} \star \mathcal{C} := (B^+A^-C^+ + A^+B^-A^+C^-A^+, B^-A^+C^- + A^-B^+A^-C^+A^-)$$

is an ideal of $\mathcal{A}$. Now it is straightforward to verify that the lattice $\mathcal{L}(\mathcal{A})$ of all ideals of $\mathcal{A}$ endowed with the $\star$-product defined above is an algebraic lattice. Moreover, $\mathcal{A}$ is a semiprime (prime) associative pair if and only if $(\mathcal{L}(\mathcal{A}), \star)$ is a semiprime (prime) lattice.

(4) Finally, let (T, P) be a Jordan triple systems, and B and C ideals of T. By [16], the product $B \star C := P_B C + P_T P_B C$ is an ideal of T, and we have again that the lattice $\mathcal{L}(T)$ of all ideals of T is an algebraic lattice relative to this product. Moreover, semiprimeness (primeness) of T is equivalent to semiprimeness (primeness) of $\mathcal{L}(T)$. Similar results hold for Jordan pairs.

2. Annihilators in semiprime algebraic lattices

There is a natural notion of annihilator for elements of a semiprime algebraic lattice satisfying properties similar to those of the annihilators of ideals of semiprime algebraic systems (algebras, triples or pairs).

Lemma 2.1. *Let* $(L, \leq, \star)$ *be a semiprime algebraic lattice, and* $x, y \in L$. *Then* $x \star y = 0$ *if and only if* $x \wedge y = 0$.

Proof. If $x \star y = 0$ then, by (1.4), $(x \wedge y)^2 \leq x \star y = 0$, which implies $x \wedge y = 0$ by semiprimeness of $(L, \leq, \star)$. Conversely, it follows from (1.2) that $x \wedge y = 0$ implies $x \star y = 0$. □

Let $(L, \leq, \star)$ be a semiprime algebraic lattice, and let $x \in L$. It follows from (1.3) that if $\{y_\alpha\}$ is a family of elements of L satisfying the equivalent conditions of Lemma 2.1, relative to x, then $\bigvee_\alpha y_\alpha$ also satisfies these conditions. Hence there exists a largest element $y \in L$ such that $x \wedge y = 0$. We will write $x^\perp$ to denote this element and call it the *annihilator* of x. The mapping $x \to x^\perp$ satisfies the usual properties of annihilators in semiprime algebras.

A consequence of Lemma 2.1 is that primeness is equivalent to semiprimeness plus the finite intersection property: $b \wedge c = 0$ implies $b = 0$ or $c = 0$.

Proposition 2.2. *Let $(L, \leq, \star)$ be a semiprime algebraic lattice. We have*

(i) $x \leq y$ *implies* $y^\perp \leq x^\perp$,
(ii) $x \leq x^{\perp\perp}$,
(iii) $x \leq y^\perp$ *if and only if* $y \leq x^\perp$,
(iv) $x^\perp = x^{\perp\perp\perp}$,
(v) $(\bigvee_\alpha x_\alpha)^\perp = \bigwedge_\alpha x_\alpha^\perp$,
(vi) $(x^2)^\perp = x^\perp$.

Proof. (i) By (1.1), $x \leq y$ implies $y^\perp \star x \leq y^\perp \star y = 0$, so $y^\perp \leq x^\perp$.

(ii) is a direct consequence of the definition of annihilator.

(iii) and (iv) follow from (i) and (ii).

(v) By (i), $x_\beta \leq \bigvee_\alpha x_\alpha$ implies $(\bigvee_\alpha x_\alpha)^\perp \leq x_\beta^\perp$, and hence $(\bigvee_\alpha x_\alpha)^\perp \leq \bigwedge_\alpha x_\alpha^\perp$. Conversely, by (1.3), $(\bigwedge_\alpha x_\alpha^\perp) \star (\bigvee_\beta x_\beta) \leq \bigvee_\beta ((\bigwedge_\alpha x_\alpha^\perp) \star x_\beta)$ but $(\bigwedge_\alpha x_\alpha^\perp) \star x_\beta = 0$ since $(\bigwedge_\alpha x_\alpha^\perp) \star x_\beta \leq x_\beta^\perp \star x_\beta = 0$ by (1.1).

(vi) Since $x^2 \leq x$ by (1.2), it follows from (i) that $x^\perp \leq (x^2)^\perp$. Conversely, by using (1.2) and (1.4), we obtain $((x^2)^\perp \star x)^2 \leq (x^2)^\perp \star x \leq (x^2)^\perp$. Again by (1.2), $(x^2)^\perp \star x \leq x$, which implies by (1.4) $((x^2)^\perp \star x)^2 \leq x^2$. Thus $((x^2)^\perp \star x)^2 \leq (x^2)^\perp \wedge x^2 = 0$, and hence $(x^2)^\perp \star x = 0$ by semiprimeness of L. Therefore $(x^2)^\perp \leq x^\perp$, as required. □

Remarks. (1) Let A be an alternative algebra. For any ideal B of A the left annihilator $lan(B) = \{a \in A : aB = 0\}$ and the right annihilator $ran(B) = \{a \in A : Ba = 0\}$ are two-sided ideals of A [21, p.168]. If A is semiprime then $lan(B) = ran(B)$ coincides with the annihilator $B^\perp$ defined in terms of the algebraic lattice $(\mathcal{L}(A), \star)$ of Example 1. There is a similar description for the annihilator $\mathcal{B}^\perp$ of an ideal $\mathcal{B} = (B^+, B^-)$ of a semiprime associative pair.

(2) Let V be a Jordan pair. Following [15, p.104] the annihilator of any subset $X \subset V^\sigma$ is the set $ann(X) \subset V^{-\sigma}$ of all $b \in V^{-\sigma}$ satisfying

$$D(a, b) = D(b, a) = Q(a)Q(b) = Q(b)Q(a) = Q(a)b = Q(b)a = 0$$

for all $a \in X$. Note that the last four conditions are superfluous in case V has no 2-torsion, because of $2Q(a)b = D(a, b)a$ and $2Q(a)Q(b) = D(a, b)^2 - D(Q(a)b, b)$. Recall that a Jordan pair V is nondegenerate if $Q_a = 0$ implies $a = 0$. Clearly nondegenerate Jordan pairs are semiprime. Moreover, it follows from [16, Proposition 1.7], that if $I = (I^+, I^-)$ is an ideal of a nondegenerate Jordan pair V, then $I^\perp = (ann(I^-), ann(I^+))$ and $ann(I^\sigma) = \{b \in V^{-\sigma} : Q_b I^\sigma = 0\}$. A similar result of course holds for ideals of a nondegenerate Jordan algebra or Jordan triple system. Moreover, for a subset X of a linear Jordan algebra J, $ann(X)$ coincides with the Zelmanov annihilator [17, (1.3')]. For further results on annihilators in linear Jordan algebras the reader is also referred to [4].

Let L be an arbitrary lattice. For each $a \in L$, the subset $[a,\, 1] := \{y \in L : a \leq y\}$ is a sublattice of L, called the *quotient of L by a*.

Proposition 2.3. *Let $(L, \leq, \star)$ be an algebraic lattice, and $a \in L$. Then*

(i) *the quotient lattice $[a,\, 1]$ endowed with the new product $u \diamond v := (u \star v) \vee a$ is an algebraic lattice.*

Moreover, if L is semiprime then

(ii) *$([a^\perp,\, 1], \diamond)$ is a semiprime algebraic lattice.*

Proof. The proof of (i) is straightforward, so we need only to show (ii). Let $u \in [a^\perp,\, 1]$ be such that $u \diamond u = a^\perp$, i.e., $u \star u^* \leq a^\perp$. Then by (1.4) $(u \star a)^2 \leq u^2 \leq a^\perp$. Hence $(u \star a)^2 \leq a \wedge a^\perp = 0$, which implies $u \star a = 0$ by semiprimeness of L, which implies $u = a^\perp$ as required. □

Note that if A is a nonassociative algebra and I is an ideal of A, then the lattice $\mathcal{L}(A/I)$ of all ideals of A/I is isomorphic (as algebraic lattice) to $[I,\, A]$. The same is true for associative pairs and Jordan systems.

3. Uniform elements in semiprime algebraic lattices

Let L be an algebraic lattice. An element $u \in L$, $u \neq 0$, will be called *uniform* if for any nonzero elements b and c in L with $b \vee c \leq u$, $b \wedge c \neq 0$. Clearly if u is uniform then any nonzero v in L such that $v \leq u$ is uniform as well.

Proposition 3.1. *Let L be a semiprime algebraic lattice. Then*

(i) *a nonzero element u in L is uniform if and only if its annihilator $u^\perp$ is maximal among all annihilators $x^\perp$ for $0 \neq x \in L$,*

(ii) *if u is uniform then $([u^\perp,\, 1], \diamond)$ is a prime algebraic lattice,*

(iii) *for each uniform element u of L there exists a unique maximal uniform element m of L such that $u \leq m$, namely $m = u^{\perp\perp}$,*

(iv) *if m_i and m_j are maximal uniform elements of L with $m_i \neq m_j$, then $m_i \wedge m_j = 0$,*

(v) *the family $\{m_i\}_{i\in I}$ of all maximal uniform elements of L is direct, i.e.,*

$$m_j \wedge (\bigvee_{i\neq j} m_i) = 0.$$

Suppose additionally that the lattice L is modular. Then

(vi) *if $([u^\perp,\, 1], \diamond)$ is prime, then u is uniform.*

Proof. (i) Suppose that u is a uniform element of L. We claim first that $b^\perp = u^\perp$ for every nonzero element $b \leq u$. Since $b \leq u$ we have $u^\perp \leq b^\perp$ by Proposition 2.2(i). On the other hand, if $b^\perp \not\leq u^\perp$ then $b^\perp \wedge u \neq 0$ and hence $b \wedge b^\perp = b \wedge (u \wedge b^\perp) \neq 0$ by uniformity of u, which is a contradiction, so the claim is proved. Now let c be a nonzero element of L such that $u^\perp \leq c^\perp$. If $c \wedge u = 0$ then $c \leq u^\perp \leq c^\perp$ and hence $c = 0$, which is a contradiction, so $c \wedge u \neq 0$. Then we have by the first part of the proof that $c^\perp \leq (c \wedge u)^\perp = u^\perp$, and hence $c^\perp = u^\perp$, so $u^\perp$ is maximal.

Suppose now that $u^\perp$ is maximal among all the annihilators $d^\perp$, d a nonzero element of L. Let b and c be nonzero elements of L such that $b \vee c \leq u$. By Proposition 2.2(i), $u^\perp \leq b^\perp$ and hence $u^\perp = b^\perp$ by maximality of $u^\perp$. Now,

$b \wedge c = 0$ would imply $c \leq b^\perp = u^\perp$, and hence $c \leq u^\perp \wedge u = 0$, which is a contradiction.

(ii) Let $u \in L$ be uniform and take $u^\perp \neq b$ and $u^\perp \neq c$ in $[u^\perp, 1]$. Then both $b \wedge u$ and $c \wedge u$ are nonzero, and hence $b \wedge c \wedge u \neq 0$ by uniformity of u, which implies $b \wedge c \neq u^\perp$, so $([u^\perp, 1], \diamond)$ is prime.

(iii) Let $u \in L$ be uniform. By Proposition 2.2(iii), $u \leq u^{\perp\perp}$ with $u^{\perp\perp}$ uniform because, by Proposition 2.2(iv), $u^{\perp\perp\perp} = u^\perp$ which is maximal by (i). Now let b be a uniform element of L such that $u \leq b$. Then, by Proposition 2.2(i), $b^\perp \leq u^\perp$ which implies $b^\perp = u^\perp$ by maximality of $b^\perp$. Hence, by Proposition 2.2(iii) again, $b \leq b^{\perp\perp} = u^{\perp\perp}$, as required.

(iv) Let m_i and m_j be maximal uniform elements of L with $m_i \neq m_j$. Then $m_i \wedge m_j = 0$. Otherwise, $m_i^\perp \leq (m_i \wedge m_j)^\perp$ with $m_i \wedge m_j \neq 0$ would imply, by (i), $m_i^\perp = (m_i \wedge m_j)^\perp = m_j^\perp$, and hence, by (iii),

$$m_i = m_i^{\perp\perp} = m_j^{\perp\perp} = m_j,$$

which is a contradiction.

(v) Let $\{m_i\}_{i \in I}$ be the family of all maximal uniform elements of L. For each m_j we have, by (1.5),

$$m_j \star \left(\bigvee_{i \neq j} m_i \right) = \bigvee_{i \neq j} (m_j \star m_i) = 0$$

by (iv), so

$$m_j \wedge \left(\bigvee_{i \neq j} m_i \right) = 0$$

by Proposition 2.1.

(vi) Suppose now that the lattice L is modular, and that $([u^\perp, 1], \diamond)$ is prime. If b and c are nonzero elements of L such that $b \vee c \leq u$, then none of them is orthogonal to u, and hence $u^\perp \lneqq b \vee u^\perp$ and $u^\perp \lneqq c \vee u^\perp$. Since $[u^\perp, 1]$ is prime, we obtain $(b \vee u^\perp) \wedge (c \vee u^\perp) \wedge u \neq 0$. Hence, by modularity of L,

$$0 \neq (b \vee u^\perp) \wedge (c \vee u^\perp) \wedge u = u \wedge (b \vee u^\perp) \wedge u \wedge (c \vee u^\perp) = b \wedge c,$$

which proves that u is uniform, as required. □

4. Essential subdirect products of prime algebraic systems

In what follows, by an *algebraic system* we will mean any one of the following algebraic structures: a nonassociative (associative, alternative or linear Jordan) algebra, an associative pair, or a Jordan system (quadratic Jordan algebra, Jordan triple, or Jordan pair).

Essential subdirect products play an important role in the general theory of rings of quotients [12]. Here we prove a quite general structure theorem characterizing those algebraic systems which are an essential subdirect product of prime algebraic systems. This theorem is a key tool to reduce questions about semiprime algebraic systems to corresponding questions relative to prime ones (see [6] and [7]). Associated with essential subdirect products appear uniform ideals. First we recall that an ideal I of an algebraic system A is called an essential ideal if I has nonzero intersection with any nonzero ideal of A.

A subdirect product of a collection $\{A_\alpha\}$ of algebraic systems is any subsystem A of the full direct product $\prod A_\alpha$ such that the canonical projections $\pi_\alpha : A \to A_\alpha$ are onto. An *essential subdirect product* is any subdirect product of the A_α which contains an essential ideal of the full direct product $\prod A_\alpha$. If A is actually contained in the direct sum $\oplus A_\alpha$, then A will be called an *essential subdirect sum.* An ideal I of a semiprime algebraic system A is called a *closed ideal* if $I = I^{\perp\perp}$, where we are using the lattice notation $I^\perp$ to denote the annihilator of I. Since the third annihilator coincides with the first one, an ideal is closed if and only if it is the annihilator of an ideal. Note that by Proposition 3.1(iii) maximal uniform ideals are closed.

Theorem 4.1. *Let A be an algebraic system. Then*

(a) *if A is an essential subdirect product of prime algebraic systems A_α, then A is semiprime and every nonzero ideal of A contains a uniform ideal.*
(b) *Conversely, if A is semiprime and every nonzero closed ideal contains a uniform ideal, then A is an essential subdirect product of the systems $A_\alpha = A/M_\alpha^\perp$ where $\{M_\alpha\}$ is the family of all maximal uniform ideals of A.*

Proof. (a) In general, any subdirect product A of a family $\{A_\alpha\}$ of semiprime algebraic systems is also semiprime. Indeed, if B is an ideal of A such that $B^2 = 0$ then, for each index α, $\pi_\alpha(B)$ is an ideal of A_α with $\pi_\alpha(B)^2 = 0$ which implies $\pi_\alpha(B) = 0$ by semiprimeness of the A_α, so $B = 0$. Let $M \subset A$ be an essential ideal of the full direct product $\prod A_\alpha$, and set $M_\alpha := M \cap A_\alpha$, where we are regarding A_α as an ideal of $\prod A_\alpha$. Then M_α is a nonzero ideal of A_α contained in A since M is an essential ideal of $\prod A_\alpha$. Actually M_α is a uniform ideal of A since M_α is uniform in A_α because A_α is prime, and any ideal of A contained in M_α is an ideal of A_α. Now if I is a nonzero ideal of A then $\pi_\alpha(I)$ is a nonzero ideal of A_α for some index α. Hence, by primeness of A_α, $0 \neq \pi_\alpha(I) \star M_\alpha = I \star M_\alpha$ (because of the compoentwise operations in $\prod A_\alpha) \subset I \cap M_\alpha$. Therefore I contains the uniform ideal $I \cap M_\alpha$.

(b) Let $\sum M_\alpha$ be the sum of all maximal uniform ideals of A, which is direct by Proposition 3.1(v). Since $(\sum M_\alpha)^\perp$ is a closed ideal, it must be zero. Otherwise $(\sum M_\alpha)^\perp$ would contain a uniform ideal, and therefore a maximal uniform ideal because it is closed, which leads to contradiction. Hence, by a standard argument, $\cap M_\alpha^\perp = (\sum M_\alpha)^\perp = 0$ implies that A is a subdirect product of the algebraic systems $A_\alpha := A/M_\alpha^\perp$ each of which is prime by Proposition 3.1(vi). Finally, $\oplus M_\alpha$ is an essential ideal of $\prod A_\alpha$ since if B is an ideal of $\prod A_\alpha$ such that

$$B \star \left(\bigoplus M_\alpha\right) = \bigoplus (B \star M_\alpha) = \bigoplus (\pi_\alpha(B) \star M_\alpha) = 0$$

then, for each α, $\pi_\alpha(B) \star M_\alpha = 0$, so $\pi_\alpha(B) = 0$ by primeness of A_α, and hence $B = 0$, which completes the proof. □

Since the third annihilator coincides with the first one, for any semiprime algebraic lattice L the ascending chain condition (acc) on annihilators is equivalent to the descending chain condition (dcc) on annihilators. Thus we will simply speak of the chain conditions on annihilators.

Corollary 4.2. *Any semiprime algebraic system satisfying the chain condition on annihilator ideals is an essential subdirect sum of a finite number of prime ones.*

Proof. We first show that any nonzero closed ideal of A contains a uniform ideal. Let B be a nonzero closed ideal of A. Denote by $\mathcal{B}$ the set of all annihilators ideals $C^\perp$, where C is a nonzero ideal of A contained in B. Then $\mathcal{B}$ contains a maximal element, say $I^\perp$. We claim that $I^\perp$ is maximal among all annihilator ideals $D^\perp$, where D is a nonzero ideal of A. Indeed, suppose that $I^\perp \subset D^\perp$. Then

$$D \subset D^{\perp\perp} \subset I^{\perp\perp} \subset B^{\perp\perp} = B$$

because B is closed. Hence $D^\perp = I^\perp$ by maximality of $I^\perp$ in $\mathcal{B}$, which proves the claim. Thus I is uniform by Proposition 3.1(i). Then by Theorem 4.1(b), A is isomorphic to the subdirect product of the $A/M_\alpha^\perp$, where $\{M_\alpha\}$ is the family of all maximal uniform ideals of A. Since the sum of all M_α is direct by Proposition 3.1(v), there are only finitely many A_α by virtue of the chain condition on annihilator ideals. □

5. Applications to extensions of Goldie's Theorems

We present in this section some applications of the above results to obtain recent extensions of Goldie's Theorems.

A new approach to a Goldie-like Theorem for alternative algebras. H. Essannouni and A. Kaidi gave in [3] a Goldie-like Theorem for alternative rings (see also [2] for a former version of this theorem by the same authors). We now provide a new proof of this result based on Slater's theorem for prime nondegenerate alternative algebras and on the fact that a semiprime alternative algebra not containing infinite direct sums of right ideals satisfies the chain conditions on the annihilators of its ideals, and hence it is an essential subdirect sum of a finite number of prime nondegenerate alternative algebras each of which does not contain infinite direct sums of right ideals. We begin by recalling that an alternative ring R, with a nonzero center Z which does not contain zero divisors of the ring R is a *Cayley-Dickson ring* if the central localization $R_1 = (Z^*)^{-1}R$ is a Cayley-Dickson algebra over the ring of quotients of the center Z. By Slater's Theorem, any prime nondegenerate alternative algebra which is not associative is a Cayley-Dickson ring.

Theorem 5.1 (H. Essannouni and A. Kaidi)**.** *Let A be a nondegenerate alternative algebra such that*

(i) *A does not contain infinite direct sums of right ideals.*
(ii) *A satisfies the acc for the right annihilators $ran(a)$, $a \in N(A)$.*

Then A has a right quotient algebra Q with denominators in $N(A)$, the nucleus of A.

Proof. (1) Consider first the case that A is prime. It follows by Slater's Theorem [21, p.194] that A is either associative or a Cayley-Dickson ring. In the first case, A is a right order in a simple artinian associative algebra Q by Goldie's First Theorem (see [12, Corollary 3.36] together with the fact that condition (ii) implies no singularity). In the second case A is a two-sided order in the Cayley-Dickson algebra $(Z^*)^{-1}A$, where Z is the center of A.

(2) A satisfies the chain conditions on annihilator ideals. Let $B_1^\perp \supsetneqq B_2^\perp \supsetneqq \dots$ be a descending chain of annihilator ideals. We construct an infinite sequence $\{I_n\}$ of nonzero right ideals as follows. For each positive integer n, take $x_n \in B_n^\perp$ such that x_n is not in $B_{n+1}^\perp$. Since $B_{n+1}^\perp = lan(B_{n+1})$, there exists $b_{n+1} \in B_{n+1}$ such

that $x_nb_{n+1} \neq 0$. Define $I_n := [x_nb_{n+1})$, the right ideal of A generated by x_nb_{n+1}. Now we show that $\sum I_n$ is direct. For each $m \geq 2$,

$$\begin{aligned}(\sum_{n\neq m} I_n) \cap I_m &= (\sum_{j=1}^{m-1} I_j + \sum_{n\geq m+1} I_n) \cap I_m \\ &\subset (B_2 + \ldots + B_m + B_{m+1}^{\perp}) \cap (B_{m+1} \cap B_m^{\perp}) \\ &\subset (\sum_{j=2}^{m} B_j + B_{m+1}^{\perp}) \cap B_{m+1}^{\perp\perp} \cap (\bigcap_{n=2}^{m} B_n^{\perp}) \\ &= (\sum_{j=2}^{m} B_j + B_{m+1}^{\perp}) \cap (B_{m+1}^{\perp} + \sum_{j=2}^{m} B_j)^{\perp} = 0.\end{aligned}$$

For $m = 1$, $I_1 \cap (\sum_{n\geq 2} I_n) \subset B_2 \cap B_2^{\perp} = 0$, and therefore the I_n form an infinite direct sum of right ideals.

(3) By (2), it follows from Corollary 4.2 that A is an essential subdirect sum of a finite number of prime alternative algebras $A_1, A_2, \ldots A_n$, where each A_i is isomorphic to $A/M_i^{\perp}$, and where $M_1, M_2, \ldots, M_n$ are the maximal uniform ideals of A.

(4) Let M be a maximal uniform ideal of A, and suppose that the prime algebra $\overline{A} = A/M^{\perp}$ is associative. Then $\overline{A}$ satisfies the following conditions: (i) $\overline{A}$ does not contain infinite direct sums of right ideals, and (ii) $\overline{A}$ satisfies the acc for the right annihilators $ran_{\overline{A}}(\overline{a})$, $a \in M$. To prove (i), let $\pi : A \to \overline{A}$ be the canonical projection, and $\sum \overline{I}_\alpha$ a direct sum of right ideals of $\overline{A}$. For each α set $I_\alpha = \pi^{-1}(\overline{I}_\alpha)$ which is a right ideal of A. First we note that the right ideals $R_\alpha = I_\alpha \cap M$ are nonzero. Indeed, $R_\alpha = 0$ would imply $I_\alpha \subset M^{\perp}$ and hence $\overline{I}_\alpha = \pi(I_\alpha) = 0$, which is a contradiction. Now we show that the sum of the R_α is direct. Indeed, for each index β,

$$R_\beta \cap (\sum_{\alpha\neq\beta} R_\alpha) \subset I_\beta \cap (\sum_{\alpha\neq\beta} I_\alpha) \cap M$$

but $I_\beta \cap (\sum_{\alpha\neq\beta} I_\alpha) \subset M^{\perp}$ because the sum $\sum \overline{I}_\alpha$ is direct. Hence

$$R_\beta \cap (\sum_{\alpha\neq\beta} R_\alpha) \subset M \cap M^{\perp} = 0.$$

To prove (ii) we note that, for $a \in M$ and $x \in A$, $x \in ran(a)$ if and only if $\overline{x} \in ran_{\overline{A}}(\overline{a})$. Clearly, if $x \in ran(a)$ then $\overline{x} \in ran_{\overline{A}}(\overline{a})$. Conversely, if $\overline{x} \in ran_{\overline{A}}(\overline{a})$ then x annihilates a modulo $M^{\perp}$, but a belongs to M. Hence x annihilates a modulo $M^{\perp} \cap M = 0$. Now it follows from (ii) that $\overline{A}$ is right nonsingular. Indeed, let $Z_r(\overline{A})$ be the right singular ideal of $\overline{A}$, and suppose that $Z_r(\overline{A}) \neq 0$. Since $\overline{A}$ is prime and M can be regarded as an ideal of $\overline{A}$, we have that $M \cap Z_r(\overline{A}) \neq 0$. Then, by (ii), we can take a nonzero element $\overline{x} \in M \cap Z_r(\overline{A})$ with $ran_{\overline{A}}(\overline{x})$ maximal in the set $\{ran_{\overline{A}}(\overline{y}) : 0 \neq \overline{y} \in M \cap Z_r(\overline{A})\}$. Let $\overline{a} \in \overline{A}$ be such that $\overline{xax} \neq 0$. Then $\overline{ax} \neq 0$ and hence there exists $0 \neq \overline{axz} \in ran_{\overline{A}}(\overline{x})$ for some $\overline{z} \in \overline{A}$, which implies $\overline{z} \in ran_{\overline{A}}(\overline{xax})$, with $\overline{z} \notin ran_{\overline{A}}(\overline{x})$, which contradicts the maximality of $ran_{\overline{A}}(\overline{x})$.

(5) Now it follows from (1) that, for each maximal uniform ideal M_i, the alternative algebra $A_i := A/M_i^{\perp}$ is a right order in a simple artinian alternative algebra Q_i, but then M_i is also an order in Q_i (if A_i is a Cayley-Dickson ring this follows

from the fact that any ideal of a Cayley-Dickson ring is again a Cayley-Dickson ring [21, p.193], while if A_i is associative this is known [7, Proposition 7.5]). Since

$$\bigoplus_{1\le i\le n} M_i \lhd A \leqslant \bigoplus_{1\le i\le n} Q_i,$$

we finally obtain that A is a right order in the semisimple artinian alternative algebra $\bigoplus_{1\le i\le n} Q_i$ with denominators in the nucleus of A, as required. □

Uniform elements and Local Goldie Theorem for associative algebras. Based on ideas from semigroup theory, Fountain and Gould [9] introduced a notion of order in a ring which need not have a unit, and gave [10] a Goldie-like characterization of two-sided orders in semisimple rings with minimum condition on principal one-sided ideals. Later, the same authors gave a new approach to this result by reducing the semiprime to the prime case [11]. The key tool in the new proof is the notion of left uniform element, that is, an element with maximal proper left annihilator.

Proposition 5.2. *Let A be a semiprime associative algebra and let u be a nonzero element in A such that $lan(u) = lan(x)$ for all $0 \neq x \in uA$. Then the ideal $I(u)$ generated by u is uniform. Hence, if A satisfies the ascending chain condition on left annihilators of elements, then any nonzero ideal of A contains a uniform ideal.*

Proof. Let B and C be nonzero ideals of A contained in $I(u)$. We must prove that $B \cap C$ is nonzero. Note first that uB and similarly uC are both nonzero by semiprimeness of A. Otherwise, $uB = 0$ would imply $u \in B^{\perp}$, and hence $B \subset I(u) \subset B^{\perp}$, which is a contradiction. Thus there exists $b \in B$ such that $ub \neq 0$, so $lan(u) = lan(ub)$, and hence, if $B \cap C$ were zero, then $C \subset B^{\perp} \subset lan(ub) = lan(u)$ which would imply that $u \in C^{\perp}$, which is a contradiction as shown above. □

Uniform ideals and Zelmanov's Theorem for Goldie Jordan algebras. Consider now a nondegenerate Jordan algebra J. For any subset M of J denote by $\mathcal{K}_J(M)$ the inner ideal generated by M. A *direct sum of inner ideals* is a family $\{I_\lambda\}_{\lambda\in\Lambda}$ of nonzero inner ideals such that for each $\mu \in \Lambda$, $I_\mu \cap \mathcal{K}_J(\sum_{\lambda\neq\mu} I_\lambda) = 0$. As in the case of an alternative algebra, we have the following result whose proof will appear in [8], and which is a key tool in the new approach [8] to Zelmanov's Theorem for Goldie Jordan algebras [19], [20].

Proposition 5.3. *Let J be a nondegenerate Jordan algebra not containing infinite direct sums of inner ideals. Then J satisfies the chain condition on the annihilator ideals, and hence J is an essential subdirect sum of a finite number of prime nondegenerate Jordan algebras J_α not containing infinite direct sums of inner ideals.*

Uniform elements and Local Goldie Theorem for Jordan algebras. Inspired by the works of Fountain and Gould, we introduced in [5] a notion of local order in a Jordan algebra which need not have a unit and proved that a Jordan algebra J is a local order in a simple Jordan algebra Q with dcc on principal inner ideals but which is not a non-artinian quadratic factor if and only if J is a prime nondegenerate Jordan algebra satisfying local Goldie conditions. In a subsequent paper we extended this result to nondegenerate Jordan algebras. Again the key tool for such an extension was a notion of uniform element.

Let J be a nondegenerate Jordan algebra. A nonzero element $u \in J$ is called *uniform* if

$$ann(u) = ann(x) \quad \text{for any} \quad 0 \neq x \in U_u J.$$

Clearly every nonzero element $u \in J$ such that $ann(u)$ is maximal is a uniform element. In particular, if J satisfies the acc on the annihilators of its elements, then every nonzero inner ideal of J contains a uniform element. Since by [6, Proposition 1] uniform elements in nondegenerate Jordan algebras generate uniform ideals, it follows that a nondegenerate Jordan algebra J satisfying the acc on the annihilators of its elements is an essential subdirect product of prime nondegenerate Jordan algebras J_i each of which contains a uniform element.

Uniform elements and Goldie Theorem for associative pairs. In a recent work [7] we proved a Goldie-like theorem for associative pairs. Again the semiprime case was reduced to the prime one via uniform ideals. Let A be a semiprime associative pair. A nonzero element $u \in A^{\pm}$ is called an l-uniform element if any two nonzero left ideals of A inside the left ideal $A^{\pm}A^{\mp}u$ generated by u have nonzero intersection. By [7, Proposition 6.1], any l-uniform element of a semprime associative pair generates a uniform ideal. Hence, if A is a semiprime associative pair such that for any $a \in A^{\pm}$, the left ideal $A^{\pm}A^{\mp}a$ does not contain infinite direct sums of left ideals (A has finite *left local Goldie dimension*), then any nonzero ideal of A contains a uniform ideal, so A is an essential subdirect product of prime associative pairs A_i each of which has also finite left local Goldie dimension.

Acknowledgments

We are indebted to the referee for the careful reading of our manuscript.

References

1. P. Crawley and R.P. Dilworth, *Algebraic theory of lattices*, Prentice-Hall, Inc., New Jersey, 1973.
2. H. Essannouni and A. Kaidi, *Semiprime alternative rings with a.c.c.*, Lecture Notes in Mathematics vol 460, 82-93. MR **89h:**17044
3. H. Essannouni and A. Kaidi, *Goldie's Theorem for alternative rings*, Proc. Amer. Math. Soc. **121** (1994), 39–45. MR **94g:**17060
4. A. Fernández López, *On annihilators in Jordan algebras*, Publicacions Matemátiques **36** (1992), 569–589. MR **94c:**17047
5. A. Fernández López and E. García Rus, *Prime Jordan algebras satisfying local Goldie conditions*, J. Algebra **174** (1996), 1024–1048. MR **96j:**17027
6. A. Fernández López and E. García Rus, *Nondegenerate Jordan algebras satisfying local Goldie conditions*, J. Algebra **182** (1996), 52–59. MR **97e:**17045
7. A. Fernández López, E. García Rus, M. Gómez Lozano and M. Siles Molina, *Goldie Theorem for associative pairs* (to appear).
8. A. Fernández López, E. García Rus and F. Montaner, *Goldie Theory for Jordan algebras* (to appear).
9. J. Fountain and V. Gould, *Orders in rings without identity*, Comm. in Algebra **18** (1990), 3085–3110. MR **91h:**16031
10. J. Fountain and V. Gould, *Orders in regular rings with minimal condition for principal right ideals*, Comm. in Algebra **19** (1991), 1501–1527. MR **93b:**16044
11. J. Fountain and V. Gould, *Orders in semiprime rings with minimal condition for principal right ideals*, Proc. Royal Soc. Edinburgh **121(A)** (1992), 375–380. MR **93h:**16033
12. K.R. Goodearl, *Ring theory: Nonsingular rings and modules* Pure and Applied Mathematics, Marcel Dekker, New York, 1976.

13. N. Jacobson, *Structure and representations of Jordan algebras,* Colloquium Publications, Vol 39, Amer. Math. Soc., Providence, 1968. MR **40:**4330
14. N. Jacobson, *Structure theory of Jordan algebras,* Lecture Notes in Mathematics, Vol 5, University of Arkansas, Fayetteville, 1981. MR **83b:**17015
15. O. Loos, *Jordan pairs,* Lecture Notes in Mathematics, vol 460, Springer-Verlag, Berlin and New York, 1975.
16. K. McCrimmon, *Strong prime inheritance in Jordan systems*, Algebras, Groups and Geometries **1** (1984), 217–234. MR **86c:**17017
17. K. McCrimmon, *The Zelmanov annihilator and nilpotence of the nil radical in quadratic Jordan algebras with chain conditions*, J. Algebra **67** (1980), 230–253. MR **82g:**17016
18. E. Neher, *Jordan triple systems by the grid approach,* Lecture Notes in Mathematics, Vol 1280, Springer, Berlin-Heidelgeberg-New York, 1987. MR **89b:**17024
19. E. Zel'manov, *Goldie's theorem for Jordan algebras*, Siberian Math. J. **28** (1987), 44–52. MR **89f:**17034
20. E. Zel'manov, *Goldie's theorem for Jordan algebras II*, Siberian Math. J. **29** (1988), 68–74. MR **89m:**17048
21. K.A. Zhevlakov, A.M Slin'ko, I.P. Shestakov, A.I. and Shirshov, *Rings that are nearly associative,*, Academic Press, New York, 1982. MR **83i:**17001

Departamento de Algebra, Geometría y Topología, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain

E-mail address: `emalfer@ccuma.sci.uma.es`