

## PRODUCTS OF CONSTANT CURVATURE SPACES WITH A BROWNIAN INDEPENDENCE PROPERTY

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ABSTRACT. The time and place Brownian motion on the product of constant curvature spaces first exits a normal ball of radius  $\epsilon$  centered at the starting point of the Brownian motion are considered. The asymptotic expansions, as  $\epsilon$  decreases to zero, for joint moments of the first exit time and place random variables are computed with error  $O(\epsilon^{10})$ . It is shown that the first exit time and place are independent random variables only if each factor space is locally flat or of dimension three.

### 1. INTRODUCTION

It is well known that, for Brownian motion started at the origin in  $\mathbb{R}^n$ , the exit time and place from a ball centered at the origin are independent random variables. Among Riemannian manifolds, this property also holds for locally harmonic spaces [KO1], [L2]. In [H1], it is shown that the product of constant curvature spaces  $S^3 \times H^3$ , which is not locally harmonic, also satisfies this independence property. In this paper, we completely determine which products of constant curvature spaces have this independence property.

Let  $M$  be an  $n$ -dimensional Riemannian manifold. Let  $B_m(\epsilon)$  denote the geodesic ball of radius  $\epsilon$  about  $m$  in  $M$ . Let  $S_m(\epsilon)$  denote the boundary of  $B_m(\epsilon)$ . Let  $(X_t)$  be Brownian motion on  $B_m(\epsilon)$ , started from  $m$ . The infinitesimal generator of  $(X_t)$  is  $\frac{1}{2}\Delta$ , where  $\Delta$  is the Laplace-Beltrami operator on functions. Let  $T(\epsilon)$  be the first exit time of  $X_t$  from  $B_m(\epsilon)$ :

$$(1.1) \quad T(\epsilon) = \inf\{t > 0 : d(m, X_t) \geq \epsilon\}.$$

Then  $X_{T(\epsilon)}$  is the first exit place of  $X_t$  from  $B_m(\epsilon)$ .

In this notation, we have the result from [H1] mentioned above.

**Theorem 1.1.** *If  $M = S^3 \times H^3$ , where  $S^3$  and  $H^3$  are a three-dimensional sphere and hyperbolic space, respectively, and  $B_m(\epsilon)$  is a normal ball, then  $T(\epsilon)$  and  $X_{T(\epsilon)}$  are independent random variables.*

As remarked in [H1], this result is easily extended to any  $M$  that is a product of spaces of constant sectional curvature where each factor is of dimension three or is locally flat (sectional curvature is zero or the dimension is one).

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We fix a point  $m \in M$  and represent all tensors with respect to normal coordinates at  $m$ . No distinction is made between upper and lower indices and we use the summation convention for any repeated indices, unless otherwise noted. Let  $R_{ijkl}$  denote the Riemann curvature tensor,  $\rho_{ij} = R_{iaja}$  denote the Ricci curvature tensor, and  $\tau = \rho_{aa}$  denote scalar curvature. The metric tensor is denoted  $g_{ij}$  and equals the Kronecker delta  $\delta_{ij}$  at  $m$ . In [KO1] and [H2] it is shown that if  $T(\epsilon)$  and  $X_{T(\epsilon)}$  are independent for all normal balls at each  $m \in M$  then  $M$  has constant scalar curvature,

$$(1.2) \quad (3\nabla_p \nabla_p \rho_{ij} - 4\rho_{ip} \rho_{jp} + 2R_{ipjq} \rho_{pq} + 2R_{ipqr} R_{jpqr})_m = (1/n)[(2\|R\|^2 - 2\|\rho\|^2)g_{ij}]_m$$

holds for every  $m \in M$ , and  $\|R\|^2 - \|\rho\|^2$  is constant on  $M$ .

It can be shown, for example, that these conditions are satisfied by any manifold  $M = M^{n_1}(\kappa_1) \times M^{n_2}(\kappa_2)$  where  $M^{n_i}(\kappa_i)$  is an  $n_i$  dimensional manifold with constant sectional curvature  $\kappa_i$  and where  $n_1 = n_2$  and  $|\kappa_1| = |\kappa_2|$  [KO2]. However, by considering higher order asymptotics, we will show that  $T(\epsilon)$  and  $X_{T(\epsilon)}$  are independent for all normal balls centered at  $m \in M$  only if  $n_i = 1$  or  $n_i = 3$  or  $\kappa_i = 0$ ,  $i = 1, 2$ . In fact, we show this result for the product of  $K$  constant curvature spaces. Thus, further information is obtained from higher order asymptotics and we conjecture that these higher order asymptotics could be used to classify more general symmetric spaces with independent first exit time and place from normal balls.

## 2. PERTURBATION METHOD AND REDUCTION PROCESS

We use the extension of the perturbation method of [GP] and [L1] described in [H2]. We will translate the problem from a normal ball about  $m \in M$  to the unit ball in  $T_m M$ , the tangent space at  $m$ . We consider values of  $\epsilon$  less than  $\epsilon_1$  where  $\epsilon_1$  is chosen such that the exponential map is non-singular in  $B_0(\epsilon_1)$ , the ball of radius  $\epsilon_1$  in  $T_m M$ . Let  $B$  and  $S$  denote the unit ball and unit sphere, respectively, in  $T_m M$ . We define  $\Phi_\epsilon : C^\infty(\overline{B}) \rightarrow C^\infty(\overline{B_m(\epsilon)})$  by

$$(2.1) \quad (\Phi_\epsilon f)(\exp_m x) \doteq f(x/\epsilon).$$

Working in the unit ball, we find it convenient to use homogeneous harmonic polynomials. Let  $\sum_{P(i_1 \dots i_k)}$  denote summation over the permutations of the indices  $i_1, \dots, i_k$ . It follows that we can define  $Q_{i_1 \dots i_k}(z)$ , a homogeneous harmonic polynomial of degree  $k$ , by the following [VK, Section 9.2.3]:

$$(2.2) \quad z_{i_1} \dots z_{i_k} + \sum_{j=1}^{[k/2]} \sum_{P(i_1 \dots i_k)} \frac{(-1)^j |z|^{2j} \delta_{i_1 i_2} \dots \delta_{i_{2j-1} i_{2j}} z_{i_{2j+1}} \dots z_{i_k}}{2^j j! (k-2j)! (n+2k-4)(n+2k-6) \dots (n+2k-2j-2)}.$$

In particular, we will use the following.

$$(2.3) \quad Q_{ij}(z) = z_i z_j - \frac{1}{n} \delta_{ij} |z|^2,$$

$$Q_{ijkl}(z) = z_i z_j z_k z_l - \frac{1}{4(n+4)} \sum_{P(ijkl)} \delta_{ij} z_k z_l |z|^2 + \frac{1}{8(n+2)(n+4)} \sum_{P(ijkl)} \delta_{ij} \delta_{kl} |z|^4.$$

For a Riemannian manifold  $M$ , the infinitesimal generator of Brownian motion is  $A = \frac{1}{2}\Delta$  which has the following homogeneous decomposition with respect to normal coordinates  $(x^j)$  [GP]:

$$(2.4) \quad A = \frac{1}{2}\Delta_{-2} + \frac{1}{2} \sum_{i=0}^{\infty} \Delta_i \quad (\Delta_{-1} = 0).$$

Here  $\Delta_{-2}$  is the ordinary Euclidean Laplacian and  $\Delta_i$  is a second order linear differential operator with polynomial coefficients of the form  $\Delta_i = q_i^j \partial_j + q_i^{jk} \partial_j \partial_k$  where  $q_i^j$  and  $q_i^{jk}$  are respectively  $(i + 1)$ -degree and  $(i + 2)$ -degree homogeneous polynomials in  $(x^j)$ .  $\Delta_i$  maps  $k$ -degree homogeneous polynomials to  $(k + i)$ -degree homogeneous polynomials.

From [H2] we have the following result.

**Theorem 2.1.** *Let  $f$  be a smooth function on  $S$ . Suppose that  $u_j$  and  $v_j$ ,  $0 \leq j \leq k$ , satisfy*

$$(2.5) \quad \begin{aligned} \Delta_{-2}u_0 &= 0 \text{ in } B, \quad u_0 = f \text{ on } S, \\ \Delta_{-2}u_j + \sum_{i=2}^j \Delta_{i-2}u_{j-i} &= 0 \text{ in } B, \quad u_j = 0 \text{ on } S \quad (1 \leq j \leq k), \\ \Delta_{-2}v_j + \sum_{i=2}^j \Delta_{i-2}v_{j-i} + 2u_j &= 0 \text{ in } B, \quad v_j = 0 \text{ on } S \quad (0 \leq j \leq k). \end{aligned}$$

Let  $\hat{u}_\epsilon$  and  $\hat{v}_\epsilon$  be defined as follows:

$$(2.6) \quad \hat{u}_\epsilon = \sum_{j=0}^k \epsilon^j u_j, \quad \hat{v}_\epsilon = \sum_{j=0}^k \epsilon^{j+2} v_j.$$

Then

$$(2.7) \quad \begin{aligned} E_m[(\Phi_\epsilon f)(X_{T(\epsilon)})] &= \hat{u}_\epsilon(0) + O(\epsilon^{k+1}), \\ E_m[T(\epsilon)(\Phi_\epsilon f)(X_{T(\epsilon)})] &= \hat{v}_\epsilon(0) + O(\epsilon^{k+3}). \end{aligned}$$

Note that (2.5) defines  $u_j$  and  $v_j$  recursively. Also, the sums in (2.5) are empty when  $j = 1$  and it follows that  $u_1$  and  $v_1$  are identically zero.

Set  $f \equiv 1$ . Then  $\hat{u}_\epsilon \equiv 1$  and Theorem 2.1 implies:

**Corollary 2.2.** *Let  $f \equiv 1$ . Suppose that  $u_j$  and  $v_j$ ,  $0 \leq j \leq k$ , satisfy (2.5) and  $\hat{u}_\epsilon$  and  $\hat{v}_\epsilon$  are defined as in (2.6). Then  $\hat{u}_\epsilon \equiv 1$  and*

$$(2.8) \quad E_m[T(\epsilon)] = \hat{v}_\epsilon(0) + O(\epsilon^{k+3}).$$

As described in [H2], the values  $u_j(0)$ ,  $1 \leq j \leq k$ , and  $v_j(0)$ ,  $0 \leq j \leq k$ , defined by solutions of (2.5) can be expressed in terms of  $u_0$  and its derivatives evaluated at 0. In the remainder of this section we outline here the procedure which is proved in detail in [H2].

Define the differential operator  $A_k^j$  as follows:

$$(2.9) \quad A_k^j \doteq \begin{cases} \Delta_k, & j = 0, \\ \Delta_{-2}A_k^{j-1} - A_k^{j-1}\Delta_{-2}, & j \geq 1. \end{cases}$$

It follows easily by induction that each  $A_k^j$  has the following properties:

**Lemma 2.3.**  $A_k^j$  is a linear differential operator, of order  $2 + j$ , with polynomial coefficients of degree  $\leq 2 + k - j$  and  $\geq 1 + k - 2j$  (where a polynomial of negative degree is understood to be the 0 polynomial). Furthermore,

$$(2.10) \quad \Delta_{-2}^m \Delta_k = \sum_{j=0}^m \binom{m}{j} A_k^j \Delta_{-2}^{m-j}.$$

For the homogeneous harmonic polynomial of degree  $k$  defined by (2.2), define the coefficients  $Q_{i_1 \dots i_k}^{j_1 \dots j_k}$  by the relation

$$(2.11) \quad Q_{i_1 \dots i_k}(z) = \sum_{j_1, \dots, j_k=1}^n Q_{i_1 \dots i_k}^{j_1 \dots j_k} z_{j_1} \dots z_{j_k}.$$

Without loss of generality, we may take  $Q_{i_1 \dots i_k}^{j_1 \dots j_k}$  to be symmetric in the indices  $j_1, \dots, j_k$ . Define the differential operator

$$(2.12) \quad Q_{i_1 \dots i_k}(\nabla) \doteq \sum_{j_1, \dots, j_k=1}^n Q_{i_1 \dots i_k}^{j_1 \dots j_k} \partial_{j_1} \dots \partial_{j_k}.$$

Define the following dimension constants for  $k \geq 0$ :

$$(2.13) \quad \phi_h^k \doteq \begin{cases} 1, & h = 0, \\ \frac{1}{2^h h! (n + 2k)(n + 2k + 2) \dots (n + 2k + 2h - 2)}, & h > 0. \end{cases}$$

Using Pizetti’s formula in  $\mathbb{R}^n$  [CH, pp. 287–289], we have the following lemma.

**Lemma 2.4.** Suppose  $\Delta_{-2}^{m+1} u = 0$  in  $B$  and  $u = 0$  on  $S$ . Then

$$(2.14) \quad \begin{aligned} u(0) &= - \sum_{h=1}^m \phi_h^0 (\Delta_{-2}^h u)(0), \\ [Q_{i_1 \dots i_k}(\nabla) u](0) &= - \sum_{h=1}^m \phi_h^k [Q_{i_1 \dots i_k}(\nabla) \Delta_{-2}^h u](0). \end{aligned}$$

By successively applying (2.5), (2.10), and (2.14), the values of  $u_j(0)$  and  $v_j(0)$  are found in terms of  $u_0$  and its derivatives evaluated at 0 as desired. These values can be expressed as follows (using the Poisson formula).

**Lemma 2.5.**

$$(2.15) \quad (\partial_{i_1} \partial_{i_2} \dots \partial_{i_k} u_0)(0) = \int_S n(n + 2) \dots (n + 2k - 2) Q_{i_1 i_2 \dots i_k}(z) f(z) d\sigma(z)$$

where  $\sigma$  is the uniform probability measure on  $S$ .

When a harmonic homogeneous polynomial  $Q_{p_1 \dots p_m}(z)$  is substituted for  $f(z)$ , we have an explicit computation for the integral.

**Lemma 2.6.**

$$(2.16) \quad \begin{aligned} \int_S n(n + 2) \dots (n + 2k - 2) Q_{i_1 \dots i_k}(z) Q_{p_1 \dots p_m}(z) d\sigma(z) \\ = \begin{cases} k! Q_{i_1 \dots i_k}^{p_1 \dots p_k}, & k = m, \\ 0, & k \neq m, \end{cases} \end{aligned}$$

where  $Q_{i_1 \dots i_k}^{p_1 \dots p_k}$  are the coefficients of  $Q_{i_1 \dots i_k}(z)$  given by (2.11).

3. DECOMPOSITION OF THE LAPLACIAN FOR CONSTANT CURVATURE SPACES

Using the reduction procedure of the preceding section a straightforward (but extremely lengthy) calculation gives as many terms as desired in the expansion of

$$(3.1) \quad E_m[T(\epsilon)(\Phi_\epsilon f)(X_{T(\epsilon)})] - E_m[T(\epsilon)]E_m[(\Phi_\epsilon f)(X_{T(\epsilon)})]$$

in powers of  $\epsilon$ . Note that if the first exit time and place are independent then this expression equals zero. In order to compute this expansion within  $O(\epsilon^{10})$  it is necessary to compute  $u_j(0)$ ,  $1 \leq j \leq 7$ , and  $v_j(0)$ ,  $0 \leq j \leq 7$ . This requires the homogeneous decomposition of the Laplacian up to  $\Delta_5$ . For a general Riemannian manifold  $M$ , this is a long and difficult calculation involving complicated curvature terms. However, if  $M$  is a manifold with constant sectional curvature, the operators  $\Delta_j$  are somewhat simpler and the computation becomes feasible.

**Proposition 3.1.** *Let  $M$  be a Riemannian manifold of dimension  $n$  with constant sectional curvature  $\kappa$ . Then the homogeneous decomposition of the Laplacian with respect to normal coordinates has  $\Delta_j = 0$  if  $j$  is odd and*

$$(3.2) \quad \begin{aligned} \Delta_0 &= \frac{\kappa}{3}\{|x|^2\Delta_{-2} - x^i x^j \partial_i \partial_j - 2(n-1)x^i \partial_i\}, \\ \Delta_2 &= \frac{\kappa^2}{45}\{3|x|^4\Delta_{-2} - 3|x|^2 x^i x^j \partial_i \partial_j - 4(n-1)|x|^2 x^i \partial_i\}, \\ \Delta_4 &= \frac{\kappa^3}{945}\{10|x|^6\Delta_{-2} - 10|x|^4 x^i x^j \partial_i \partial_j - 12(n-1)|x|^4 x^i \partial_i\}. \end{aligned}$$

*Proof.* We show this for the case  $\kappa > 0$ . Here  $M$  is locally isometric to a sphere  $S^n(\kappa)$ . The case with  $\kappa < 0$  is similar with trigonometric functions replaced by hyperbolic functions. Let  $\{x^i\}$  be normal coordinates at  $m$  and  $r = |x|$  be the geodesic distance from  $m$ . Then the Laplacian for  $S^n$  is

$$(3.3) \quad \begin{aligned} \Delta &= \frac{\partial^2}{\partial r^2} + (n-1)\sqrt{\kappa} \cot(r\sqrt{\kappa}) \frac{\partial}{\partial r} + \frac{\kappa}{\sin^2(r\sqrt{\kappa})} \Delta_{S^{n-1}} \\ &= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}} \\ &\quad + \frac{n-1}{r} (r\sqrt{\kappa} \cot(r\sqrt{\kappa}) - 1) \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{\kappa r^2}{\sin^2(r\sqrt{\kappa})} - 1 \right) \Delta_{S^{n-1}} \end{aligned}$$

where  $\Delta_{S^{n-1}}$  is the Laplacian on the  $(n-1)$ -dimensional unit sphere (the angular components). Now use the facts

$$(3.4) \quad \begin{aligned} r\partial r &= x^i \partial_i, \\ \Delta_{-2} &= \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{n-1}}. \end{aligned}$$

These imply

$$(3.5) \quad \Delta_{S^{n-1}} = r^2 \Delta_{-2} - (n-1)x^i \partial_i - x^i x^j \partial_i \partial_j.$$

Define the functions

$$(3.6) \quad \begin{aligned} \Psi_1(t^2) &\doteq \frac{1}{t^2} \left( \frac{t^2}{\sin^2 t} - 1 \right), \\ \Psi_2(t^2) &\doteq \frac{1 - t \cot t}{t^2}. \end{aligned}$$

Then we have

$$(3.7) \quad \begin{aligned} \Delta &= \Delta_{-2} + \kappa r^2 \Psi_1(\kappa r^2) \Delta_{-2} - \kappa \Psi_1(\kappa r^2) x^i x^j \partial_i \partial_j \\ &\quad - (n - 1) \kappa (\Psi_1(\kappa r^2) + \Psi_2(\kappa r^2)) x^i \partial_i. \end{aligned}$$

Finally, using the Maclaurin series for  $\Psi_1$  and  $\Psi_2$ ,

$$(3.8) \quad \begin{aligned} \Psi_1(t^2) &= \frac{1}{3} + \frac{1}{15} t^2 + \frac{2}{189} t^4 + \dots, \\ \Psi_2(t^2) &= \frac{1}{3} + \frac{1}{45} t^2 + \frac{2}{945} t^4 + \dots, \end{aligned}$$

we get the decomposition in (3.2). □

#### 4. RESULTS FOR PRODUCTS OF CONSTANT CURVATURE SPACES

Now consider the metric product manifold  $M = M_1 \times \dots \times M_K$  where, for  $1 \leq \alpha \leq K$ ,  $M_\alpha$  is a manifold of dimension  $n_\alpha$  and has constant sectional curvature  $\kappa_\alpha$ . Let  $n = \sum_{\alpha=1}^k n_\alpha$  and  $N_\beta = \sum_{\alpha=1}^\beta n_\alpha$ . Consider the normal coordinate system  $\{x^j\}_{1 \leq j \leq n}$  where  $\{x^j\}_{N_{\alpha-1}+1 \leq j \leq N_\alpha}$  is a normal coordinate system for the  $\alpha$ th factor space  $M_\alpha$ . Let  ${}_\alpha \Delta$  and  ${}_\alpha \Delta_j$  be the Laplacian and its homogeneous decomposition, respectively, with respect to these coordinates on  $M_\alpha$ . Then the Laplacian for  $M$  is

$$(4.1) \quad \Delta = \sum_{\alpha=1}^K {}_\alpha \Delta$$

and has homogeneous decomposition given by

$$(4.2) \quad \Delta_j = \sum_{\alpha=1}^K {}_\alpha \Delta_j.$$

Define the Kronecker delta for each factor by

$$(4.3) \quad {}_\alpha \delta_{ij} = \begin{cases} 1, & \text{if } N_{\alpha-1} + 1 \leq i = j \leq N_\alpha, \\ 0, & \text{otherwise.} \end{cases}$$

We use many iterations of the reduction process described in Section 2 with the decomposition of the Laplacian given above. The details of each step are too long to show here. The result of this process is the following.

**Theorem 4.1.** *Let  $M$  be given as above. Then*

$$(4.4) \quad \begin{aligned} &E_m[T(\epsilon)(\Phi_\epsilon f)(X_{T(\epsilon)})] - E_m[T(\epsilon)]E_m[(\Phi_\epsilon f)(X_{T(\epsilon)})] \\ &= \epsilon^6 \int_S B_{ij} Q_{ij}(z) f(z) d\sigma(z) + \epsilon^8 \int_S C_{ij} Q_{ij}(z) f(z) d\sigma(z) \\ &\quad + \epsilon^8 \int_S D_{ijkl} Q_{ijkl}(z) f(z) d\sigma(z) + O(\epsilon^{10}), \end{aligned}$$

where

$$(4.5) \quad B_{ij} = -\frac{\sum_{\alpha=1}^K \kappa_\alpha^2 (n_\alpha - 1)(n_\alpha - 3) {}_\alpha \delta_{ij}}{180(n + 4)^2(n + 6)}$$

and

$$\begin{aligned}
 C_{ij} = & \left( 6n(n+4) \sum_{\alpha=1}^K \kappa_\alpha^3 (n_\alpha - 1)(n_\alpha - 3)(2n_\alpha - 17)_\alpha \delta_{ij} \right. \\
 & - 7n(n-16) \sum_{\alpha=1}^K \kappa_\alpha^3 (n_\alpha - 1)^2 (n_\alpha - 3)_\alpha \delta_{ij} \\
 (4.6) \quad & - 35(3n+8) \sum_{\alpha,\beta=1}^K \kappa_\alpha^2 \kappa_\beta (n_\alpha - 1)(n_\alpha - 3)(n_\beta - 1)(n_\beta)_\alpha \delta_{ij} \\
 & \left. - 35(n+8) \sum_{\alpha,\beta=1}^K \kappa_\alpha \kappa_\beta^2 (n_\alpha - 1)(n_\beta - 1)(n_\beta - 3)(n_\beta)_\alpha \delta_{ij} \right) \\
 & / (18900n(n+4)^3(n+6)(n+8))
 \end{aligned}$$

and

$$\begin{aligned}
 D_{ijkl} = & - \frac{\sum_{\alpha=1}^K \kappa_\alpha^3 (n_\alpha - 1)(n_\alpha - 3)(\alpha \delta_{ij})(\alpha \delta_{kl})}{1890(n+8)^2(n+10)} \\
 (4.7) \quad & + \frac{\sum_{\alpha,\beta=1}^K \kappa_\alpha^2 \kappa_\beta (n_\alpha - 1)(n_\alpha - 3)(n_\beta - 1)(\alpha \delta_{ij})(\beta \delta_{kl})}{2160(n+4)^2(n+6)}.
 \end{aligned}$$

**Theorem 4.2.** *Let  $M$  be given as above and suppose  $T(\epsilon)$  and  $X_{T(\epsilon)}$  are independent for  $\epsilon$  small enough so that  $B_m(\epsilon)$  is a normal ball. Then, for each  $\alpha$ ,  $1 \leq \alpha \leq K$ ,  $\kappa_\alpha = 0$  or  $n_\alpha = 3$ .*

*Proof.* Since  $T(\epsilon)$  and  $X_{T(\epsilon)}$  are independent, the left-hand side of (4.4) is 0. Replace  $f$  in formula (4.4) with  $Q_{pq}$  and consider the  $\epsilon^6$ . By Lemma 2.6,

$$(4.8) \quad B_{pq} - \frac{B_{ij} \delta_{ij}}{n} \delta_{pq} = 0.$$

I.e., the matrix with components  $B_{pq}$  is a multiple of the identity. It follows then from (4.5) that

$$(4.9) \quad \mu \doteq \kappa_1^2(n_1 - 1)(n_1 - 3) = \kappa_2^2(n_2 - 1)(n_2 - 3) = \dots = \kappa_K^2(n_K - 1)(n_K - 3).$$

If this common value  $\mu = 0$ , then the conclusion follows since  $n_\alpha = 1$  implies  $\kappa_\alpha = 0$ .

If  $\mu \neq 0$ , replace  $f$  in formula (4.4) with  $Q_{pq}$  and consider the  $\epsilon^8$  term. By Lemma 2.6,

$$(4.10) \quad C_{pq} - \frac{C_{ij} \delta_{ij}}{n} \delta_{pq} = 0.$$

As before, the matrix  $C_{pq}$  is a multiple of the identity and substituting  $\mu$  in (4.6) it follows that

$$\begin{aligned}
 & \mu \left( 6n(n+4)(n_1+2)\kappa_1 + 7(3n+8) \sum_{\beta=1}^K \kappa_\beta(n_\beta-1)n_\beta \right) \\
 (4.11) \quad & = \mu \left( 6n(n+4)(n_2+2)\kappa_2 + 7(3n+8) \sum_{\beta=1}^K \kappa_\beta(n_\beta-1)n_\beta \right) \\
 & = \dots = \mu \left( 6n(n+4)(n_K+2)\kappa_K + 7(3n+8) \sum_{\beta=1}^K \kappa_\beta(n_\beta-1)n_\beta \right).
 \end{aligned}$$

Assuming  $\mu \neq 0$ , this reduces to

$$(4.12) \quad (n_1+2)\kappa_1 = (n_2+2)\kappa_2 = \dots = (n_K+2)\kappa_K.$$

Taking this together with (4.9) we get

$$(4.13) \quad \frac{(n_1-1)(n_1-3)}{(n_1+2)^2} = \frac{(n_2-1)(n_2-3)}{(n_2+2)^2} = \dots = \frac{(n_K-1)(n_K-3)}{(n_K+2)^2}.$$

Now  $f(x) = (x-1)(x-3)/(x+2)^2$  is strictly increasing for  $x \geq 7/4$ . Thus when  $\mu \neq 0$ , (4.13) implies that  $n_1 = n_2 = \dots = n_K$  and hence, from (4.12),  $\kappa_1 = \kappa_2 = \dots = \kappa_K$ .

Now replace  $f$  in formula (4.4) with  $Q_{pqrs}$  and consider the  $\epsilon^8$  term. By Lemma 2.6,

$$(4.14) \quad D_{ijkl}Q_{ijkl}^{pqrs} = 0.$$

When  $n_1 = n_2 = \dots = n_K$  and  $\kappa_1 = \kappa_2 = \dots = \kappa_K$ , (4.7) reduces to

$$(4.15) \quad D_{ijkl} = -\frac{\kappa_1^3(n_1-1)(n_1-3) \sum_{\alpha=1}^K (\alpha\delta_{ij})(\alpha\delta_{kl})}{1890(n+8)^2(n+10)} + \frac{\kappa_1^3(n_1-1)^2(n_1-3)(\delta_{ij})(\delta_{kl})}{2160(n+4)^2(n+6)}.$$

Since  $(\delta_{ij})(\delta_{kl})Q_{ijkl}^{pqrs} = 0$  and  $\sum_{\alpha=1}^K (\alpha\delta_{ij})(\alpha\delta_{kl})Q_{ijkl}^{pqrs} \neq 0$ , it follows in this case also that  $\kappa_1 = 0$  or  $n_1 = 3$ . □

### 5. REMARKS

These results completely determine which products of constant curvature spaces have independent exit time and place for small balls. The harmonic spaces are the only other Riemannian manifolds known to satisfy this independence property. One might expect to classify more completely manifolds with this independence property by considering curvature conditions that follow from the same level of asymptotics for a general symmetric space. The computations required would be even more complicated than the special case we consider here.

We also note that due to the large number of calculations involved in the reduction process, we have used the symbolic algebra capabilities of *Mathematica*(TM) to carry out some of the calculations. *Mathematica*(TM), is a mathematical software system developed and distributed by Wolfram Research, Inc.

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