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A REMARK ABOUT $\Lambda(p)$-SETS AND ROSENTHAL SETS

DANIEL LI

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Abstract. There exist $\Lambda(p)$-sets which are not Rosenthal sets.

Résumé. Il existe des ensembles $\Lambda(p)$ qui ne sont pas des ensembles de Rosenthal.

Introduction

In this note, we point out that the existence of $\Lambda(p)$-sets which are not Rosenthal sets follows immediately from a theorem of J. Bourgain and older results of Y. Katznelson and F. Lust-Piquard.

Recall that a subset Λ of $\mathbb{Z}$ is called a *Sidon set* if every continuous function with spectrum in Λ has an absolutely convergent Fourier series, and is called a $\Lambda(p)$*-set* ($p > 1$) if there is a q with $1 < q < p$ such that the norms L^p and L^q are equivalent on the subspace generated by the trigonometric polynomials with spectrum in Λ. Every Sidon set is a $\Lambda(p)$-set for every $p > 1$ (see [20]). Y. Meyer ([18]) said that Λ is a *Riesz set* if every measure with spectrum in Λ is absolutely continuous; it is well-known that every $\Lambda(p)$-set is a Riesz set ([20], Theorem 5.1). There is a fourth interesting class: Λ is a *Rosenthal set* if every bounded measurable function with spectrum in Λ is equal a.e. to a continuous function. It is clear that every Sidon set is a Rosenthal set, and H.P. Rosenthal constructed non-Sidon sets with the above property ([19]; see also [1]). Since every Rosenthal set is a Riesz set ([11]), it is natural to ask whether every $\Lambda(p)$-set is a Rosenthal set. It might be thought that for a reflexive subspace X of $L^1(\mathbb{T})$ the subspace $X \cap \mathcal{C}$ of $\mathcal{C}(\mathbb{T})$ does not contain c_0. It is not true.

Before explaining why, let us give a Banach space argument, due to F. Lust-Piquard ([15], Theorem 3.1), for the fact that every Rosenthal set is a Riesz set. If Λ is a Rosenthal set, then so is $(-\Lambda)$. Then, as a separable dual space, $\mathcal{C}_{(-\Lambda)}(\mathbb{T}) = L^\infty_{(-\Lambda)}(\mathbb{T})$ has no subspace isomorphic to c_0. For every measure μ with spectrum in Λ, the convolution operator $C_{\check{\mu}}$ from $L^\infty(\mathbb{T})$ into itself is then weakly compact (since its image is $\mathcal{C}_{(-\Lambda)}(\mathbb{T})$ and $L^\infty(\mathbb{T})$ has Pełczynski's property V : see [9], page 116). This operator is the adjoint of C_μ from L^1 into itself, which is then also weakly compact. So μ is absolutely continuous ([10], Theorem 12, page 75, and page 90).

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MAIN RESULT

Theorem. *There exist sets $\Lambda \subseteq \mathbb{Z}$ which are $\Lambda(p)$-sets for all $p > 1$, but which are not Rosenthal sets*

Proof. The construction is made randomly.

The set $\mathbb{N}$ of the integers will be divided into the subsets

$$N_k = \{2^k + 1, \cdots, 2^{k+1}\} \qquad (k \geq 1).$$

Independently for $k \geq 1$, independent $(0, 1)$ random variables ξ_n are chosen for $n \in N_k$ in such a way that $P(\xi_n = 1) = k/2^k$. Then $\Lambda = \Lambda(\omega)$ is defined by

$$\Lambda = \bigcup_{k \geq 1} \Lambda_k,$$

where

$$\Lambda_k = \Lambda_k(\omega) = \{n \in N_k \ ; \ \xi_n(\omega) = 1\} \ .$$

Since $n(k/2^k) > k$ for $n \in N_k$, Proposition 8.2 (i) of [7] asserts that Λ is almost surely homogeneously distributed (actually, Proposition 2.1 (2) of [6] would suffice for our purpose).

We recall the

Definition. Let Λ be a subset of $\mathbb{N}$, ordered in the natural way in a sequence $(\lambda_k)_{k \geq 1}$. It is said to be *homogeneously distributed* if for every $x \in \mathbb{T} \setminus \{0\}$ we have

$$\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} e^{2\pi i \lambda_k x} = 0.$$

This terminology is used in [5] ; in [14], such a sequence is said to be *Hartman uniformly distributed*, and J. Bourgain called it an *ergodic sequence* in [7], because ergodic sequences are those for which there is a mean ergodic theorem (see [3]; connected results can be found in [2], and [4]). Such sequences are dense in the Bohr group.

Now, for such a set Λ, the upper semi-continuous function $\mathbb{1}_{\{0\}}$ defines, by Lebesgue's dominated convergence theorem, an element in $\mathcal{C}_\Lambda^{\perp\perp}$. By a perturbation argument due to A. Pełczynski (see [21], page 446), and Bessaga-Pełczynski theorem, we obtain that $\mathcal{C}_\Lambda$ contains a subspace isomorphic to c_0.

Remark. This last result is a particular case of a result of F. Lust-Piquard ([16], lemme 4, and [15], Proposition 3.1) : if $\mathcal{C}_\Lambda^{\perp\perp}$ contains a non-zero element ℓ which defines an element of $c_0(\mathbb{T})$, then $\mathcal{C}_\Lambda$ contains a subspace isomorphic to c_0. A beautiful application of this fact is in [17]; it is proven there, in particular, that the set of squares $\{n^2 \ ; \ n \geq 1\}$ is not a Rosenthal set; recall that this set is not $\Lambda(4)$ ([20], 4.6), but it is not known whether it is $\Lambda(2)$.

To finish the proof, it remains to use [13], §2, which says that this random set $\Lambda = \Lambda(\omega)$ is also almost surely a $\Lambda(p)$-set for all $p > 1$.

Since no proof of this later fact has been published, we shall sketch one.

As usual, for $p > 2$, by Littlewood-Paley theory, it suffices to show that Λ_k is a $\Lambda(p)$-set with a constant $C(p)$ (defined by $\|f\|_p \leq C(p)\|f\|_2$) independent of k.

Moreover, it suffices to do it for even integers $p = 2s$. By [20], Theorem 4.5 (b), it suffices to see that the number of ways $r_s(\Lambda_k, j)$ to write j with s elements of Λ_k is a bounded function of j and k. We only check this for $s = 2$.

By definition

$$\sum_{j=1}^{\infty} r_2(\Lambda_k, j) z^j = \left[\sum_{n \in N_k} \xi_n z^n \right]^2$$

so

$$r_2(\Lambda_k, j) = \sum_{m+n=j} \xi_n \xi_m$$

for $j \in M_k = N_k + N_k = \{2^{k+1} + 2, \cdots, 2^{k+2}\}$, and 0 otherwise. More precisely, for even j in M_k, we have

$$r_2(\Lambda_k, j) = \xi_{j/2} + 2 \sum_{n=2^k+1}^{\frac{j}{2}-1} \xi_n \xi_{j-n}$$

and for odd j

$$r_2(\Lambda_k, j) = 2 \sum_{n=2^k+1}^{\frac{j-1}{2}} \xi_n \xi_{j-n}.$$

Then, we have

$$\begin{aligned}
&\mathbf{P}(\sup_{j \geq 1} r_2(\Lambda_k, j) \geq 3) \\
&\quad \leq \sum_{h=2^k+2}^{2^{k+1}} \sum_{n=2^k+1}^{h-1} \mathbf{P}(\xi_h = 1, \xi_n = 1, \xi_{2h-n} = 1) \\
&\qquad + \sum_{h=2^k+2}^{2^{k+1}} \sum_{\substack{\{n_1, n_2\} \subseteq \\ \{2^k+1, \cdots, h-1\}}} \mathbf{P}(\xi_{n_1} = 1, \xi_{n_2} = 1, \xi_{2h-n_1} = 1, \xi_{2h-n_2} = 1) \\
&\qquad + \sum_{l=2^k+1}^{2^{k+1}-1} \sum_{\substack{\{n_1, n_2\} \subseteq \\ \{2^k+1, \cdots, l\}}} \mathbf{P}(\xi_{n_1} = 1, \xi_{n_2} = 1, \xi_{2l+1-n_1} = 1, \xi_{2l+1-n_2} = 1) \\
&\quad = \frac{2^k(2^k-1)}{2} \left(\frac{k}{2^k} \right)^3 + \sum_{h=2^k+2}^{2^{k+1}} \binom{h-1-2^k}{2} \left(\frac{k}{2^k} \right)^4 \\
&\qquad\qquad + \sum_{l=2^k+1}^{2^{k+1}-1} \binom{l-1-2^k}{2} \left(\frac{k}{2^k} \right)^4 \\
&\quad \leq \frac{k^3}{2^k} + \frac{k^4}{2^k}\ ,
\end{aligned}$$

and so, by the Borel-Cantelli lemma, almost surely, for enough large k, we have $r_2(\Lambda_k, j) \leq 2$. □

Remark. Consider the Bohr group $b\mathbb{Z}$, (i.e. the dual group of the circle group with the discrete topology), and suppose that its Haar measure $\tilde{m}$ is the weak-star limit of a *sequence* of probability measures ν_n carried by finite subsets of Λ. Their Fourier transforms are then trigonometric polynomials p_n with spectrum in Λ, which converge pointwise to 1 if $x = 0$ and to 0 when $x \neq 0$. By Lebesgue's dominated convergence theorem, we are in the setting of F. Lust-Piquard's result, and we can conclude that $\mathcal{C}_\Lambda$ contains a subspace isomorphic to c_0. Sets containing homogeneously distributed sequences are examples of such sets, and these sets are FC^+-sets (see [6], Lemma 2.2) and are dense in the Bohr group.

In [12], Lemma 1.2, Y. Katznelson observed that Λ is dense in $b\mathbb{Z}$ if and only if $\tilde{m}$ belongs to the weak-star closure of the probability measures carried by Λ. Since Lebesgue's dominated convergence theorem fails for convergence along filters, we cannot get the same conclusion. Whether $\mathcal{C}_\Lambda$ contains ℓ^n_∞ uniformly (i.e. Λ is not a Sidon set, by Bourgain-Milman theorem [8]) when Λ is dense in $b\mathbb{Z}$ is an old open problem.

REFERENCES

1. R. Blei, *On trigonometric series associated with separable, translation invariant subspaces of $L^\infty(G)$.*, Trans. Amer. Math. Soc. **173** (1972), 491-499. MR **47:**2269
2. J. R. Blum - R. Cogburn, *On ergodic sequences of measures*, Proc. Amer. Math. Soc. **51** (1975), 359-365. MR **51:**8736
3. J. Blum - B. Eisenberg, *Generalized summing sequences and the mean ergodic theorem*, Proc. Amer. Math. Soc. **42** (1974), 423-429. MR **48:**8749
4. J. R. Blum - B. Eisenberg - L.-S. Hahn, *Ergodic theory and the measure of sets in the Bohr group*, Acta Sci. Math. Szeged **34** (1973), 17-24. MR **51:**10536
5. M. Boshernitzan, *Homogeneously Distributed Sequences and Poincaré Sequences of Integers of Sublacunary Growth*, Monat. Math. **96** (1983), 173-181. MR **85k:**11034
6. J. Bourgain, *Ruzsa's problem on sets of recurrence*, Israël J. Math. **59** (1987), 150-166. MR **89d:**11012
7. J. Bourgain, *On the maximal ergodic theorem for certain subsets of the integers*, Israël J. Math **61** (1988), 39-72. MR **89f:**28037a
8. J. Bourgain - V. Milman, *Dichotomie du cotype pour les espaces invariants*, CRAS Paris, Série I, 300 (1985) 263-266. MR **86i:**43010
9. J. Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Math n°92, Springer (1984). MR **85i:**46020
10. J. Diestel - J.J. Uhl Jr., *Vector Measures*, Math. Surveys n°15. MR **56:**12216
11. R.E. Dressler - L. Pigno, *Rosenthal sets and Riesz sets*, Duke Math. J. **41** (1974), 675-677. MR **58:**23352
12. Y. Katznelson, *Sequences of integers dense in the Bohr group*, Proc. Royal Inst. Techn. Stockholm (1973), 79-86.
13. Y. Katznelson, *Suites aléatoires d'entiers.*, Lecture Notes in Math 336, Springer-Verlag Berlin (1973) 148-152. MR **53:**1176
14. L. Kuipers - H. Niederreiter, *Uniform Distribution of Sequences*, Wiley-Interscience, New-York, 1974. MR **54:**7415
15. F. Lust-Piquard, *Propriétés géométriques des sous-espaces invariants par translation de $L^1(G)$ et $\mathcal{C}(G)$.*, Sémin. Géom. Espaces Banach. Exposé n°26 (1977-78). Ecole Polytechnique. MR **80h:**46024
16. F. Lust-Piquard, *Eléments ergodiques et totalement ergodiques dans $L^\infty(\Gamma)$.*, Studia Math. **69** (1981), 191-225. MR **84h:**43003
17. F. Lust-Piquard, *Bohr local properties of $\mathcal{C}_\Lambda(\mathbb{T})$.*, Colloq. Math. **58** (1989), 29-38. MR **91c:**43009
18. Y. Meyer, *Spectres des mesures et mesures absolument continues*, Studia Math. **30** (1968), 87-99. MR **37:**3281
19. H.P. Rosenthal, *On trigonometric Series associated with weak* closed subspaces of continuous functions*, Journ. Math. Mech. **17** (1967), 485-490. MR **35:**7064

20. W. Rudin, *Trigonometric Series with Gaps.*, Journal of Math. and Mech. **9** (1960), 203-227. MR **22:**6972
21. I. Singer, *Bases in Banach Spaces I.*, Springer Verlag Berlin-Heidelberg-New-York (1970). MR **45:**7451

Analyse Harmonique, Université Paris-Sud, Mathématiques, Bâtiment 425, 91405 Orsay, France
E-mail address: `daniel.li@math.u-psud.fr`

Equipe d'Analyse, Université Paris VI, 4 Place Jussieu, Boîte 186, 75252 Paris cedex 05, France
Current address: Université d'Artois, Faculté Jean Perrin, rue Jean Souvraz, SP 18, 62307 Lens Cedex, France
E-mail address: `li@poincare.univ-artois.fr`