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WHEN DO CONNECTED SPACES HAVE NICE CONNECTED PREIMAGES?

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ABSTRACT. We prove that every connected Tychonoff space is an open monotone continuous image of a connected strictly σ-discrete left-separated Tychonoff space. For wide classes of connected spaces it is established that they have a finer Hausdorff strictly σ-discrete connected topology. Another result is that a finer Tychonoff connected strictly σ-discrete topology exists for any Tychonoff topology with a countable network. We show that there are Tychonoff connected spaces with countable network which are not continuous images of connected second countable spaces. It is established also that every connected Tychonoff space $\mathcal{X}$ is an open retract of a connected homogeneous Tychonoff space, while it is not always possible to find a finer connected homogeneous topology on $\mathcal{X}$.

0. INTRODUCTION

There are quite a few theorems about representing topological spaces as continuous images of spaces with additional properties. For example, it is trivial that any topological space X is a continuous image of a metrizable space, namely a discrete one. But sometimes it is far from trivial to represent X as a continuous image of a metric (or "good" in some another sense) space if we want to preserve some properties or some algebraic structure on X. Any Tychonoff X with a countable network is a continuous image of a second countable Tychonoff space [ArPo, Ch. II, Problem 148], but if X is a topological group and we want to represent X as a continuous homomorphic image of a second countable topological group, then it is not always possible [PeSh].

In this paper we are interested in preserving connectedness of spaces in quest of a preimage with additional properties. We construct a connected Tychonoff space with a countable network which cannot be represented as a continuous image of a connected space with a countable base.

We also establish an analogue of the theorem of Junnila [Ju] which states that every topological space is an open image of a strictly σ-discrete Tychonoff space. We prove that if X is a Tychonoff connected space then there exists a Tychonoff connected left separated strictly σ-discrete space which maps openly onto X. The

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method of proof is essentially different from the one of Junnila, because he gets a hereditarily paracompact preimage which can never be connected.

We show that any ω-resolvable connected space has a finer connected strictly σ-discrete Hausdorff topology. This implies, in particular, that the same is true for every countably compact or sequential Hausdorff space.

It is much harder to prove existence of a finer connected σ-discrete Tychonoff topology. This we managed to do only for connected Tychonoff spaces X of power and weight $\leq 2^\omega$ and with ω_1 a precaliber for X. In particular, any separable Tychonoff connected space of power 2^ω has a finer connected strictly σ-discrete Tychonoff topology.

The last group of results are related to V.V.Uspenskij's theorem [Us]: every connected Tychonoff space is an open retract of a connected homogeneous Tychonoff space, while it is not always possible to strengthen the topology of a connected second countable Tychonoff space to a homogeneous connected topology.

1. Notation and terminology

All spaces are meant to be Hausdorff. A space is called (strictly) σ-discrete, if it is a union of countably many of its (closed) discrete subspaces. If X is a space, then a family $\mathcal{P}$ of (not necessarily open) subsets of X is called a network in X, if any open subset of X is a union of some subfamily of $\mathcal{P}$. A space X is countably tight (or has countable tightness) if for every $A \subset X$ and every $x \in \overline{A}$ there is a countable $B \subset A$ such that $x \in \overline{B}$. We say that X is sequential if for every subset A of X if $A \neq \overline{A}$, then there is a convergent sequence $S \subset A$ with $\overline{S} \backslash A \neq \emptyset$. A family $\mathcal{B}$ of non-empty open subsets of a space X is called a π-base of X if each non-empty open subset of X contains an element from $\mathcal{B}$. A space has countable π-weight if it has a countable π-base. All maps are considered to be continuous if the opposite is not stated explicitly. A well ordering of a set P is called minimal if every initial segment of P has a cardinality $< |P|$. A space X is called left-separated if there is a well-order $<$ on X such that for every $x \in X$ the set $I_x = \{y \in X : y < x\}$ is closed in X. The space $\mathbb{R}$ is the real line with the natural topology; $I = [0, 1]$ is the relevant subspace of $\mathbb{R}$. The end of a proof is denoted by the symbol $\square$. All other notions are standard and can be found in [En].

2. Looking for better connected preimages of connected spaces

It is known, that not every connected Tychonoff space can be represented as a continuous image of a connected sequential space. The relevant example is $\beta\mathbb{R}$ [En, 6.1.E(a)]. Let us prove a simple fact that will have important consequences.

2.1. Proposition. *Let X be an infinite connected space which contains a point x_0 such that there is no convergent sequence (a countable set) S in $X \backslash \{x_0\}$ with $x_0 \in \overline{S}$. Then X cannot be represented as continuous image of a sequential (resp. countably tight) connected space.*

Proof. Suppose, on the contrary, that Y is such a preimage of X under a continuous map f. Consider the set $F = f^{-1}(x_0)$. It is a proper closed subspace of Y. Since space Y is connected and F is not open, there is a convergent sequence (or a countable set) P in $Y \backslash F$ such that $\overline{P} \cap F \neq \emptyset$. Let $S = f(P)$. Then S is a convergent sequence (resp. a countable set) in $X \backslash \{x_0\}$ with $x_0 \in \overline{S}$ — a contradiction. $\square$

2.2. Example. There is a connected compact linearly ordered space which cannot be represented as a continuous image of a connected space of countable tightness.

Proof. The long segment L [En, 3.12.18] is a compact linearly ordered connected space in which the point ω_1 is not in the closure of any countable subset of $L\setminus\{\omega_1\}$. □

It is known [ArPo, Ch.2 Problem 148], that any Tychonoff (Hausdorff) space with a countable network is a continuous image of a second countable Tychonoff space. The following example shows that it is not possible to add connectedness to the hypothesis and to the conclusion of this theorem.

2.3. Example. There is a connected Tychonoff space with countable network which is not representable as a continuous image of a sequential (and hence of a second countable) connected space.

Proof. There is a far point y in $\beta\mathbb{R}\setminus\mathbb{R}$, that is, a point which is not in the closure of any discrete subset of $\mathbb{R}$ [vDa]. The space $\mathbb{R}\cup\{y\}$ with the topology induced from $\beta\mathbb{R}$ has a countable network, but there is no convergent sequence from $\mathbb{R}$ to y. Now apply Proposition 2.1. □

2.4. Corollary. *There exist connected Tychonoff spaces with countable network which do not have finer connected second countable (or even sequential) topology.*

In the realm of Hausdorff spaces there exist countable connected spaces. Every such space has a countable network, but it is not necessarily a continuous image of a sequential connected space.

2.5. Example. There is a countable connected space X which is not a continuous image of a connected sequential space. In particular, X does not have a finer second countable (or even sequential) connected topology.

Proof. Consider a dense in itself countable Tychonoff space Y of countable π-weight (see Section 1 for the definition of countable π-weight) without non-trivial convergent sequences. One can obtain such a Y from rationals announcing all nowhere dense subsets closed. It follows from Theorem 2.2 of [WaWi] that there exists a connected Hausdorff countable extension X of Y with Y open in X. Therefore no point of Y is a limit of a non-trivial convergent sequence in X and we can use 2.1 to conclude that X is not a continuous image of a sequential connected space. □

Now let us prove a positive result. It will be an analogue of the theorem of Junnila [Ju], who established that every regular topological space is an open continuous image of a strictly σ-discrete Tychonoff space. It turns out that connectedness of a space can be preserved in the process of constructing a σ-discrete open preimage.

2.6. Theorem. *Let X be a connected T_i-space ($i = 2, 3, 3\frac{1}{2}$). Then there exists a left-separated connected strictly σ-discrete space T_i-space Y, such that X is an open continuous image of Y.*

Proof. Let $|X| = \tau \geq \omega$. Then the space X^τ is connected, has the same separation axioms as X, and maps openly and continuously onto X. Therefore it suffices to prove our theorem for $Z = X^\tau$. It is clear that $|Z| = 2^\tau$.

Take a set A with $|A| = 2^\tau$. For each $\alpha \in A$ choose a set $T_\alpha \subset A$ such that $|T_\alpha| = 2^\tau$, $\bigcup\{T_\alpha : \alpha \in A\} = A$ and $T_\alpha \cap T_\beta = \emptyset$ if $\alpha \neq \beta$. There are only 2^τ points in all finite faces of the space I^A. Denote the set of those points by $\mathcal{F}$.

It is possible to enumerate $E = Z \times \mathcal{F} \times \mathcal{F}$ with the indices of A. Let $E = \{e_\alpha : \alpha \in A\}$ be such an indexing. For each α we have $e_\alpha = (z_\alpha, p_\alpha, q_\alpha)$, where $z_\alpha \in Z,\ p_\alpha, q_\alpha \in \mathcal{F}$. For every $p \in \mathcal{F}$ denote by $S(p)$ the finite set of indices which determine the face the point p belongs to.

Define a subset Y of the space $Z \times I^A$ in the following way:
(1) $Y = \{y_\alpha : \alpha \in A\}$;
(2) the first coordinate of y_α is z_α, where z_α is the first coordinate of $e_\alpha = (z_\alpha, p_\alpha, q_\alpha)$;
(3) $y_\alpha(\beta) = \begin{cases} p_\alpha(\beta), & \text{if } \beta \in S(p_\alpha), \\ 1, & \text{if } \beta \in T_\alpha \backslash S(p_\alpha), \\ 0, & \text{if } \beta \notin T_\alpha \cup S(p_\alpha). \end{cases}$

Let us prove that the space Y (with the topology induced from $Z \times I^A$) has all properties we promised. Let $f : Y \to Z$ be the natural projection. Then f is a surjective map, because for any $z \in Z$ there is an $\alpha \in A$ such that $z_\alpha = z$ and consequently $f(y_\alpha) = z$.

Now fix a $z \in Z$. Then for each $p \in \mathcal{F}$ the set $\{\alpha \in A : z_\alpha = z \text{ and } p_\alpha = p\}$ has the power 2^τ. This observation will help us to prove that $f^{-1}(z)$ is connected and dense in $\{z\} \times I^A$. The density is immediate, because $f^{-1}(z)$ covers all finite faces of $\{z\} \times I^A$. Therefore f is an open map [ArPo, Ch.2 Problem 340].

Every real-valued continuous map on $F_z = f^{-1}(z) \subset \{z\} \times I^A$ depends on countably many coordinates, so to prove that F_z is connected it suffices to establish that the projection of F_z to any countable face of I^A is connected.

Let B be a countable subset of A and $\pi_B : I^A \to I^B$ the relevant projection. We are going to show that $\pi_B(F_z)$ contains the σ-product $S_B = \{y \in I^B : |\{\alpha \in B : y(\alpha) \neq 0\}| < \omega\}$ in I^B. It is well known (and easy to prove) that S_B is connected, so that $\pi_B(F_z)$ would have a dense connected subspace.

Fix a $y \in S_B$. Then $|\{\alpha : y|_{S(y)} = p_\alpha\}| = 2^\tau$. The sets T_α are disjoint, so there exists a $\beta \in A$ such that $y|_{S(y)} = p_\beta$ and $T_\beta \cap B = \emptyset$. Thus, $y_\beta(B \backslash S(y)) = \{0\}$ so that $\pi_B(y_\beta) = y$ and we proved connectedness of F_z.

Since Y is an open monotone preimage of the connected space Z, we can conclude that Y is a connected space [En, Theorem 6.1.29].

Choose a subspace H of Y with $|H| < 2^\tau$. Then it is closed and discrete in Y. Indeed, for any $\alpha \in A$ there is an index $\beta \in T_\alpha \backslash \bigcup\{S(y) : y \in H\}$. The open set $U = \{y \in Y : y(\beta) > 0\}$ contains y_α and intersects at most one element of H.

Now let $<$ be any minimal well-ordering of Y. Then for every $y \in Y$ the set $I_y = \{z \in Y : z < y\}$ has the cardinality less than 2^τ and by the remark above it is closed. This proves that the space Y is left-separated.

For every natural number n consider the set $Y_n = \{y_\beta \in Y : |S(p_\beta)| = n\}$. Take any $\alpha \in A$ and different points $\beta_1, \dots, \beta_{n+1} \in T_\alpha \backslash S(p_\alpha)$. The set $U = \{y \in Y : y(\beta_i) > \frac{1}{2}\}$ is an open neighbourhood of y_α and intersects at most one element from Y_n. Thus, Y_n is closed and discrete for every n. As $Y = \cup\{Y_n : n \in \omega\}$, it follows that Y is strictly σ-discrete. It is clear from the method of constructing Y that it satisfies the same axioms of separation as X does. $\square$

Now that we have a strictly σ-discrete connected preimage for every connected space, how about a finer connected topology with this property? This question turned out not to be so easy. Recall that a space is called ω-resolvable if it has an infinite family of disjoint dense subsets.

2.7. Theorem. *Suppose that X is a connected space such that there exists a family $\{D_n : n \in \omega\}$ with the following properties:* (1) *$D_n \subset X$, $D_n \subset D_{n+1}$ for each $n \in \omega$;*

(2) *the interior of D_n is empty for every $n \in \omega$;*

(3) *$\bigcup\{D_n : n \in \omega\} = X$.*

Then X has a finer (Hausdorff!) connected σ-discrete topology.

Proof. Denote by $\mathcal{T}$ the topology of X. Fix a family $\{D_n : n \in \omega\}$ of subsets of X as in the hypothesis of the theorem and let $\mathcal{F} = \{X \setminus D_n : n \in \omega\}$.

The family $\mathcal{F}$ is a filter base of dense subsets of X. There exists an ultrafilter $\mathcal{H}$ of dense subsets of X that contains $\mathcal{F}$. Generate a new topology $\mathcal{T}_1$ on X by taking the family $\mathcal{T} \cup \mathcal{H}$ as a pseudobase for $\mathcal{T}_1$. We claim that $\mathcal{T}_1$ is the promised topology. It is clear that $\mathcal{T} \subset \mathcal{T}_1$. Hence the space $Y = (X, \mathcal{T}_1)$ is Hausdorff. It is known that the process of expanding a connected topology by an ultrafilter of dense sets yields a connected topology [GuReSt], so that Y is connected.

Take any $n \in \omega$. Every subset of D_n is closed in X because its complement contains an element of $\mathcal{F}$ and hence belongs to $\mathcal{H}$. Therefore D_n is closed and discrete in X and consequently the space Y is σ-discrete which completes the proof. □

2.8. Corollary. *If X is a connected ω-resolvable space, then it has a finer Hausdorff strictly σ-discrete connected topology.*

Proof. Indeed, if the family $\{B_n : n \in \omega\}$ consists of disjoint dense subspaces of X, then the sets $D_n = B_0 \cup \ldots \cup B_n$ are as required in Theorem 2.7. □

Recall that a space X is called k-space if $A \subset X$ is closed in X as soon as $A \cap K$ is closed for every compact subset K of X. N.V.Velichko proved [Ve] that every k-space is ω-resolvable. Thus we can conclude that

2.9. Corollary. *Every connected k-space (in particular, every connected sequential space) has a finer connected Hausdorff strictly σ-discrete topology.*

W.Comfort and S.García-Ferreira proved [CoGa], that every countably compact space is ω-resolvable. Hence we have

2.10. Corollary. *Any countably compact connected space has a finer connected Hausdorff strictly σ-discrete topology.*

It is clear that it is more difficult to construct finer Tychonoff connected strictly σ-discrete topologies than the Hausdorff ones. Our results in this direction are more modest.

We will need the following lemma which was proved in [STTWW].

2.11. Lemma. *Let X, Y be connected spaces and S a dense subset of the product $\Pi = X \times Y$ with $\pi_X(S) = X$, where $\pi_X : X \times Y \to X$ is the projection. If U and V are non-empty disjoint open sets in $X \times Y$ with $S \subset U \cup V$ and $\Phi = cl_\Pi U \cap cl_\Pi V$, then the set $\pi_X(\Phi)$ has a non-empty interior in X. Further, if both U and V are regular open in $X \times Y$ then there exists a non-empty open set W in X such that $W \subset \pi_X(\Phi)$ and $U \cap \pi_X^{-1}(x) \neq \emptyset \neq V \cap \pi_X^{-1}(x)$ for all $x \in W$.*

Recall that ω_1 is a precaliber of a space X if any uncountable family γ of non-empty open subsets of X has an uncountable subfamily μ with the finite intersection property ($\equiv \cap\mu_1 \neq \emptyset$ for any finite $\mu_1 \subset \mu$).

2.12. Theorem. *Suppose that X is a Tychonoff connected space such that $|X| \le 2^\omega$ and $w(X) \le 2^\omega$. If ω_1 is a precaliber of X, then X has a finer Tychonoff connected strictly σ-discrete topology.*

Proof. It follows from the connectedness of X that $|U| = 2^\omega$ for each non-empty open $U \subset X$. We shall need an auxiliary space M, which is a simpler version of the one constructed in Theorem 2.6 for $\tau = \omega$.

Take a set A with $|A| = 2^\omega$. For each $\alpha \in A$ choose a set $T_\alpha \subset A$ such that $|T_\alpha| = 2^\omega$, $\bigcup\{T_\alpha : \alpha \in A\} = A$ and $T_\alpha \cap T_\beta = \emptyset$ if $\alpha \ne \beta$. There are only 2^ω points in all finite faces of the space I^A. Denote the set of those points by $\mathcal{F}$.

It is possible to enumerate $E = \mathcal{F} \times \mathcal{F}$ using the set A as an index set. Let $E = \{e_\alpha : \alpha \in A\}$ be such an indexing. For each α we have $e_\alpha = (p_\alpha, q_\alpha)$, where $p_\alpha, q_\alpha \in \mathcal{F}$. For every $p \in \mathcal{F}$ denote by $S(p)$ the finite set of indices which determine the face the point p belongs to.

Define a subset M of the space I^A in the following way:

(1) $M = \{y_\alpha : \alpha \in A\}$;

(2) $y_\alpha(\beta) = \begin{cases} p_\alpha(\beta), & \text{if } \beta \in S(p_\alpha), \\ 1, & \text{if } \beta \in T_\alpha \setminus S(p_\alpha), \\ 0, & \text{if } \beta \notin T_\alpha \cup S(p_\alpha). \end{cases}$

The same reasoning as in 2.6 shows that the space M (with the topology induced from I^A) is connected and strictly σ-discrete. Moreover, if P is a subset of M with $|P| < 2^\omega$, then P is closed and discrete — this too, was proved in 2.6.

We claim that

($*$) $M \setminus P$ is connected for every $P \subset M$ such that $|P| < 2^\omega$.

It suffices to prove that the projection of $M \setminus P$ to any countable face contains the σ-product of this face.

Let B be a countable subset of A and $\pi_B : I^A \to I^B$ the relevant projection. We are going to show that $\pi_B(M \setminus P)$ contains the σ-product $S_B = \{y \in I^B : |\{\alpha \in B : y(\alpha) \ne 0\}| < \omega\}$ in I^B.

Fix a $y \in S_B$. Then $|\{\alpha : y|_{S(y)} = p_\alpha \text{ and } y_\alpha \in M \setminus P\}| = 2^\omega$. The sets T_α are disjoint, so there exists a $\beta \in A$ such that $y_\beta \in M \setminus P$, $y|_{S(y)} = p_\beta$ and $T_\beta \cap B = \emptyset$. Thus, $y_\beta(B \setminus S(y)) = \{0\}$ so that $\pi_B(y_\beta) = y$ and we proved connectedness of $M \setminus P$.

Next, we are going to construct a discontinuous injection $f : X \to M$ such that the graph $F = \{(x, f(x)) : x \in X\}$ of the function f is connected in the product $X \times M$. This will enable us to show that X has the desired topology. Indeed, the projection of F to M is an injection and therefore F is strictly σ-discrete. On the other hand, the projection of F onto X is a continuous bijection. Hence F can be considered as X with a finer topology. As we proved, this topology is a Tychonoff strictly σ-discrete connected one.

Observe first that ω_1 is a precaliber of M, because M is dense in I^A. Therefore ω_1 is a precaliber of $X \times M$, due to the fact that precalibers are multiplicative [Arh]. If ω_1 is a precaliber of Z, then the Souslin number $c(Z)$ of the space Z is countable [Arh]. Thus, $c(X \times M) = \omega$.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases in X and M respectively such that the cardinality of each of them does not exceed continuum.

Any nonempty regular open set in $X \times M$ can be represented as the interior of the closure of the union of a disjoint (and hence countable) family of sets of the form $U \times V$, where $U \in \mathcal{B}$ and $V \in \mathcal{C}$. This shows that $X \times M$ has no more than 2^ω regular open sets.

Let $\{U_\alpha : \alpha \in A\}$ be an enumeration of all regular open proper sets in $X \times M$. Here we call a regular open set $U \subset X \times M$ proper if $\emptyset \neq U \subset \overline{U} \neq X \times M$. Let $X = \{x_\alpha : \alpha \in A\}$ be any enumeration of the set X with the indices of A. From now on we assume A to be a minimally well-ordered set.

Suppose that $\alpha \in A$ and that for each $\beta < \alpha$ we have a subset $X_\beta \subset X$ and $f_\beta : X_\beta \to M$ with the following properties:
(i) $\{x_\gamma : \gamma < \beta\} \subset X_\beta$ and $|X_\beta| < 2^\omega$ for every $\beta < \alpha$;
(ii) f_β is an injection for all $\beta < \alpha$;
(iii) $X_\gamma \subset X_\beta$ and $f_\beta|_{X_\gamma} = f_\gamma$ if $\gamma < \beta$;

The set $g = \bigcup\{f_\beta : \beta < \alpha\}$ is an injection from $T = \bigcup\{X_\beta : \beta < \alpha\}$ to M. Consider the set U_α. If there exists a point (x, m) in the boundary of U_α such that $x \notin \bigcup\{X_\beta : \beta < \alpha\}$ and $m \notin g(T)$, then let $X_\alpha = T \cup \{x\} \cup \{x_\alpha\}$. Define f_α to be g on T and $f_\alpha(x) = m$. If $x_\alpha = x$ or $x_\alpha \in T$, then the definition of X_α and f_α is over. If not, then let $f_\alpha(x_\alpha) = z$, where z is any point from $M \backslash (g(T) \cup \{m\})$. It is clear that for X_α and f_α the properties (i)-(iii) hold as well.

If the boundary of U_α does not have a point (x, m) as above, take

$$(x, m) \in U_\alpha \backslash ((T \times M) \cup (X \times g(T))),$$

which is possible because $|T| < 2^\omega$. Let $X_\alpha = T \cup \{x\} \cup \{x_\alpha\}$. Define f_α to be g on T and $f_\alpha(x) = m$. If $x_\alpha = x$ or $x_\alpha \in T$, then the definition of X_α and f_α is over. If not, then let $f_\alpha(x_\alpha) = z$, where z is any point from $M \backslash (g(T) \cup \{m\})$. It is clear that for X_α and f_α the properties (i)-(iii) hold as well.

Now, consider the map $f = \cup\{f_\alpha : \alpha \in A\}$. By the inductive construction, conditions (i)-(iii) imply that f is an injection of X to M. We are going to show that the graph F of f is connected.

Observe first of all that F is dense in $X \times M$. Indeed, each open set V of $X \times M$ contains a closure of a proper regular open set U_α for some $\alpha \in A$. In the construction of f at the α-th step we added a point (x, m) from the closure of U_α to the graph F. Therefore $(x, m) \in V$ which proves that F is dense.

Now if $F = F_1 \cup F_2$ where F_1 and F_2 are open and disjoint in F, then there exist open regular disjoint sets U and V with $U \cap F = F_1$ and $V \cap F = F_2$. There is an $\alpha \in A$ such that $U = U_\alpha$. Denote by B the boundary of the set U_α. Apply Lemma 2.11 to conclude that there is a non-empty open set $W \subset \pi_X(B)$ such that $\pi_X^{-1}(x)$ intersects both U and V for each $x \in W$.

The set T, constructed at the α-th step of the induction, has the power less than 2^ω. Therefore there exists a point $y \in W \backslash T$. Note that the subspace $F_y = \pi_X^{-1}(y)$ is homeomorphic to M and $U_1 = F_y \cap U$ is a proper open subset of F_y. Since the space M is connected, the boundary H of U_1 in F_y is non-empty and is of cardinality 2^ω. Otherwise $F_y \backslash H$ would be disconnected which is not possible by the observation $(*)$ about M. The set $\pi_M(H)$ has the power of continuum as well, so that there exists a point $n \in \pi_M(H) \backslash f(T)$.

It is clear that $H \subset B$. Therefore the point (y, n) belongs to B. This implies that at the α-th step of our inductive construction the first case took place. Thus, some point (x, m) (which, in general, could be different from (y, n)) was chosen in B at the α-th step of the induction. It is obvious that (x, m) belongs to $B \cap F$ and this is a contradiction with $B \cap F = \emptyset$. Hence F is connected. ☐

2.13. Corollary. *Let X be a connected Tychonoff space with a countable network (in particular, a second countable connected Tychonoff space). Then X has a finer connected strictly σ-discrete Tychonoff topology.*

2.14. Corollary. *Let X be a connected separable Tychonoff space of cardinality 2^ω. Then X has a finer connected strictly σ-discrete Tychonoff topology.*

Proof. Any separable regular space X has weight $\leq 2^\omega$. It is evident that ω_1 is a precaliber of X. Now apply Theorem 2.12. □

V.V.Uspensky proved in [Us] that for every topological space X there is a homogeneous space Y such that $X \times Y$ is homeomorphic to Y. Therefore every space is an open retract of a homogeneous space. A minor modification of V.V.Uspensky's proof yields the following theorem.

2.15. Theorem. *Let X be a connected Tychonof space. Then there exists a connected homogeneous Tychonoff space Y such that $X \times Y$ is homeomorphic to Y.*

Proof. Let $\tau = |X|$. Take a set A of power τ^+ and consider the following subspace of X^A:

$$Y = \{f \in X^A : |f^{-1}(x)| = \tau^+ \text{ for every } x \in X\}.$$

It is evident that $X \times Y$ is homeomorphic to Y. For any $f, g \in Y$ it is easy to find a bijection of A onto itself such that the relevant homeomorphism of X^A takes f to g and Y onto Y [Us] which proves that Y is homogeneous.

Now we prove that Y is connected. Suppose not. Then there is a continuous surjection $p : Y \to \{0, 1\}$. The function p depends on not more than τ coordinates [Ar], so there is a $B \subset A$ with $|B| \leq \tau$ such that $\pi_B(Y)$ is disconnected ($\pi_B : X^A \to X^B$ is the natural projection). But it is easy to see that $\pi_B(Y) = X^B$ which is a contradiction because X^B is connected. □

2.16. Corollary. *Every connected Tychonoff space is an open retract of a connected homogeneous Tychonoff space.*

As there is a homogeneous connected preimage for every connected space, it is natural to ask whether any connected space has a finer homogeneous connected topology. The following proposition shows that it is not the case.

2.17. Proposition. *Let X be an infinite connected space which has a dispersion point. Then X does not have a finer connected homogeneous topology.*

Proof. Indeed, if y is a dispersion point of X, then it is a dispersion point in any finer topology. Therefore in any finer homogeneous topology all points of X would be dispersion points. But one of the consequences of Kuratowski's theorem [Ko, Section 1.5] says that in a connected space there is at most one dispersion point. Therefore this finer topology is not connected. □

2.18. Corollary. *There are connected second countable Tychonoff spaces which do not have a finer connected homogeneous topology.*

3. Unsolved problems

The author managed to prove or disprove less than one tenth of possible analogues of classical theorems on good preimages. In what follows the list of problems is given that the author was unable to solve while working on this paper.

The first three problems, as well as Problem 3.6 and 3.7 are motivated by the following results (see [En, 4.2.D]):
— every sequential space is a quotient image of a metric space;
— every Fréchet space is a hereditarily quotient image of a metric space;
— every first countable space is an open image of a metric space;

The rest of the problems stated below outline some ways of improving the results of this paper.

3.1. Problem. Let X be a Tychonoff sequential (or Fréchet) connected space. Is it true that X is a continuous (or a quotient, or hereditarily quotient for the Fréchet case) image of a metric connected space?

3.2. Problem. Let X be a Tychonoff first countable connected space. Is it true that X is a continuous (or an open continuous) image of a metric connected space?

3.3. Problem. Is any (Tychonoff) connected space a continuous (an open continuous) image of a normal (paracompact) connected space?

3.4. Problem. Let X be a Tychonoff sequential (or Fréchet, or first countable) connected space with a countable network. Is it true that X is a continuous image of a second countable connected space? Is it true that there exists a finer Tychonoff second countable connected topology on X?

3.5. Problem. Does every Hausdorff (or Tychonoff) connected space have a finer σ-discrete connected Hausdorff topology?

3.6. Problem. Is it true that every Hausdorff maximal connected space is strictly σ-discrete?

3.7. Problem. Does every Tychonoff connected space have a finer σ-discrete connected Tychonoff topology?

3.8. Problem. Does every compact (or pseudocompact, or countably compact) Tychonoff space have a finer σ-discrete Tychonoff topology?

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