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PROJECTIVE MODULES AND HILBERT SPACES WITH A NEVANLINNA-PICK KERNEL

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Abstract. In this paper we solve a mapping problem for a particular class of Hilbert modules over an algebra multipliers of a diagonal Nevanlinna-Pick (NP) kernel. In this case, the regular representation provides a multiplier norm which induces the topology on the algebra. In particular, we show that, in an appropriate category, a certain class of Hilbert modules are projective. In addition, we establish a commutant lifting theorem for diagonal NP kernels.

1. Introduction

Douglas and Paulsen [DP89] define a Hilbert module as a Hilbert space $\mathcal{H}$ together with a continuous action of a function algebra $\mathbb{A}$. The notion of a Hilbert module allows some interesting questions in operator theory to be expressed via certain natural algebraic constructions. We consider below both a reproducing kernel Hilbert space $H^2(k)$, where k is an NP kernel, which for expository purposes we assume to be diagonal, and the Hilbert modules over the algebra of bounded multipliers of $H^2(k)$. The regular representation allows the identification of an element, $f \in H^2(k)$, with a (possibly unbounded) multiplication operator, M_f, with symbol f. The algebra of bounded multipliers then consists of those multipliers with finite operator norm.

The full power of the commutant lifting theorem [FF90] is that given contractions T, and T' acting on Hilbert spaces $\mathcal{H}$, and $\mathcal{H}'$, respectively, and a contractive intertwining $A : \mathcal{H} \to \mathcal{H}'$, there exists a contractive intertwining of the minimal isometric dilations of T and T'. In the sequel, we obtain a commutant lifting theorem when $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert modules over the algebra of multipliers of $H^2(k)$. This lifting theorem will follow from the solution of a mapping problem which naturally arises in homological constructions. Indeed, it is the solution of this mapping problem which shows certain Hilbert modules to be projective, in an appropriate category.

2. Results

The operator of multiplication by z on a reproducing kernel Hilbert space, where the kernel k is a Nevanlinna-Pick kernel (NP for short), enjoys many of the properties of the unilateral shift operator [Agl90a]. In this paper we establish a commutant

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lifting theorem for diagonal NP kernels. Explicitly, our kernel has the form

$$k(z,\zeta) = \sum_0^\infty a_n z^n \overline{\zeta}^n, \tag{2.1}$$

where $a_0 = 1$, and $a_n > 0$. We also assume k has a positive radius of convergence about (0,0) and

$$\frac{a_j}{a_{j+1}} \leq C^2. \tag{2.2}$$

When $a_n = 1$ for each n, k is the Szego kernel; when $a_n = n+1$, k is the Bergman kernel; and when $a_n = \frac{1}{n+1}$, k is the Dirichlet kernel.

Since $k(0,0) = 1$, near $(0,0)$ we have

$$\frac{1}{k}(z,\zeta) = 1 - \sum_{n=1}^\infty b_n z^n \overline{\zeta}^n. \tag{2.3}$$

We note for future use that, for $n \geq 1$,

$$a_n = \sum_{s=1}^n b_s a_{n-s}. \tag{2.4}$$

In this context, k is an *NP kernel* if $b_n \geq 0$ for each $n \geq 1$. It is easy to verify that the Szego kernel is an NP kernel and that the Bergman kernel is not. To show that the Dirichlet kernel is an NP kernel requires some effort [SS62] [Agl90b]. Let $H^2(k)$ denote the Hilbert space (equals complex separable or finite dimensional Hilbert space) obtained by closing up analytic polynomials in the inner product determined by

$$\langle z^s, z^t \rangle = \begin{cases} \frac{1}{a_s}, & \text{if } s = t; \\ 0, & \text{if } s \neq t. \end{cases} \tag{2.5}$$

Let $k_\ell(z) = a_\ell \zeta^\ell$. Since $\langle z^s, k_t \rangle = \delta_{s,t}$, and polynomials are dense in $H^2(k)$, $\{k_\ell\}$ is a dual basis to $\{z^n\}$.

The condition (2.2) implies the operator S_k on $H^2(k)$ defined by $S_k z^n = z^{n+1}$ is bounded with $\|S_k\| \leq C$. Standard computations show that

$$S_k^* k_\ell = k_{\ell-1}, \tag{2.6}$$

where we interpret $k_s = 0$, if $s < 0$.

We will state the result in the language of modules. Given a Hilbert space $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ denote the bounded operators on $\mathcal{H}$. Each $J \in \mathcal{L}(\mathcal{H})$ determines a module over $\mathbb{C}[z]$ by

$$p \cdot h = P(J)h, \tag{2.7}$$

for $p \in \mathbb{C}[z]$ and $h \in H$. We denote this module $(J, \mathcal{H})$. For our purposes, a module $(T, \mathcal{M})$ is a **-submodule* of $(J, \mathcal{H})$ if $\mathcal{M}$ is a subspace of $\mathcal{H}$ which is invariant for J^* and $T = P_\mathcal{M} J$, where $P_\mathcal{M}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. Since, for $p \in \mathbb{C}[z]$, $p(T)P_\mathcal{M} = P_\mathcal{M} p(J)$,

$$P_\mathcal{M} : (J, \mathcal{H}) \mapsto (T, \mathcal{M}) \tag{2.8}$$

is a module homomorphism. In what follows $\mathcal{H}^2(k) = \bigoplus_{j=0}^\infty H^2(k)$ and $\mathcal{S}_k = \bigoplus_{j=0}^\infty S_k$.

Theorem 2.1. *Let k be a diagonal NP kernel. If $(C,\mathcal{M})$ is a *-submodule of $(\mathcal{S}_k,\mathcal{H}^2(k))$ and if $f:(\mathcal{S}_k,\mathcal{H}^2(k))\mapsto(C,\mathcal{M})$ is a bounded module homomorphism, then there exists a module homomorphism $F:(\mathcal{S}_k,\mathcal{H}^2(k))\mapsto(\mathcal{S}_k,\mathcal{H}^2(k))$ such that $P_{\mathcal{M}}F=f$ and $\|F\|=\|f\|$. Diagrammatically,*

$$\begin{tikzcd} & \mathcal{H}^2(k) \arrow[dl, dotted, "F"'] \arrow[d, "f"] & \\ \mathcal{H}^2(k) \arrow[r, "P_{\mathcal{M}}"] & \mathcal{M} \arrow[r] & 0 \end{tikzcd}$$

Theorem 2.1 prepares the way for the study of the homological algebra of *-submodules of $(\mathcal{S}_k,\mathcal{H}^2(k))$, an investigation pursued by the first author in forthcoming work. Indeed, let $\mathcal{C}$ be the category whose objects are *-submodules of $\mathcal{H}^2(k)$, and morphisms are bounded module maps. Let $\mathcal{E}$ be the class of all sequences

$$\cdots \longrightarrow \mathcal{M}' \xrightarrow{\mu'} \mathcal{M} \xrightarrow{\mu} \mathcal{M}'' \longrightarrow \cdots$$

in which each morphism is a partial isometry, and for μ' and μ successive morphisms in the sequence we have image(μ') =kernel(μ). Decree the elements of $\mathcal{E}$ to be the exact sequences of $\mathcal{C}$, and define an object $\mathcal{P}\in\mathcal{C}$ to be projective if for every diagram with bottom row exact,

$$\begin{tikzcd} & \mathcal{P} \arrow[d, "f"] \arrow[dr, "0"] & \\ \mathcal{M}' \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{M}'' \end{tikzcd} \tag{2.9}$$

there exists a bounded module map $F:\mathcal{P}\to\mathcal{M}'$ making the diagram commute. In this setting we have the following

Corollary 2.2. *Let $\mathcal{C},\mathcal{E},\mathcal{H}^2(k)$ be as above. $\mathcal{H}^2(k)$ is projective in $\mathcal{C}$.*

The notion of an NP kernel and a version of Theorem 2.1 suitable for Nevanlinna-Pick interpolation is due to Agler [Agl90b]. Douglas and Paulsen [DP89], Carlson and Clark [CC95], and Ferguson [Fer96] have produced results for k, the Szego kernel. Choosing k to be the Szego kernel in Theorem 2.1 gives commutant lifting [FS70] for contractions C and C' whose minimal isometric dilations have no unitary summand as can be seen immediately from the following corollary.

Corollary 2.3. *If $\mathcal{M}$ and $\mathcal{M}'$ are invariant for $\mathcal{S}_k^*$, X is a bounded operator from $\mathcal{M}$ to $\mathcal{M}'$, $C=P_{\mathcal{M}}\mathcal{S}_kP_{\mathcal{M}}$, $C'=P_{\mathcal{M}'}\mathcal{S}_kP_{\mathcal{M}'}$, and $XC=C'X$, then there exists $Y\in\mathcal{L}(\mathcal{H}^2(k))$ such that $Y^*|_{\mathcal{M}}=X^*$, $\|Y\|=\|X\|$, and $Y\mathcal{S}_k=\mathcal{S}_kY$.*

The remainder of the paper is devoted to proofs of Theorem 2.1 and Corollaries 2.2 and 2.3.

3. The proofs

Throughout we assume k is an NP kernel and a_j and b_j are as in the introduction. For expository purposes, we make use of tensor product notation. Fix a (separable)

Hilbert space $\mathcal{H}$. We identify $\mathcal{H}^2(k)$ with $\mathcal{H} \otimes H^2(k)$ and $\mathcal{S}_k$ with $I \otimes S_k$, where I is the identity on $\mathcal{H}$. In particular, $g \in \mathcal{H}^2(k)$ can be represented as

$$g = \sum g_n \otimes z^n, \tag{3.1}$$

for some sequence g_n from $\mathcal{H}$, where the series converges in norm; and

$$\mathcal{S}_k g = \sum g_n \otimes z^{n+1}. \tag{3.2}$$

Lemma 3.1. *Let $A : \mathcal{H} \mapsto \mathcal{M}$. Suppose $(T, \mathcal{N})$ is a *-submodule of $(\mathcal{S}_k, \mathcal{H}^2(k))$ and $(C, \mathcal{M})$ is a *-submodule of $(T, \mathcal{N})$. If*

$$\sum_{j=0}^{N} a_j (C^j A)(C^j A)^* \leq I$$

for all N, and if $T^j A = TC^{j-1}A$ for all $j \geq 1$, then

$$\sum_{j=1}^{M} a_j (T^j A)(T^j A)^* \leq I,$$

for all M.

Proof. For a fixed M we have, using (2.6) repeatedly,

$$\langle (\sum_{j=1}^{M} b_j S_k^j S_k^{*j}) k_s, k_t \rangle = \sum_{j=1}^{M} b_j \langle k_{s-j}, k_{t-j} \rangle. \tag{3.3}$$

If $s \neq t$ then the sum is 0. In the case $s = t$, (3.3) equals $\sum_{j=1}^{M} b_j a_{s-j}$ which is less than $\sum_{j=1}^{s} b_j a_{s-j} = a_s$, where we interpret $a_\ell = 0$ for $\ell < 0$. It follows that

$$\sum_{j=1}^{M} b_j S_k^j S_k^{*j} \leq I, \tag{3.4}$$

for each M. It is immediate that (3.4) holds with S_k replaced by $\mathcal{S}_k$. With $T = P_{\mathcal{N}} \mathcal{S}_k$ we have

$$T^j T^{*j} = P_{\mathcal{N}} \mathcal{S}_k^j \mathcal{S}_k^{*j} P_{\mathcal{N}}. \tag{3.5}$$

Since $b_j \geq 0$, it now follows from (3.5) that (3.4) holds with T in place of S_k.

We can now estimate,

$$
\begin{aligned}
I &\geq \sum_{s=1}^{M} b_s T^s T^{*s} \\
&\geq \sum_{s=1}^{M} b_s T^s \Big(\sum_{j=0}^{M-s} a_j C^j A A^* C^{*j} \Big) T^{*s} \\
&= \sum_{s=1}^{M} \sum_{j=0}^{M-s} b_s a_j T^{j+s} A A^* T^{*(j+s)} \\
&= \sum_{n=1}^{M} \sum_{s=1}^{n} b_s a_{n-s} T^n A A^* T^{*n} \\
&= \sum_{n=1}^{M} a_n T^n A A^* T^{*n}.
\end{aligned}
$$

□

The strategy is to use Lemma 3.1 and the Parrott Lemma [Par70] to do one step extensions. Accordingly, let q denote the least natural number such that $\mathcal{H} \otimes [k_q] = \{h \otimes k_q : h \in \mathcal{H}\}$ is not a subspace of $\mathcal{M}$. Let $\mathcal{N}$ denote the closure of the span of the subspaces $\mathcal{M}$ and $\mathcal{H} \otimes [k_q]$. Since $\mathcal{S}_k^*$ maps $\mathcal{H} \otimes [k_q]$ into $\mathcal{H} \otimes [k_{q-1}] \subset \mathcal{M}$, $\mathcal{N}$ is invariant for $\mathcal{S}_k^*$. Thus, with $T = P_{\mathcal{N}} \mathcal{S}_k$, $(T, \mathcal{N})$ is a *-submodule of $(\mathcal{S}_k, \mathcal{H}^2(k))$ and $(C, \mathcal{M})$ is a *-submodule of $(T, \mathcal{N})$. Also observe that, as the range of T^* is in $\mathcal{M}$,

$$T(\mathcal{N} \ominus \mathcal{M}) = \{0\}. \tag{3.6}$$

Lemma 3.2. *There exists a module homomorphism $F : (\mathcal{S}_k, \mathcal{H}^2(k)) \mapsto (T, \mathcal{N})$ such that $P_{\mathcal{M}} F = f$, and $\|F\| \leq \|f\|$.*

Proof. Without loss of generality, we may assume that $\|f\| = 1$. Given a Hilbert space $\mathcal{K}$ and a bounded operator $G : \mathcal{H}^2(k) \mapsto \mathcal{K}$, define $G_j : \mathcal{H} \mapsto \mathcal{K}$ by $G_j h = G(h \otimes z^j)$. The matrix of G^* with respect to the orthogonal decomposition of $\mathcal{H}^2(k) = \bigoplus (\mathcal{H} \otimes [z^j])$ is then

$$G = \begin{pmatrix} G_0^* & a_1 G_1^* & a_2 G_2^* & \dots \end{pmatrix}^t. \tag{3.7}$$

Thus $\|G\| \leq I$ if and only if

$$I \geq G G^* = \sum a_j G_j G_j^*. \tag{3.8}$$

Since $f \mathcal{S}_k^j = C^j f$, we have $f_j = C^j f_0$. Hence from (3.8),

$$I \geq \sum a_j (C^j f_0)(C^j f_0)^*. \tag{3.9}$$

To obtain $P_{\mathcal{M}} F = f$, with respect to the orthogonal decomposition

$$\mathcal{N} = (\mathcal{N} \ominus \mathcal{M}) \oplus \mathcal{M},$$

our desired F must have the form

$$F_j = \begin{pmatrix} g_j \\ f_j \end{pmatrix}, \tag{3.10}$$

for some $g : \mathcal{H}^2(k) \mapsto \mathcal{N} \ominus \mathcal{M}$. The condition that F is a module homomorphism is equivalent to $F\mathcal{S}_k^n = T^n F$, for $n \geq 1$. Consequently, given F_0, F is a module homomorphism if and only if

$$F_j = T^j F_0$$

(assuming F is a bounded operator). Notice that (3.6) implies $F_j = T^j f_0$, for $j \geq 1$. In particular, for $j \geq 1$, g_j is already determined. The only choice is of g_0.

The matrix of F with respect to the orthogonal decomposition of $\mathcal{H}^2(k)$ as $(\mathcal{H} \otimes [1]) \oplus (\mathcal{H} \otimes [1])^\perp$ and $\mathcal{N}$ as $(\mathcal{N} \ominus \mathcal{M}) \oplus \mathcal{M}$ has the form

$$F = \begin{pmatrix} g_0 & a \\ b & c \end{pmatrix} \tag{3.11}$$

where g_0 is to be chosen,

$$\begin{pmatrix} b & c \end{pmatrix} = f,$$

and

$$\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \sqrt{a_1} F_1 & \sqrt{a_2} F_2 & \dots \end{pmatrix}. \tag{3.12}$$

We will show that the operator in (3.12) has norm at most one. Since $\|f\| \leq 1$, from the Parrott Lemma there exists g_0 such that $\|F\| \leq 1$. This will then complete the proof of Theorem 2.1. Now the operator of (3.12) has norm at most one if and only if

$$I \geq \sum_{j=1}^{M} a_j F_j F_j^* \tag{3.13}$$

holds for all M. Using $F_j = T^j f_0$ and $T^j f_0 = TC^{j-1} f_0$ for $j \geq 1$, and (3.9), we see (3.13) holds by Lemma 3.1. □

Theorem 2.1 is now established by repeatedly applying Lemma 3.2. We omit the details. Let

$$\mathcal{N}' \xrightarrow[P_{\mathcal{N}}]{} \mathcal{N} \longrightarrow \mathcal{N}''$$

be a member of the class $\mathcal{E}$ of exact sequences. To see that $\mathcal{H}^2(k)$ is projective as described above we consider the following

$$\begin{tikzcd}
 & & \mathcal{H}^2(k) \arrow[dl, dotted, "F"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
\mathcal{H}^2(k) \arrow[r, "P_{\mathcal{N}'}"'] & \mathcal{N}' \arrow[r, "P_{\mathcal{N}}"'] & \mathcal{N} \arrow[r] & \mathcal{N}''.
\end{tikzcd}$$

Hence Theorem 2.1 now implies we can complete the diagram with a map $F : \mathcal{H}^2(k) \to \mathcal{H}^2(k)$, by letting $\mathcal{M} = \mathcal{N}$, $g = f$, and $\mathcal{N}'' = 0$. Corollary 2.2 follows since $\hat{F} := P_{\mathcal{N}'} \circ F : \mathcal{H}^2(k) \to \mathcal{N}'$ solves the following diagram

$$\begin{tikzcd}
 & \mathcal{H}^2(k) \arrow[dl, dotted, "\widehat{F}"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
\mathcal{N}' \arrow[r] & \mathcal{N} \arrow[r] & \mathcal{N}''.
\end{tikzcd}$$

From the usual diagram

$$\begin{tikzcd}
& \mathcal{H}^2(k) \arrow[ddl, dotted, "Y"'] \arrow[d, "P_{\mathcal{M}}"] & \\
& \mathcal{M} \arrow[d, "X"] & \\
\mathcal{H}^2(k) \arrow[r, "P_{\mathcal{M}'}"] & \mathcal{M}' \arrow[r] & 0
\end{tikzcd}$$

it is seen that Corollary 2.3 is an immediate consequence of Theorem 2.1.

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