ON THE RECURSIVE SEQUENCE $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$

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Abstract. We show that every positive solution of the equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n = 0, 1, \ldots,$$

where $A \in (0, \infty)$, converges to a period two solution.

1. Introduction

Our aim in this paper is to establish that every positive solution of the equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n = 0, 1, \ldots,$$

where

$$x_{-2}, x_{-1}, x_0, A \in (0, \infty),$$

converges to a period two solution. This confirms conjecture 2.4.2 in [1]. More precisely, we show that each of the two subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of every positive solution of Eq. (1) converges to a finite limit.

One of the main ingredients in our proof, and an interesting result in its own right, is to show that every positive solution of the equation

$$y_{n+1} = A + \frac{y_n}{y_{n-1}}, \quad n = 0, 1, \ldots,$$

where

$$y_{-1}, y_0, A \in (0, \infty),$$

converges to its equilibrium $\bar{y} = 1 + A$. From this it follows easily that the equilibrium $1 + A$ of Eq. (3) is globally asymptotically stable.

It follows from the work in [2] that every positive solution of Eq. (1) is bounded and persists. It was also observed in [2] that for every real number $q$, Eq. (1) possesses the period two solution

$$q, \frac{A + 1}{q}, q, \frac{A + 1}{q}, \ldots$$
We say that a solution \( \{x_n\} \) of a difference equation is bounded and persists if there exist positive constants \( P \) and \( Q \) such that
\[
P \leq x_n \leq Q \quad \text{for} \quad n = -1, 0, \ldots.
\]

A positive semicycle of a solution \( \{y_n\} \) of Eq. (3) consists of a “string” of terms \( \{y_l, y_{l+1}, \ldots, y_m\} \), all greater than or equal to the equilibrium \( \bar{y} \), with \( l \geq -1 \) and \( m \leq \infty \), and such that

either \( l = -1 \), or \( l > -1 \) and \( y_{l-1} < \bar{y} \)

and

either \( m = \infty \), or \( m < \infty \) and \( y_{m+1} < \bar{y} \).

A negative semicycle of a solution \( \{y_n\} \) of Eq. (3) consists of a “string” of terms \( \{y_l, y_{l+1}, \ldots, y_m\} \), all less than the equilibrium \( \bar{y} \), with \( l \geq -1 \) and \( m \leq \infty \), and such that

either \( l = -1 \), or \( l > -1 \) and \( y_{l-1} \geq \bar{y} \)

and

either \( m = \infty \), or \( m < \infty \) and \( y_{m+1} \geq \bar{y} \).

The first semicycle of a solution starts with the term \( y_{-1} \), and is positive if \( y_{-1} \geq \bar{y} \) and negative if \( y_{-1} < \bar{y} \).

2. Global asymptotic stability of Eq. (3)

Let \( \{x_n\} \) be a positive solution of Eq. (1), and set
\[
y_n = x_n x_{n-1}.
\]
Then clearly, \( \{y_n\} \) satisfies Eq. (3), and condition (4) holds. The main result in this section is the following.

**Theorem 1.** Assume that (4) holds. Then the positive equilibrium \( \bar{y} = 1 + A \) of Eq. (3) is globally asymptotically stable.

Before we establish the above theorem we need the following result.

**Lemma 1.** Let \( \{y_n\} \) be a nontrivial positive solution of Eq. (3). Then the following statements are true.

(a) \( \{y_n\} \) oscillates about the equilibrium \( \bar{y} = 1 + A \) with semicycles of length two or three.

(b) The extreme in a semicycle occurs in the first or second term.

(c) For \( n > 2 \),
\[
A < y_n < A + \frac{A + 1}{A}.
\]

**Proof.** We will first show that every positive semicycle, except possibly the first, has two or three terms. The case for the negative semicycles is similar and is omitted.

Let \( y_N \geq \bar{y} \) be the first term in a positive semicycle, other than the first positive semicycle. Then \( y_{N-1} < \bar{y} \), and
\[
y_{N+1} = A + \frac{y_N}{y_{N-1}} > A + \frac{y_N}{\bar{y}} = A + \frac{y_N}{1 + A} \geq \bar{y}.
\]

Now if \( y_{N+1} > y_N \), then
\[
y_{N+2} = A + \frac{y_{N+1}}{y_N} > A + 1 = \bar{y}.
\]
Also
\[ y_{n+2} = A + \frac{y_{n+1}}{y_N} \leq A + \frac{y_{n+1}}{\bar{y}} = A + \frac{y_{n+1}}{1 + A} < y_{n+1}. \]
Thus \( \bar{y} < y_{n+2} < y_{n+1} \), and so
\[ y_{n+3} = A + \frac{y_{n+2}}{y_{n+1}} < A + 1 = \bar{y}, \]
and the positive semicycle has length three. If \( y_{n+1} \leq y_N \), then
\[ y_{n+2} = A + \frac{y_{n+1}}{y_N} \leq A + 1 = \bar{y}, \]
and the positive semicycle has length at most three, with length equal to two unless \( y_{n+1} = y_N \).

From this it is clear that every solution \( \{y_n\} \) of Eq. (3) oscillates about \( \bar{y} = 1 + A \). It is also clear from the above that the extreme in a semicycle occurs in the first or second term.

From Eq. (3) one can see that \( y_n > A \) for \( n > 0 \). Let \( y_N \), where \( N > 2 \), be the first term in a positive semicycle. Then \( A < y_{N-1} < \bar{y} \) and \( A < y_{N-2} < \bar{y} \), giving
\[ y_N = A + \frac{y_{N-1}}{y_{N-2}} < A + \frac{\bar{y}}{A} = A + \frac{1 + A}{A}. \]
Now
\[ y_{n+1} = A + \frac{y_N}{y_{n-1}} = A + \frac{A + \frac{y_{n-1}}{y_{n-2}}}{y_{n-1}} = A + \frac{A}{y_{n-2}} + \frac{1}{y_{n-2}} < A + \frac{1 + A}{A}. \]
The proof is complete. \( \square \)

Proof of Theorem 1. It is easy to see (by linearized stability analysis) that \( \bar{y} = 1 + A \) is locally asymptotically stable. So it remains to show that if \( \{y_n\} \) is a nontrivial solution of Eq. (3), then
\[ \lim_{n \to \infty} y_n = 1 + A. \]
To this end, define the sequences \( \{L_n\} \) and \( \{U_n\} \) as follows:
\[ L_1 = A, \quad U_1 = A + \frac{1 + A}{A}, \]
and for \( n = 1, 2, \ldots \)
\[ L_{n+1} = A + \frac{1 + A}{U_n} \quad \text{and} \quad U_{n+1} = A + \frac{1 + A}{L_{n+1}}. \]
Now, it can be seen that \( \{U_n\} \) and \( \{L_n\} \) are sequences of upper and lower bounds for the semicycles of the solutions \( \{y_n\} \) of Eq. (3). Also
\[ L_{n+1} = A + \frac{1 + A}{A + \frac{1 + A}{L_n}} \]
and
\[ U_{n+1} = A + \frac{1 + A}{A + \frac{1 + A}{U_n}}. \]
From this it follows that
\[ L_1 < L_2 < \ldots < L_n < L_{n+1} < \ldots < \bar{y} < \ldots < U_{n+1} < U_n < \ldots < U_2 < U_1. \]
Thus
\[ \lim_{n \to \infty} L_n = L \leq \bar{y} \quad \text{and} \quad \lim_{n \to \infty} U_n = U \geq \bar{y}. \]
But since the only solution to
\[ y = A + \frac{1 + A}{A + \frac{1 + A}{y}} \]
is \( y = \bar{y} \), it must be true that \( U = L = \bar{y} \). The proof is complete. \( \square \)

3. Periodic character of the solutions of Eq. (1)

It follows from Eq. (1) that
\[ x_{2n+1} = \frac{A}{x_{2n}} + \frac{1}{x_{2n}}, \quad n = 0, 1, \ldots, \] (5)
and
\[ x_{2n+2} = \frac{A}{x_{2n+1}} + \frac{1}{x_{2n}}, \quad n = 0, 1, \ldots, \] (6)
and so
\[ x_{2n+2} = \frac{A}{x_{2n+1} + \frac{1}{x_{2n}}} + \frac{1}{x_{2n} + \frac{1}{x_{2n-1}}}, \quad n = 1, 2, \ldots. \] (7)

The following result summarizes some of the properties of the solutions of Eq. (1)

Lemma 2. Let \( \{x_n\} \) be a positive solution of Eq. (1). Then the following statements are true.

(i) For \( N \geq 0 \), let
\[ m_N = \min \{x_{2N-2}, x_{2N}, x_{2N+2}\} \]
and
\[ M_N = \max \{x_{2N-2}, x_{2N}, x_{2N+2}\}. \]
Then
\[ m_N \leq x_{2n} \leq M_N \text{ for } n \geq N. \]

(ii) There exist positive numbers \( m \) and \( M \) such that \( m \leq x_n \leq M \) for \( n = 0, 1, \ldots. \)

(iii) \[ \lim_{n \to \infty} \frac{x_{2n}}{x_{2n-2}} = 1. \]

Proof. To prove (i), note that the function
\[ f(x, y, z) = \frac{A}{x} + \frac{1}{y} + \frac{1}{z} \]
is increasing in \( x, y, \) and \( z \). Thus
\[ x_{2N+4} = \frac{A}{x_{2N+2}} + \frac{1}{x_{2N}} + \frac{1}{x_{2N-2}} \leq \frac{A}{M_N} + \frac{1}{M_N} + \frac{1}{M_N} = M_N \]
and by induction, \( x_{2n} \leq M_N \) for \( n \geq N \). Similarly \( x_{2n} \geq m_N \) for \( n \geq N \). The proof of (ii) follows directly from (i), since (i) implies that the sequences of even and odd terms of Eq. (3) are bounded. To prove (iii), note that \( y_{2n} = x_{2n}x_{2n-1} \) converges to \( 1 + A \), and so
\[ \frac{y_{2n}}{y_{2n-1}} = \frac{x_{2n}x_{2n-1}}{x_{2n-1}x_{2n-2}} = \frac{x_{2n}}{x_{2n-2}} \]
converges to 1. Thus
\[ \lim_{n \to \infty} \frac{x_{2n}}{x_{2n-2}} = 1. \]
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The main result in this paper is the following:

**Theorem 2.** Let $\{x_n\}$ be a positive solution of Eq. (1). Then there exist positive constants $L_E$ and $L_O$ such that $L_EL_O = 1 + A$ and

$$\lim_{n \to \infty} x_{2n} = L_E \text{ and } \lim_{n \to \infty} x_{2n+1} = L_O.$$ 

**Proof.** It follows from Eq. (5) and Lemma 2 that

$$\left| \frac{x_{2n}}{x_{2n-2}} - 1 \right| \geq \left| \frac{x_{2n} - x_{2n-2}}{M} \right|.$$ 

Hence

$$\lim_{n \to \infty} \left| x_{2n} - x_{2n-2} \right| = 0.$$ 

It is now clear from Lemma 2 (i) and (iii) that $\lim_{n \to \infty} x_{2n}$ exists and is a positive number $L_E$. From Eq. (5) it then follows that $\lim_{n \to \infty} x_{2n+1}$ also exists and is a positive number $L_O$. Finally, from Eq. (5) we see that

$$L_EL_O = 1 + A,$$

and the proof is complete.

**References**


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