

ON THE RECURSIVE SEQUENCE $x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}$

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ABSTRACT. We show that every positive solution of the equation

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n = 0, 1, \dots,$$

where $A \in (0, \infty)$, converges to a period two solution.

1. INTRODUCTION

Our aim in this paper is to establish that every positive solution of the equation

$$(1) \quad x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}, \quad n = 0, 1, \dots,$$

where

$$(2) \quad x_{-2}, x_{-1}, x_0, A \in (0, \infty),$$

converges to a period two solution. This confirms conjecture 2.4.2 in [1]. More precisely, we show that each of the two subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ of every positive solution of Eq. (1) converges to a finite limit.

One of the main ingredients in our proof, and an interesting result in its own right, is to show that every positive solution of the equation

$$(3) \quad y_{n+1} = A + \frac{y_n}{y_{n-1}}, \quad n = 0, 1, \dots,$$

where

$$(4) \quad y_{-1}, y_0, A \in (0, \infty),$$

converges to its equilibrium $\bar{y} = 1 + A$. From this it follows easily that the equilibrium $1 + A$ of Eq. (3) is globally asymptotically stable.

It follows from the work in [2] that every positive solution of Eq. (1) is bounded and persists. It was also observed in [2] that for every real number q , Eq. (1) possesses the period two solution

$$q, \frac{A+1}{q}, q, \frac{A+1}{q}, \dots$$

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We say that a solution $\{x_n\}$ of a difference equation is bounded and persists if there exist positive constants P and Q such that

$$P \leq x_n \leq Q \quad \text{for } n = -1, 0, \dots$$

A positive semicycle of a solution $\{y_n\}$ of Eq. (3) consists of a “string” of terms $\{y_l, y_{l+1}, \dots, y_m\}$, all greater than or equal to the equilibrium \bar{y} , with $l \geq -1$ and $m \leq \infty$, and such that

$$\text{either } l = -1, \text{ or } l > -1 \text{ and } y_{l-1} < \bar{y}$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \text{ and } y_{m+1} < \bar{y}.$$

A negative semicycle of a solution $\{y_n\}$ of Eq. (3) consists of a “string” of terms $\{y_l, y_{l+1}, \dots, y_m\}$, all less than the equilibrium \bar{y} , with $l \geq -1$ and $m \leq \infty$, and such that

$$\text{either } l = -1, \text{ or } l > -1 \text{ and } y_{l-1} \geq \bar{y}$$

and

$$\text{either } m = \infty, \text{ or } m < \infty \text{ and } y_{m+1} \geq \bar{y}.$$

The first semicycle of a solution starts with the term y_{-1} , and is positive if $y_{-1} \geq \bar{y}$ and negative if $y_{-1} < \bar{y}$.

2. GLOBAL ASYMPTOTIC STABILITY OF EQ. (3)

Let $\{x_n\}$ be a positive solution of Eq. (1), and set

$$y_n = x_n x_{n-1}.$$

Then clearly, $\{y_n\}$ satisfies Eq. (3), and condition (4) holds. The main result in this section is the following.

Theorem 1. *Assume that (4) holds. Then the positive equilibrium $\bar{y} = 1 + A$ of Eq. (3) is globally asymptotically stable.*

Before we establish the above theorem we need the following result.

Lemma 1. *Let $\{y_n\}$ be a nontrivial positive solution of Eq. (3). Then the following statements are true.*

(a) $\{y_n\}$ oscillates about the equilibrium $\bar{y} = 1 + A$ with semicycles of length two or three.

(b) The extreme in a semicycle occurs in the first or second term.

(c) For $n > 2$,

$$A < y_n < A + \frac{A + 1}{A}.$$

Proof. We will first show that every positive semicycle, except possibly the first, has two or three terms. The case for the negative semicycles is similar and is omitted. Let $y_N \geq \bar{y}$ be the first term in a positive semicycle, other than the first positive semicycle. Then $y_{N-1} < \bar{y}$, and

$$y_{N+1} = A + \frac{y_N}{y_{N-1}} > A + \frac{y_N}{\bar{y}} = A + \frac{y_N}{1 + A} \geq \bar{y}.$$

Now if $y_{N+1} > y_N$, then

$$y_{N+2} = A + \frac{y_{N+1}}{y_N} > A + 1 = \bar{y}.$$

Also

$$y_{N+2} = A + \frac{y_{N+1}}{y_N} \leq A + \frac{y_{N+1}}{\bar{y}} = A + \frac{y_{N+1}}{1+A} < y_{N+1}.$$

Thus $\bar{y} < y_{N+2} < y_{N+1}$, and so

$$y_{N+3} = A + \frac{y_{N+2}}{y_{N+1}} < A + 1 = \bar{y},$$

and the positive semicycle has length three. If $y_{N+1} \leq y_N$, then

$$y_{N+2} = A + \frac{y_{N+1}}{y_N} \leq A + 1 = \bar{y},$$

and the positive semicycle has length at most three, with length equal to two unless $y_{N+1} = y_N$.

From this it is clear that every solution $\{y_n\}$ of Eq. (3) oscillates about $\bar{y} = 1 + A$. It is also clear from the above that the extreme in a semicycle occurs in the first or second term.

From Eq. (3) one can see that $y_n > A$ for $n > 0$. Let y_N , where $N > 2$, be the first term in a positive semicycle. Then $A < y_{N-1} < \bar{y}$ and $A < y_{N-2} < \bar{y}$, giving

$$y_N = A + \frac{y_{N-1}}{y_{N-2}} < A + \frac{\bar{y}}{A} = A + \frac{1+A}{A}.$$

Now

$$y_{N+1} = A + \frac{y_N}{y_{N-1}} = A + \frac{A + \frac{y_{N-1}}{y_{N-2}}}{y_{N-1}} = A + \frac{A}{y_{N-1}} + \frac{1}{y_{N-2}} < A + \frac{1+A}{A}.$$

The proof is complete. □

Proof of Theorem 1. It is easy to see (by linearized stability analysis) that $\bar{y} = 1 + A$ is locally asymptotically stable. So it remains to show that if $\{y_n\}$ is a nontrivial solution of Eq. (3), then

$$\lim_{n \rightarrow \infty} y_n = 1 + A.$$

To this end, define the sequences $\{L_n\}$ and $\{U_n\}$ as follows:

$$L_1 = A, \quad U_1 = A + \frac{1+A}{A},$$

and for $n = 1, 2, \dots$

$$L_{n+1} = A + \frac{1+A}{U_n} \quad \text{and} \quad U_{n+1} = A + \frac{1+A}{L_{n+1}}.$$

Now, it can be seen that $\{U_n\}$ and $\{L_n\}$ are sequences of upper and lower bounds for the semicycles of the solutions $\{y_n\}$ of Eq. (3). Also

$$L_{n+1} = A + \frac{1+A}{A + \frac{1+A}{L_n}}$$

and

$$U_{n+1} = A + \frac{1+A}{A + \frac{1+A}{U_n}}.$$

From this it follows that

$$L_1 < L_2 < \dots < L_n < L_{n+1} < \dots < \bar{y} < \dots < U_{n+1} < U_n < \dots < U_2 < U_1.$$

Thus

$$\lim_{n \rightarrow \infty} L_n = L \leq \bar{y} \quad \text{and} \quad \lim_{n \rightarrow \infty} U_n = U \geq \bar{y}.$$

But since the only solution to

$$y = A + \frac{1 + A}{A + \frac{1+A}{y}}$$

is $y = \bar{y}$, it must be true that $U = L = \bar{y}$. The proof is complete. □

3. PERIODIC CHARACTER OF THE SOLUTIONS OF EQ. (1)

It follows from Eq. (1) that

$$(5) \quad x_{2n+1} = \frac{A}{x_{2n}} + \frac{1}{x_{2n-2}}, \quad n = 0, 1, \dots,$$

and

$$(6) \quad x_{2n+2} = \frac{A}{x_{2n+1}} + \frac{1}{x_{2n-1}}, \quad n = 0, 1, \dots,$$

and so

$$(7) \quad x_{2n+2} = \frac{A}{\frac{A}{x_{2n}} + \frac{1}{x_{2n-2}}} + \frac{1}{\frac{A}{x_{2n-2}} + \frac{1}{x_{2n-4}}}, \quad n = 1, 2, \dots$$

The following result summarizes some of the properties of the solutions of Eq. (1)

Lemma 2. *Let $\{x_n\}$ be a positive solution of Eq. (1). Then the following statements are true.*

(i) *For $N \geq 0$, let*

$$m_N = \min\{x_{2N-2}, x_{2N}, x_{2N+2}\}$$

and

$$M_N = \max\{x_{2N-2}, x_{2N}, x_{2N+2}\}.$$

Then

$$m_N \leq x_{2n} \leq M_N \quad \text{for } n \geq N.$$

(ii) *There exist positive numbers m and M such that $m \leq x_n \leq M$ for $n = 0, 1, \dots$*

(iii)

$$\lim_{n \rightarrow \infty} \frac{x_{2n}}{x_{2n-2}} = 1.$$

Proof. To prove (i), note that the function

$$f(x, y, z) = \frac{A}{\frac{A}{x} + \frac{1}{y}} + \frac{1}{\frac{A}{y} + \frac{1}{z}}$$

is increasing in x, y , and z . Thus

$$x_{2N+4} = \frac{A}{\frac{A}{x_{2N+2}} + \frac{1}{x_{2N}}} + \frac{1}{\frac{A}{x_{2N}} + \frac{1}{x_{2N-2}}} \leq \frac{A}{\frac{A}{M_N} + \frac{1}{M_N}} + \frac{1}{\frac{A}{M_N} + \frac{1}{M_N}} = M_N$$

and by induction, $x_{2n} \leq M_N$ for $n \geq N$. Similarly $x_{2n} \geq m_N$ for $n \geq N$. The proof of (ii) follows directly from (i), since (i) implies that the sequences of even and odd terms of Eq. (3) are bounded. To prove (iii), note that $y_{2n} = x_{2n}x_{2n-1}$ converges to $1 + A$, and so

$$\frac{y_{2n}}{y_{2n-1}} = \frac{x_{2n}x_{2n-1}}{x_{2n-1}x_{2n-2}} = \frac{x_{2n}}{x_{2n-2}}$$

converges to 1. Thus

$$\lim_{n \rightarrow \infty} \frac{x_{2n}}{x_{2n-2}} = 1.$$

□

The main result in this paper is the following:

Theorem 2. *Let $\{x_n\}$ be a positive solution of Eq. (1). Then there exist positive constants L_E and L_O such that $L_E L_O = 1 + A$ and*

$$\lim_{n \rightarrow \infty} x_{2n} = L_E \text{ and } \lim_{n \rightarrow \infty} x_{2n+1} = L_O.$$

Proof. It follows from Eq. (5) and Lemma 2 that

$$\left| \frac{x_{2n}}{x_{2n-2}} - 1 \right| \geq \left| \frac{x_{2n} - x_{2n-2}}{M} \right|.$$

Hence

$$\lim_{n \rightarrow \infty} |x_{2n} - x_{2n-2}| = 0.$$

It is now clear from Lemma 2 (i) and (iii) that $\lim_{n \rightarrow \infty} x_{2n}$ exists and is a positive number L_E . From Eq. (5) it then follows that $\lim_{n \rightarrow \infty} x_{2n+1}$ also exists and is a positive number L_O . Finally, from Eq. (5) we see that

$$L_E L_O = 1 + A,$$

and the proof is complete. □

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