HYERS-ULAM-RASSIAS STABILITY OF JENSEN’S EQUATION
AND ITS APPLICATION

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Abstract. The Hyers-Ulam-Rassias stability for the Jensen functional equation is investigated, and the result is applied to the study of an asymptotic behavior of the additive mappings; more precisely, the following asymptotic property shall be proved: Let $X$ and $Y$ be a real normed space and a real Banach space, respectively. A mapping $f : X \to Y$ satisfying $f(0) = 0$ is additive if and only if $\|2f[(x + y)/2] - f(x) - f(y)\| \to 0$ as $\|x\| + \|y\| \to \infty$.

1. Introduction

In 1940, S. M. Ulam [9] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms:

Let $G_1$ be a group and let $G_2$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by D. H. Hyers [2] under the assumption that $G_1$ and $G_2$ are Banach spaces. In 1978, Th. M. Rassias [6] generalized the result of Hyers as follows:

Let $f : X \to Y$ be a mapping between Banach spaces and let $0 \leq p < 1$ be fixed. If $f$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \geq 0$ and for all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p$$

for all $x \in X$. If, in addition, $f(tx)$ is continuous in $t$ for each fixed $x \in X$, then $A$ is linear.
Taking this fact into account, the additive functional equation \( f(x+y) = f(x) + f(y) \) is said to have the Hyers-Ulam-Rassias stability on \((X,Y)\). This terminology is also applied to the case of other functional equations. For more detailed definitions of such terminology one can refer to [1] and [3].

Throughout this paper, let \( X \) and \( Y \) be a real normed space and a real Banach space, respectively. According to Theorem 6 in [5], a mapping \( f : X \to Y \) satisfying \( f(0) = 0 \) is a solution of the Jensen functional equation

\[
2f \left( \frac{x+y}{2} \right) = f(x) + f(y)
\]

if and only if it satisfies the additive Cauchy equation \( f(x+y) = f(x) + f(y) \). Hence, the most general continuous solution of the Jensen’s equation in \( \mathbb{R} \) is \( f(x) = ax + b \), where \( a \) and \( b \) are arbitrary constants.

The first result on the stability of Jensen’s equation was obtained by Z. Kominek (see [4]). In fact, he proved the following theorem:

**Theorem.** Let \( D \) be a subset of \( \mathbb{R}^n \) with non-empty interior. Assume that there exists an \( x_0 \) in the interior of \( D \) such that \( D_0 = D - x_0 \) satisfies the condition \((1/2)D_0 \subset D_0 \). Let a mapping \( f : D \to Y \) satisfy the inequality

\[
\left\| 2f \left( \frac{x+y}{2} \right) - f(x) - f(y) \right\| \leq \delta,
\]

for some \( \delta \geq 0 \) and for all \( x, y \in D \). Then there exist a mapping \( F : \mathbb{R}^n \to Y \) and a constant \( K > 0 \) such that

\[
2F \left( \frac{x+y}{2} \right) = F(x) + F(y)
\]

for all \( x, y \in \mathbb{R}^n \), and

\[
\| f(x) - F(x) \| \leq K
\]

for all \( x \in D \).

In section 2 of the present paper, using ideas from the papers of Th. M. Rassias [6] and D. H. Hyers [2], the Hyers-Ulam-Rassias stability of Jensen’s equation will be investigated, i.e., the stability of the functional inequality

\[
(1) \quad \left\| 2f \left( \frac{x+y}{2} \right) - f(x) - f(y) \right\| \leq \delta + \theta \left( \|x\|^p + \|y\|^p \right)
\]

for the case of \( p \geq 0 \) \((p \neq 1)\) shall be proved. Moreover, by using the same mapping which was constructed by Th. M. Rassias and P. Šemrl [7], we shall show that the inequality (1) is not stable for the case when \( p = 1 \). In section 3, the Hyers-Ulam stability for Jensen’s equation on a restricted domain will be treated, and the result applied to the study of an interesting asymptotic behavior of the additive mappings—more precisely, we prove that a mapping \( f : X \to Y \) satisfying \( f(0) = 0 \) is additive if and only if

\[
(2) \quad \left\| 2f \left( \frac{x+y}{2} \right) - f(x) - f(y) \right\| \to 0 \quad \text{as} \quad \|x\| + \|y\| \to \infty.
\]

It should be remarked here that F. Skof [8] has also proved an asymptotic property of the additive mappings. Indeed, she proved that a mapping \( f : X \to Y \) is additive if and only if

\[
(3) \quad \| f(x+y) - f(x) - f(y) \| \to 0 \quad \text{as} \quad \|x\| + \|y\| \to \infty.
\]
2. Hyers-Ulam-Rassias stability

First, we prove the Hyers-Ulam-Rassias stability of the Jensen’s equation. Assume that \( \delta \geq 0 \) and \( \theta \geq 0 \) are fixed.

**Theorem 1.** Let \( p > 0 \) be given with \( p \neq 1 \). Suppose a mapping \( f : X \to Y \) satisfies the inequality (1) for all \( x, y \in X \). Further, assume \( f(0) = 0 \) and \( \delta = 0 \) in (1) for the case of \( p > 1 \). Then there exists a unique additive mapping \( F : X \to Y \) such that

\[
\| f(x) - F(x) \| \leq \delta + \| f(0) \| + \frac{\theta}{2^{1-p} - 1} \| x \|^p \quad \text{for} \ p < 1
\]

or

\[
\| f(x) - F(x) \| \leq \frac{2^{p-1} - 1}{2^{p-1} - 1} \theta \| x \|^p \quad \text{for} \ p > 1,
\]

for all \( x \in X \).

**Proof.** If we put \( y = 0 \) in (1), then we have

\[
\| 2f\left( \frac{x}{2} \right) - f(x) \| \leq \delta + \| f(0) \| + \theta \| x \|^p
\]

for all \( x \in X \). By induction on \( n \), we prove that

\[
\| 2^n f(2^n x) - f(x) \| \leq (\delta + \| f(0) \|) \sum_{k=1}^{n} 2^{-k} + \theta \| x \|^p \sum_{k=1}^{n} 2^{-(1-p)k}
\]

for the case when \( 0 < p < 1 \). By substituting \( 2x \) for \( x \) in (6) and dividing by 2 both sides of (6) we see the validity of (7) for \( n = 1 \). Now, assume that the inequality (7) holds true for some \( n \in \mathbb{N} \). If we replace \( x \) in (6) by \( 2^{n+1} x \) and divide both sides of (6) by 2, then it follows from (7) that

\[
\| 2^{-(n+1)} f(2^{n+1} x) - f(x) \| \leq 2^{-n} \| 2^{-1} f(2^{n+1} x) - f(2^n x) \| + \| 2^{-n} f(2^n x) - f(x) \|
\]

\[
\leq (\delta + \| f(0) \|) \sum_{k=1}^{n+1} 2^{-k} + \theta \| x \|^p \sum_{k=1}^{n+1} 2^{-(1-p)k}.
\]

This completes the proof of the inequality (7). Let’s define

\[
F(x) = \lim_{n \to \infty} 2^{-n} f(2^n x)
\]

for all \( x \in X \). The definition (8) is available because \( Y \) is a Banach space and the sequence \( \{ 2^{-n} f(2^n x) \} \) is a Cauchy sequence for all \( x \in X \). For \( n > m \) we use (7) to get

\[
\| 2^{-n} f(2^n x) - 2^{-m} f(2^m x) \|
\]

\[
= 2^{-m} \| 2^{-(n-m)} f(2^{n-m} \cdot 2^m x) - f(2^m x) \|
\]

\[
\leq 2^{-m} \left( \delta + \| f(0) \| + \frac{2^{mp}}{2^{1-p} - 1} \theta \| x \|^p \right)
\]

\[
\to 0 \quad \text{as} \ m \to \infty.
\]
Let $x, y \in X$ be arbitrary. It then follows from (8) and (1) that
\[
\|F(x + y) - F(x) - F(y)\| = \lim_{n \to \infty} 2^{-(n+1)} \left\| 2f \left( \frac{2^{n+1}(x + y)}{2} \right) - f(2^{n+1}x) - f(2^{n+1}y) \right\|
\leq \lim_{n \to \infty} 2^{-(n+1)} \left[ \delta + \theta 2^{(n+1)p} (\|x\|^p + \|y\|^p) \right]
= 0.
\]
Hence, $F$ is an additive mapping, and the inequality (7) and the definition (8) imply the validity of (4).

Now, let $G : X \to Y$ be another additive mapping which satisfies the inequality (4). Then, it follows from (4) that
\[
\|F(x) - G(x)\| = 2^{-n} \|F(2^n x) - G(2^n x)\|
\leq 2^{-n} \left( \|F(2^n x) - f(2^n x)\| + \|f(2^n x) - G(2^n x)\| \right)
\leq 2^{-n} \left( 2\delta + 2\|f(0)\| + \frac{2\theta}{21-p-1} 2^{np}\|x\|^p \right)
\]
for all $x \in X$ and for any $n \in \mathbb{N}$. Since the right-hand side of (9) tends to 0 as $n \to \infty$, we conclude that $F(x) = G(x)$ for all $x \in X$, which proves the uniqueness of $F$.

For the case when $p > 1$ and $\delta = 0$ in the functional inequality (1) we can analogously prove the inequality
\[
\|2^n f(2^{-n} x) - f(x)\| \leq \theta \|x\|^p \sum_{k=0}^{n-1} 2^{-(p-1)k}
\]
instead of (7). The rest of the proof for this case goes through in the similar way. \qed

Remark 1. The proof of the Hyers-Ulam-Rassias stability of Jensen’s equation for the case of $p = 0$ is similar to that of Theorem 1: If a mapping $f : X \to Y$ satisfies the inequality (1) with $\theta = 0$ for all $x, y \in X$, then there exists a unique additive mapping $F : X \to Y$ satisfying (4) with $\theta = 0$.

Remark 2. Let $p \in [0, 1)$ be given. By substituting $x + y$ for $x$ and putting $y = 0$ in (1) we get
\[
\|2f \left( \frac{x + y}{2} \right) - f(x + y)\| \leq \delta + \|f(0)\| + \theta \|x\|^p + \|y\|^p.
\]
This inequality, together with (1), yields
\[
\|f(x + y) - f(x) - f(y)\| \leq 2\delta + \|f(0)\| + 2\theta \|x\|^p + \|y\|^p
\]
for all $x, y \in X$. According to D. H. Hyers [2] and Th. M. Rassias [6] there exists a unique additive mapping $F : X \to Y$ such that
\[
\|f(x) - F(x)\| \leq 2\delta + \|f(0)\| + \frac{2\theta}{1-2^{p-1}} \|x\|^p, \quad x \in X,
\]
which is by no means attractive in comparison with (4).
Remark 3. We also remark that the ideas from the proof of Theorem 1 cannot be applied to the proof of the stability of (1) for the case of \( p < 0 \). An essential process in the proof of Theorem 1 was to put \( y = 0 \) in the inequality (1) which is impossible for the case when \( p < 0 \). The Hyers-Ulam-Rassias stability problem for the case of \( p < 0 \) remains still as an open problem.

Th. M. Rassias and P. Šemrl have constructed in their paper [7] a continuous real-valued mapping to show that the functional inequality

\[
\|f(x + y) - f(x) - f(y)\| \leq \theta (\|x\| + \|y\|)
\]

is not stable in the sense of D. H. Hyers, S. M. Ulam and Th. M. Rassias. By using the result of [7], we prove in the following theorem that the mapping constructed by Rassias and Šemrl serves as a counterexample to Theorem 1 for the case \( p = 1 \).

**Theorem 2.** The continuous real-valued mapping defined by

\[
f(x) = \begin{cases} x \log_2(x + 1) & \text{for } x \geq 0, \\ x \log_2 |x - 1| & \text{for } x < 0 \end{cases}
\]

satisfies the inequality

\[
(10) \quad \left| 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right| \leq 2 (|x| + |y|),
\]

for all \( x, y \in \mathbb{R} \), and the range of \( |f(x) - a(x)|/|x| \) for \( x \neq 0 \) is unbounded for each additive mapping \( a : \mathbb{R} \to \mathbb{R} \).

**Proof.** It follows from [7] that the mapping \( f \) satisfies the inequality

\[
(11) \quad |f(x + y) - f(x) - f(y)| \leq |x| + |y|
\]

for all \( x, y \in \mathbb{R} \). By substituting \( x/2 \) and \( y/2 \) for \( x \) and \( y \) in (11), respectively, and multiplying both sides by 2, we have

\[
(12) \quad \left| 2f \left( \frac{x + y}{2} \right) - 2f \left( \frac{x}{2} \right) - 2f \left( \frac{y}{2} \right) \right| \leq |x| + |y|
\]

for any \( x, y \in \mathbb{R} \). If we put \( x = y \) and divide both sides in (12) by 2, then we get

\[
(13) \quad \left| f(x) - 2f \left( \frac{x}{2} \right) \right| \leq |x|
\]

for \( x \in \mathbb{R} \). By using (12) we obtain

\[
\left| 2f \left( \frac{x + y}{2} \right) - 2f \left( \frac{x}{2} \right) - 2f \left( \frac{y}{2} \right) \right| = \left| 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) + f(x) - 2f \left( \frac{x}{2} \right) + f(y) - 2f \left( \frac{y}{2} \right) \right| 
\]

\[
\leq |x| + |y|
\]

for \( x, y \in \mathbb{R} \). The validity of (10) follows immediately from (13) and (14). It is well-known that if an additive mapping \( a : \mathbb{R} \to \mathbb{R} \) is continuous at a point, then \( a(x) = cx \), where \( c \) is a real number. It is trivial that \( |f(x) - cx|/|x| \to \infty \) as \( x \to \infty \) for any real number \( c \), and that the range of \( |f(x) - a(x)|/|x| \) for \( x \neq 0 \) is also unbounded for every non-continuous additive mapping \( a : \mathbb{R} \to \mathbb{R} \), because the graph of the mapping \( a \) is everywhere dense in \( \mathbb{R}^2 \).
3. Hyper-Ulam Stability on a Restricted Domain

The Hyers-Ulam stability for Jensen's equation on a restricted domain is investigated, and the result is applied to the study of an interesting asymptotic property of additive mappings.

**Theorem 3.** Let \( d > 0 \) and \( \delta \geq 0 \) be given. Assume that a mapping \( f : X \to Y \) satisfies the functional inequality

\[
\left\| 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right\| \leq \delta
\]

for all \( x, y \in X \) with \( \|x\| + \|y\| \geq d \). Then there exists a unique additive mapping \( F : X \to Y \) such that

\[
\|f(x) - F(x)\| \leq 5\delta + \|f(0)\|
\]

for all \( x \in X \).

**Proof.** Suppose \( \|x\| + \|y\| < d \). If \( x = y = 0 \), we can choose a \( z \in X \) such that \( \|z\| = d \). Otherwise, let \( z = (1 + d/\|x\|) x \) for \( \|x\| \geq \|y\| \) or \( z = (1 + d/\|y\|) y \) for \( \|x\| < \|y\| \). It is then obvious that

\[
\|x - z\| + \|y + z\| \geq d; \quad \|2z\| + \|x - z\| \geq d; \quad \|y\| + \|2z\| \geq d;
\]

\[
\|y + z\| + \|z\| \geq d; \quad \|x\| + \|z\| \geq d.
\]

From (15), (17) and the relation

\[
2f \left( \frac{x + y}{2} \right) - f(x) - f(y) = 2f \left( \frac{x + y}{2} \right) - f(x - z) - f(y + z)
\]

\[
- \left[ 2f \left( \frac{x + z}{2} \right) - f(2z) - f(x - z) \right]
\]

\[
+ \left[ 2f \left( \frac{y + 2z}{2} \right) - f(y) - f(2z) \right]
\]

\[
- \left[ 2f \left( \frac{y + 2z}{2} \right) - f(2z) - f(y + z) - f(z) \right]
\]

\[
+ \left[ 2f \left( \frac{x + z}{2} \right) - f(x) - f(z) \right]
\]

we get

\[
\left\| 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \right\| \leq 5\delta.
\]

In view of (15) and (18), the mapping \( f \) satisfies the inequality (18) for all \( x, y \in X \). Therefore, it follows from (18) and Theorem 1 that there exists a unique additive mapping \( F : X \to Y \) which satisfies the inequality (16) for all \( x \in X \).

By using the result of Theorem 3 we now prove an asymptotic behavior of the additive mappings.

**Corollary 4.** Suppose a mapping \( f : X \to Y \) satisfies the condition \( f(0) = 0 \). Then \( f \) is additive if and only if the asymptotic condition (2) holds true.
Proof. On account of (2), there exists a sequence \((\delta_n)\), monotonically decreasing to 0, such that
\[
\left\| 2f \left( \frac{x+y}{2} \right) - f(x) - f(y) \right\| \leq \delta_n \tag{19}
\]
for all \(x, y \in X\) with \(\|x\| + \|y\| \geq n\). It then follows from (19) and Theorem 3 that there exists a unique additive mapping \(F_n : X \to Y\) such that
\[
\left\| f(x) - F_n(x) \right\| \leq 5\delta_n \tag{20}
\]
for all \(x \in X\). Let \(\ell, m \in \mathbb{N}\) satisfy \(m \geq \ell\). Obviously, it follows from (20) that
\[
\left\| f(x) - F_m(x) \right\| \leq 5\delta_m \leq 5\delta_\ell
\]
for all \(x \in X\), since \((\delta_n)\) is a monotonically decreasing sequence. The uniqueness of \(F_n\) implies \(F_m = F_\ell\). Hence, by letting \(n \to \infty\) in (20), we conclude that \(f\) is additive. The reverse assertion is trivial. \(\Box\)

References