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HYERS-ULAM-RASSIAS STABILITY OF JENSEN'S EQUATION AND ITS APPLICATION

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Abstract. The Hyers-Ulam-Rassias stability for the Jensen functional equation is investigated, and the result is applied to the study of an asymptotic behavior of the additive mappings; more precisely, the following asymptotic property shall be proved: Let X and Y be a real normed space and a real Banach space, respectively. A mapping $f : X \to Y$ satisfying $f(0) = 0$ is additive if and only if $\|2f\,[(x+y)/2] - f(x) - f(y)\| \to 0$ as $\|x\| + \|y\| \to \infty$.

1. Introduction

In 1940, S. M. Ulam [9] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then a homomorphism $H : G_1 \to G_2$ exists with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by D. H. Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias [6] generalized the result of Hyers as follows:

> Let $f : X \to Y$ be a mapping between Banach spaces and let $0 \le p < 1$ be fixed. If f satisfies the inequality
> $$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$
> for some $\theta \ge 0$ and for all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that
> $$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$
> for all $x \in X$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

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Taking this fact into account, the additive functional equation $f(x+y) = f(x)+f(y)$ is said to have the Hyers-Ulam-Rassias stability on (X, Y). This terminology is also applied to the case of other functional equations. For more detailed definitions of such terminology one can refer to [1] and [3].

Throughout this paper, let X and Y be a real normed space and a real Banach space, respectively. According to Theorem 6 in [5], a mapping $f : X \to Y$ satisfying $f(0) = 0$ is a solution of the Jensen functional equation

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$$

if and only if it satisfies the additive Cauchy equation $f(x+y) = f(x)+f(y)$. Hence, the most general continuous solution of the Jensen's equation in $\mathbb{R}$ is $f(x) = ax+b$, where a and b are arbitrary constants.

The first result on the stability of Jensen's equation was obtained by Z. Kominek (see [4]). In fact, he proved the following theorem:

Theorem. *Let D be a subset of $\mathbb{R}^n$ with non-empty interior. Assume that there exists an x_0 in the interior of D such that $D_0 = D - x_0$ satisfies the condition $(1/2)D_0 \subset D_0$. Let a mapping $f : D \to Y$ satisfy the inequality*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \delta,$$

for some $\delta \geq 0$ and for all $x, y \in D$. Then there exist a mapping $F : \mathbb{R}^n \to Y$ and a constant $K > 0$ such that

$$2F\left(\frac{x+y}{2}\right) = F(x) + F(y)$$

for all $x, y \in \mathbb{R}^n$, and

$$\|f(x) - F(x)\| \leq K$$

for all $x \in D$.

In section 2 of the present paper, using ideas from the papers of Th. M. Rassias [6] and D. H. Hyers [2], the Hyers-Ulam-Rassias stability of Jensen's equation will be investigated, i.e., the stability of the functional inequality

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \leq \delta + \theta\left(\|x\|^p + \|y\|^p\right) \tag{1}$$

for the case of $p \geq 0$ $(p \neq 1)$ shall be proved. Moreover, by using the same mapping which was constructed by Th. M. Rassias and P. Šemrl [7], we shall show that the inequality (1) is not stable for the case when $p = 1$. In section 3, the Hyers-Ulam stability for Jensen's equation on a restricted domain will be treated, and the result applied to the study of an interesting asymptotic behavior of the additive mappings—more precisely, we prove that a mapping $f : X \to Y$ satisfying $f(0) = 0$ is additive if and only if

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\| \to 0 \quad \text{as} \quad \|x\| + \|y\| \to \infty. \tag{2}$$

It should be remarked here that F. Skof [8] has also proved an asymptotic property of the additive mappings. Indeed, she proved that a mapping $f : X \to Y$ is additive if and only if

$$\|f(x+y) - f(x) - f(y)\| \to 0 \quad \text{as} \quad \|x\| + \|y\| \to \infty. \tag{3}$$

2. Hyers-Ulam-Rassias stability

First, we prove the Hyers-Ulam-Rassias stability of the Jensen's equation. Assume that $\delta \geq 0$ and $\theta \geq 0$ are fixed.

Theorem 1. *Let $p > 0$ be given with $p \neq 1$. Suppose a mapping $f : X \to Y$ satisfies the inequality (1) for all $x, y \in X$. Further, assume $f(0) = 0$ and $\delta = 0$ in (1) for the case of $p > 1$. Then there exists a unique additive mapping $F : X \to Y$ such that*

$$\|f(x) - F(x)\| \leq \delta + \|f(0)\| + \frac{\theta}{2^{1-p} - 1}\|x\|^p \quad (\text{for } p < 1) \tag{4}$$

or

$$\|f(x) - F(x)\| \leq \frac{2^{p-1}}{2^{p-1} - 1}\theta\|x\|^p \quad (\text{for } p > 1), \tag{5}$$

for all $x \in X$.

Proof. If we put $y = 0$ in (1), then we have

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\| \leq \delta + \|f(0)\| + \theta\|x\|^p \tag{6}$$

for all $x \in X$. By induction on n, we prove that

$$\left\|2^{-n}f\left(2^n x\right) - f(x)\right\| \leq (\delta + \|f(0)\|)\sum_{k=1}^{n} 2^{-k} + \theta\|x\|^p \sum_{k=1}^{n} 2^{-(1-p)k} \tag{7}$$

for the case when $0 < p < 1$. By substituting $2x$ for x in (6) and dividing by 2 both sides of (6) we see the validity of (7) for $n = 1$. Now, assume that the inequality (7) holds true for some $n \in \mathbb{N}$. If we replace x in (6) by $2^{n+1}x$ and divide both sides of (6) by 2, then it follows from (7) that

$$\begin{aligned} &\left\|2^{-(n+1)}f\left(2^{n+1}x\right) - f(x)\right\| \\ &\leq 2^{-n}\left\|2^{-1}f\left(2^{n+1}x\right) - f(2^n x)\right\| + \left\|2^{-n}f\left(2^n x\right) - f(x)\right\| \\ &\leq (\delta + \|f(0)\|)\sum_{k=1}^{n+1} 2^{-k} + \theta\|x\|^p \sum_{k=1}^{n+1} 2^{-(1-p)k}. \end{aligned}$$

This completes the proof of the inequality (7). Let's define

$$F(x) = \lim_{n\to\infty} 2^{-n}f\left(2^n x\right) \tag{8}$$

for all $x \in X$. The definition (8) is available because Y is a Banach space and the sequence $\{2^{-n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$: For $n > m$ we use (7) to get

$$\begin{aligned} &\left\|2^{-n}f(2^n x) - 2^{-m}f(2^m x)\right\| \\ &= 2^{-m}\left\|2^{-(n-m)}f(2^{n-m} \cdot 2^m x) - f(2^m x)\right\| \\ &\leq 2^{-m}\left(\delta + \|f(0)\| + \frac{2^{mp}}{2^{1-p} - 1}\theta\|x\|^p\right) \\ &\to 0 \text{ as } m \to \infty. \end{aligned}$$

Let $x, y \in X$ be arbitrary. It then follows from (8) and (1) that

$$\begin{aligned}
&\|F(x+y) - F(x) - F(y)\| \\
&= \lim_{n\to\infty} 2^{-(n+1)} \left\| 2f\left(\frac{2^{n+1}(x+y)}{2}\right) - f(2^{n+1}x) - f(2^{n+1}y) \right\| \\
&\le \lim_{n\to\infty} 2^{-(n+1)} \left[\delta + \theta 2^{(n+1)p} \left(\|x\|^p + \|y\|^p\right)\right] \\
&= 0.
\end{aligned}$$

Hence, F is an additive mapping, and the inequality (7) and the definition (8) imply the validity of (4).

Now, let $G : X \to Y$ be another additive mapping which satisfies the inequality (4). Then, it follows from (4) that

$$\begin{aligned}
\|F(x) - G(x)\| &= 2^{-n} \|F(2^n x) - G(2^n x)\| \\
&\le 2^{-n} \left(\|F(2^n x) - f(2^n x)\| + \|f(2^n x) - G(2^n x)\|\right) \\
&\le 2^{-n} \left(2\delta + 2\|f(0)\| + \frac{2\theta}{2^{1-p} - 1} 2^{np} \|x\|^p\right) \qquad (9)
\end{aligned}$$

for all $x \in X$ and for any $n \in \mathbb{N}$. Since the right-hand side of (9) tends to 0 as $n \to \infty$, we conclude that $F(x) = G(x)$ for all $x \in X$, which proves the uniqueness of F.

For the case when $p > 1$ and $\delta = 0$ in the functional inequality (1) we can analogously prove the inequality

$$\left\|2^n f\left(2^{-n}x\right) - f(x)\right\| \le \theta \|x\|^p \sum_{k=0}^{n-1} 2^{-(p-1)k}$$

instead of (7). The rest of the proof for this case goes through in the similar way. □

Remark 1. The proof of the Hyers-Ulam-Rassias stability of Jensen's equation for the case of $p = 0$ is similar to that of Theorem 1: If a mapping $f : X \to Y$ satisfies the inequality (1) with $\theta = 0$ for all $x, y \in X$, then there exists a unique additive mapping $F : X \to Y$ satisfying (4) with $\theta = 0$.

Remark 2. Let $p \in [0, 1)$ be given. By substituting $x + y$ for x and putting $y = 0$ in (1) we get

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x+y)\right\| \le \delta + \|f(0)\| + \theta\left(\|x\|^p + \|y\|^p\right).$$

This inequality, together with (1), yields

$$\|f(x+y) - f(x) - f(y)\| \le 2\delta + \|f(0)\| + 2\theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. According to D. H. Hyers [2] and Th. M. Rassias [6] there exists a unique additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le 2\delta + \|f(0)\| + \frac{2\theta}{1 - 2^{p-1}} \|x\|^p, \quad x \in X,$$

which is by no means attractive in comparison with (4).

Remark 3. We also remark that the ideas from the proof of Theorem 1 cannot be applied to the proof of the stability of (1) for the case of $p < 0$. An essential process in the proof of Theorem 1 was to put $y = 0$ in the inequality (1) which is impossible for the case when $p < 0$. The Hyers-Ulam-Rassias stability problem for the case of $p < 0$ remains still as an open problem.

Th. M. Rassias and P. Šemrl have constructed in their paper [7] a continuous real-valued mapping to show that the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\left(\|x\| + \|y\|\right)$$

is not stable in the sense of D. H. Hyers, S. M. Ulam and Th. M. Rassias. By using the result of [7], we prove in the following theorem that the mapping constructed by Rassias and Šemrl serves as a counterexample to Theorem 1 for the case $p = 1$.

Theorem 2. *The continuous real-valued mapping defined by*

$$f(x) = \begin{cases} x\log_2(x+1) & \text{for } x \geq 0, \\ x\log_2|x-1| & \text{for } x < 0 \end{cases}$$

satisfies the inequality

$$\left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right| \leq 2\left(|x| + |y|\right), \tag{10}$$

for all $x, y \in \mathbb{R}$, and the range of $|f(x) - a(x)|/|x|$ for $x \neq 0$ is unbounded for each additive mapping $a : \mathbb{R} \to \mathbb{R}$.

Proof. It follows from [7] that the mapping f satisfies the inequality

$$|f(x+y) - f(x) - f(y)| \leq |x| + |y| \tag{11}$$

for all $x, y \in \mathbb{R}$. By substituting $x/2$ and $y/2$ for x and y in (11), respectively, and multiplying both sides by 2, we have

$$\left|2f\left(\frac{x+y}{2}\right) - 2f\left(\frac{x}{2}\right) - 2f\left(\frac{y}{2}\right)\right| \leq |x| + |y| \tag{12}$$

for any $x, y \in \mathbb{R}$. If we put $x = y$ and divide both sides in (12) by 2, then we get

$$\left|f(x) - 2f\left(\frac{x}{2}\right)\right| \leq |x| \tag{13}$$

for $x \in \mathbb{R}$. By using (12) we obtain

$$\begin{aligned} &\left|2f\left(\frac{x+y}{2}\right) - 2f\left(\frac{x}{2}\right) - 2f\left(\frac{y}{2}\right)\right| \\ &\quad = \left|2f\left(\frac{x+y}{2}\right) - f(x) - f(y) + f(x) - 2f\left(\frac{x}{2}\right) + f(y) - 2f\left(\frac{y}{2}\right)\right| \\ &\quad \leq |x| + |y| \qquad (14) \end{aligned}$$

for $x, y \in \mathbb{R}$. The validity of (10) follows immediately from (13) and (14). It is well-known that if an additive mapping $a : \mathbb{R} \to \mathbb{R}$ is continuous at a point, then $a(x) = cx$, where c is a real number. It is trivial that $|f(x) - cx|/|x| \to \infty$ as $x \to \infty$ for any real number c, and that the range of $|f(x) - a(x)|/|x|$ for $x \neq 0$ is also unbounded for every non-continuous additive mapping $a : \mathbb{R} \to \mathbb{R}$, because the graph of the mapping a is everywhere dense in $\mathbb{R}^2$. □

3. Hyers-Ulam stability on a restricted domain

The Hyers-Ulam stability for Jensen's equation on a restricted domain is investigated, and the result is applied to the study of an interesting asymptotic property of additive mappings.

Theorem 3. *Let $d > 0$ and $\delta \geq 0$ be given. Assume that a mapping $f : X \to Y$ satisfies the functional inequality*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \delta \tag{15}$$

for all $x, y \in X$ with $\|x\| + \|y\| \geq d$. Then there exists a unique additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq 5\delta + \|f(0)\| \tag{16}$$

for all $x \in X$.

Proof. Suppose $\|x\| + \|y\| < d$. If $x = y = 0$, we can choose a $z \in X$ such that $\|z\| = d$. Otherwise, let $z = (1 + d/\|x\|)\,x$ for $\|x\| \geq \|y\|$ or $z = (1 + d/\|y\|)\,y$ for $\|x\| < \|y\|$. It is then obvious that

$$\begin{array}{l} \|x - z\| + \|y + z\| \geq d; \quad \|2z\| + \|x - z\| \geq d; \quad \|y\| + \|2z\| \geq d; \\ \|y + z\| + \|z\| \geq d; \quad \|x\| + \|z\| \geq d. \end{array} \tag{17}$$

From (15), (17) and the relation

$$\begin{aligned} 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \quad &= \quad 2f\left(\frac{x+y}{2}\right) - f(x-z) - f(y+z) \\ &\quad - \left[2f\left(\frac{x+z}{2}\right) - f(2z) - f(x-z)\right] \\ &\quad + \left[2f\left(\frac{y+2z}{2}\right) - f(y) - f(2z)\right] \\ &\quad - \left[2f\left(\frac{y+2z}{2}\right) - f(y+z) - f(z)\right] \\ &\quad + \left[2f\left(\frac{x+z}{2}\right) - f(x) - f(z)\right] \end{aligned}$$

we get

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq 5\delta. \tag{18}$$

In view of (15) and (18), the mapping f satisfies the inequality (18) for all $x, y \in X$. Therefore, it follows from (18) and Theorem 1 that there exists a unique additive mapping $F : X \to Y$ which satisfies the inequality (16) for all $x \in X$. □

By using the result of Theorem 3 we now prove an asymptotic behavior of the additive mappings.

Corollary 4. *Suppose a mapping $f : X \to Y$ satisfies the condition $f(0) = 0$. Then f is additive if and only if the asymptotic condition* (2) *holds true.*

Proof. On account of (2), there exists a sequence (δ_n), monotonically decreasing to 0, such that

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \delta_n \tag{19}$$

for all $x, y \in X$ with $\|x\| + \|y\| \ge n$. It then follows from (19) and Theorem 3 that there exists a unique additive mapping $F_n : X \to Y$ such that

$$\|f(x) - F_n(x)\| \le 5\delta_n \tag{20}$$

for all $x \in X$. Let $\ell, m \in \mathbb{N}$ satisfy $m \ge \ell$. Obviously, it follows from (20) that

$$\|f(x) - F_m(x)\| \le 5\delta_m \le 5\delta_\ell$$

for all $x \in X$, since (δ_n) is a monotonically decreasing sequence. The uniqueness of F_n implies $F_m = F_\ell$. Hence, by letting $n \to \infty$ in (20), we conclude that f is additive. The reverse assertion is trivial. □

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