TRIANGULAR EXTENSION SPECTRUM
OF WEIGHTED SHIFTS

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Abstract. A necessary and sufficient condition for a complex number to be
in the triangular extension spectrum of a weighted backward shift is obtained.
It is shown that the triangular extension spectrum of a weighted backward
shift is always a closed annulus when it is not empty. Moreover, for any given
closed annulus, there exists a weighted backward shift with the annulus as its
triangular extension spectrum.

A bounded operator $T$ acting on a separable Hilbert space $H$ is called triangular
if there exists an orthonormal basis $\{e_n\}_{1}^{\infty}$ for $H$ such that $Te_n$ is in the span of
$\{e_1, e_2, \ldots, e_n\}$ for each $n$. A vector $x$ is called an algebraic vector
for an operator $T$ if there exists a nonzero polynomial $p(z)$ such that
$p(T)x = 0$. Let $\mathcal{E}_T$ be the set of algebraic vectors for an operator $T$. It can be shown that $T$ is triangular
if and only if $\mathcal{E}_T$ is dense in $H$. $T$ is called algebraic if there exists a nonzero
polynomial $p(z)$ so that $p(T) = 0$. An operator $A$ is called semitriangular if $A$ is
an extension of a triangular operator by a finite rank operator, or equivalently,
the norm closure of the set of algebraic vectors has finite codimension in $H$. The
finite codimension is called the index of semitriangularity of $A$. Semitriangular
operators have been proved to be very useful in constructing counter-examples.
Several longstanding open questions were answered negatively in [W], which led to
the study of semitriangular operators; see [HLP], [HLW], [LW1], [LW2].

Let $B(H)$ be the set of all bounded linear operators on $H$, $\sigma(T)$ denote
the spectrum of $T$ in $B(H)$, $\sigma_e(T)$ denote the essential spectrum of $T$, and let $\mathbb{N}$ be the
set of natural numbers. For any bounded sequence of complex numbers $\{s_n\}_{1}^{\infty}$, we
define the corresponding weighted backward shift $S$ with respect to $\{e_n\}_{1}^{\infty}$ such that
$Se_1 = 0, Se_2 = s_1 e_1, Se_3 = s_2 e_2, \ldots, Se_{i+1} = s_i e_i, \ldots$. Clearly, $\|S\| = \sup \{|s_i|\}$.
Throughout this note, we always let $\mathbb{N}_0 = \{i \in \mathbb{N} : s_i = 0\}$.

The triangular extension spectrum was first introduced in [HLP] and it was
shown that the triangular extension spectrum plays a very important role in studying
semitriangular operators and algebraic extensions of triangular operators.

Definition 1. Let $T \in B(H)$ be a triangular operator. The triangular extension
spectrum of $T$, denoted by $\sigma_\Delta(T)$, is defined to be the set of all $t \in \mathbb{C}$ such that
there exists a \( b \in B(\mathbb{C}, H) (=H) \) with the property that
\[
A = \begin{pmatrix} T & b \\ 0 & t \end{pmatrix}
\]
has index of semitriangularity 1 in \( H \oplus \mathbb{C} \). Let \( \rho_\Delta(T) \) denote the complement of \( \sigma_\Delta(T) \) in \( \mathbb{C} \).

**Lemma 2 ([HLP]).** If \( T \) is a triangular operator, then \( t \in \sigma_\Delta(T) \) if and only if \( \mathcal{E}_T + \text{Ran}(T-tI) \neq H \).

It follows from the above lemma that \( \sigma_\Delta(T) \subseteq \sigma(T) \). In fact, it is shown that \( \sigma_\Delta(T) \) is always a compact subset of \( \sigma_c(T) \) ([HLP]). Theorem 2.7 of [HLW] implies \( \sigma_\Delta(T) = \emptyset \) if and only if \( T \) is algebraic.

**Lemma 3 ([HLW]).** If \( \mathcal{E}_T + \text{Ran}(T-tI) = H \), then there exists a positive integer \( k \) such that \( \ker(T-tI)^k + \text{Ran}(T-tI) = H \).

**Lemma 4.** Let \( S \) be a weighted backward shift. If \( S \) is not algebraic, then
\[
0 \in \sigma_\Delta(S) \text{ if and only if } \lim_{i \to \infty} |s_i| = 0.
\]

**Proof.** “\( \Rightarrow \)” Case 1: \( \mathbb{N}_0 \) is infinite.

Suppose \( \mathbb{N}_0 = \{n_1, n_2, \ldots \} \) such that \( n_1 < n_2 < n_3 < \cdots \). It is easy to see that \( S \) is algebraic if and only if \( S \) is nilpotent if and only if \( \mathbb{N}_0 \) is infinite and \( \sup\{n_{j+1} - n_j\} \) is finite. Let \( H_0 = [e_{n_1}, e_{n_2}, \ldots, e_{n_j}, \ldots] \), where \( [\cdot] \) denotes the closed linear span. Since \( Se_{n_{j+1}} = e_{n_j}, \text{Ran}(S) \subseteq H_0^\perp \). Since \( S \) is not nilpotent, thus \( \forall k \in \mathbb{N}, \exists m_k \) such that \( S^k e_{m_k} \neq 0 \). Suppose \( n_{j-1} < m_k \leq n_{j_k} \); then \( S^k e_{m_k} \neq 0 \). Let \( b = \sum_{i=1}^{\infty} \frac{1}{i} e_i \). We show that \( b \not\in \ker(S^k) + \text{Ran}(S) \) for any \( k \), which, by Lemma 2 and Lemma 3, implies \( 0 \in \sigma_\Delta(S) \).

Suppose \( b = u + Sv \), where \( u \in \text{Ker}(S^k) \). Let \( u = \sum_{i=1}^{\infty} u_i e_i \).

Since \( \text{Ran}(S) \subseteq H_0^\perp \), we have \( u_i = \frac{1}{i} \).

Since \( S^k u = \sum_{i=1}^{\infty} u_i S^k e_i = 0 \) and \( \langle S^k e_i, S^k e_j \rangle = 0 \), for all \( i \neq j \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product, we have \( u_i S^k e_i = 0 \), for all \( i \), in particular, \( u_i S^k e_i = 0 \), i.e. \( S^k e_i = 0 \), thus \( S^k e_i = 0 \), for all \( i \), which is a contradiction.

Case 2: \( \mathbb{N}_0 \) is finite.

Let \( P_i \) be the orthogonal projections of \( H \) onto \( [e_1, e_2, \ldots, e_n] \) for \( n = 1, 2, 3, \ldots \). We need to show that \( \ker(S^k) + \text{Ran}(S) \neq H \). Since \( \ker(S^i) \subseteq \ker(S^{i+1}) \) for all \( i \), we only need to show that \( \ker(S^k) + \text{Ran}(S) \neq H \) for \( k \) large enough. Without loss of generality, we assume \( k \) is large enough so that \( s_i \neq 0 \) for all \( i > k \). For any \( x = \sum_{i=1}^{\infty} x_i e_i \in \ker(S) \), \( Sx = \sum_{i=1}^{\infty} x_i s_i e_i = 0 \). Since \( \mathbb{N}_0 \) is finite, we have all but finitely many \( x_i \)'s are zero. Therefore there exists an \( N \) such that \( \ker(S) \subseteq P_N H \), so \( \ker(S^k) \subseteq P_M H \), where \( M = N + k \). If \( \ker(S^k) + \text{Ran}(S) = H \), then \( \text{Ran}(P_M^\perp S) = P_M^\perp \text{Ran}(S) = P_M^\perp H \). Note that \( P_M^\perp S = P_M^\perp SP_{M+1}^\perp \) and \( s_i \neq 0 \), for \( i > M \). Hence \( P_M^\perp SP_{M+1}^\perp \) is a 1-1 and onto map from \( P_M^\perp H \) to \( P_M^\perp H \). Therefore \( P_M^\perp SP_{M+1}^\perp \) is invertible by the open mapping theorem, but since \( \lim_{i \to \infty} |s_i| = 0 \), there exists a subsequence \( s_{n_j} \) such that \( s_{n_j} \to 0 \) as \( j \to \infty \). Thus \( P_M^\perp SP_{M+1}^\perp e_{n_j} = s_{n_j} e_{n_j} \to 0 \) as \( j \to \infty \), contradicting the fact that \( P_M^\perp SP_{M+1}^\perp \) is invertible.

“\( \Leftarrow \)” If \( \lim_{i \to \infty} |s_i| \neq 0 \), then there exist an \( \varepsilon_0 > 0 \) and a \( k_0 \in \mathbb{N} \) such that \( |s_i| > \varepsilon_0 \), for all \( i > k_0 \). For any \( x \in P_{k_0}^\perp H \), \( x = \sum_{i=k_0+1}^{\infty} x_i e_i \), let \( y = \sum_{i=k_0+1}^{\infty} \frac{1}{i} x_i e_i + 1 \); then \( Sy = x \), i.e. \( \text{Ran}(S) \supseteq P_{k_0}^\perp H \). Since \( P_{k_0}^\perp H \subseteq \ker(S^k) \) for \( k \) large enough, we have
Lemma 2 and Lemma 3, we have $\left|\left|\left|\left|\left| S \right|\right|\right|\right|$. From Lemma 7, $\dim \ker(S)$.

Proof. We only need to show that $\sigma(S) \subseteq \sigma(S)$. Let $t \in \sigma(S)$. If $t = 0$, then $t \in \sigma(S)$ by Lemma 4. If $t \neq 0$, then $\ker(S-tI) = 0$. By Lemma 2 and Lemma 3, $t \in \sigma(S)$ if and only if if $\ker(S-tI) = 0$, which is equivalent to $t \in \sigma(S)$ by the open mapping theorem, since $\ker(S-tI) = 0$. □

Lemma 6 ([S]). If $S$ is a weighted backward shift with spectral radius $r$, then $\sigma(S) = \{t: |t| \leq r\}$.

The following is essentially Theorem 8 of [S].

Lemma 7. Let $S$ be a weighted backward shift. Suppose that $\mathbb{N}_0$ is finite with $k_0 = \max\{i: i \in \mathbb{N}_0\}$. If $t \neq 0$, then $S-tI$ is not 1-1 if and only if

$$\sum_{n=k_0+1}^{\infty} \left| \frac{s_{n-k_0}}{s_n \cdots s_{k_0+1}} \right|^2 < \infty.$$  

In this case, $\dim \ker(S-tI) = 1$.

Lemma 8. If $t \in \sigma(S)$ and $t \neq 0$, then $t \in \rho(S) \iff \ker(S-tI) = H$.

Proof. From Lemma 7, $\dim \ker(S-tI)$ is at most one, so for any $k$, $\dim \ker(S-tI)^k$ is finite (in fact at most $k$). For any natural number $n$ and $y \in [e_1, \ldots, e_n]$, we can solve $(S-tI)x = y$ for $x$. Thus $\ker(S-tI)$ is dense in $H$. Combining the above with Lemma 2 and Lemma 3, we have $t \in \rho(S) \iff \exists k, \ker(S-tI)^k + \ker(S-tI) = H \Rightarrow \ker(S-tI) = H$. □

Corollary 9. If $S$ is a weighted backward shift and $S$ is not algebraic, then $\sigma(S) = \sigma_e(S)$.

Proof. This follows immediately from Lemma 4 and Lemma 8. □

Lemma 10 ([HLP]). Suppose that $A \in B(H)$ has a closed range and let $P$ be the orthogonal projection of $H$ onto $\ker(A)$. Then there exists an $\epsilon > 0$ such that $\ker(A) \subseteq \ker(A + \lambda P)$ for all $\lambda$ with $|\lambda| < \epsilon$.

Corollary 11. If $\ker(S-tI) = H$, then there exists an $\epsilon > 0$ such that $\ker(S-zI) = H$ for all $z$ with $|z-t| < \epsilon$.

Lemma 12. Suppose that $\mathbb{N}_0$ is finite with $k_0 = \max\{i: i \in \mathbb{N}_0\}$. If $t \in \sigma(S)$ and $t \neq 0$, then $t \in \rho(S)$ if and only if there exist an $M$ and a $z_0$ with $|z_0| > |t|$ such that

$$\left| \frac{z_0^n}{s_k \cdots s_{k+n}} \right| \leq M, \quad \forall k > k_0, n > 0.$$  

Proof. “$\Rightarrow”$ If $t \in \rho(S)$, and $t \neq 0$, then by Lemma 8, $\ker(S-tI) = H$. Since $t \in \sigma(S)$ and $S-tI$ is surjective, $t \not\in \partial \sigma(S)$. Combining with Corollary 11, we can find a disk (centered at $t$) contained in $\sigma(S)$ with the property that $\ker(S-tI) = H$ for all $z$ in the disk. Since the disk is contained in the spectrum, we have $S-tI$ is
not 1-1 for all \( z \) in the disk. In particular, there exists a \( z_0 \) with \( |z_0| > |t| \) so that \( \operatorname{Ran}(S - z_0 I) = H \) and \( S - z_0 I \) is not 1-1. By Lemma 7,
\[
\sum_{n=0}^{\infty} \frac{|z_2^n|^{2n}}{|s_0 + n + \cdots + s_{k+1}|^2} < \infty.
\]
Hence \( S - z_0 I \) is onto and 1-1 from \( \ker(S - z_0 I)^\perp \) to \( H \), so it has a bounded inverse \( B \) from \( H \) to \( \ker(S - z_0 I)^\perp \).
For any \( k > k_0 \), let \( x^{(k)} = B e_k \), write \( x^{(k)} = \sum_{i=1}^{\infty} s_i^{(k)} e_i \); then \( \|x^{(k)}\| \leq \|B\| \), in particular, \( |x_i^{(k)}| \leq \|B\| \) for all \( i \). Let
\[
M = \left\{ 1 + \left( \sum_{n=0}^{\infty} \frac{|z_2^n|^{2n}}{|s_0 + n + \cdots + s_{k+1}|^2} \right)^{1/2} \right\} \|B\|.
\]
Since \( (S - z_0 I)x^{(k)} = (S - z_0 I)Be_k = e_k \), we have
\[
s_k x_{k+1}^{(k)} - z_0 x_i^{(k)} = 0, \quad i \neq k, \quad \text{and} \quad s_k x_{k+1}^{(k)} - z_0 x_k^{(k)} = 1.
\]
From the above, we can obtain
\[
x_i^{(k)} = \frac{z_0}{s_k^{k-1}} x_i^{(k)} - \frac{z_0^{k-k_0-1}}{s_k^{k-1} \cdots s_{k+1}} x_k^{(k)}.
\]
Now for any positive integer \( n \),
\[
x_i^{(k)} = \frac{z_0}{s_k^{k+n}} x_i^{(k)} - \frac{z_0^{n+k-k_0-1}}{s_k^{k+n} \cdots s_{k+1}} x_k^{(k)}.
\]
Thus
\[
\frac{z_0^n}{s_k^{k+n} \cdots s_k} x_i^{(k)} - x_{k+1}^{(k)} + \frac{z_0^{n+k-k_0-1}}{s_k^{k+n} \cdots s_{k+1}} x_k^{(k)}.
\]
Therefore
\[
\left| \frac{z_0^n}{s_k^{k+n} \cdots s_k} x_i^{(k)} \right| \leq \left| x_{k+1}^{(k)} \right| + \left| \frac{z_0^{n+k-k_0-1}}{s_k^{k+n} \cdots s_{k+1}} x_k^{(k)} \right| \leq M.
\]
\("\leq"\) Note that when \( n = 1 \), \( (\ast\ast) \) implies \( |\frac{z_0}{s_k^{k+1} s_k}| \leq M, \forall k > k_0 \), so
\[
\left| \frac{1}{s_k} \right| \leq M \left| \frac{s_k+1}{z_0} \right| \leq M \frac{\|S\|}{z_0}, \forall k > k_0.
\]
Let \( H_1 = \{ e_{k_0+1}, \ldots \} \). We only need to show \( \operatorname{Ran}(S - tI) \supset H_1 \).
First, we define a bounded linear operator \( R \) on \( H_1 \) such that \( R e_i = \frac{1}{t} e_{i+1}, \forall i \geq k_0 + 1 \). It follows that \( \|R\| \leq M \) and \( SR = I_{H_1} \). Since \( (z_0 R)^n e_i = \frac{z_0^n}{s_{i+n-1} \cdots s_i} e_{i+n} \), we have \( \|(z_0 R)^n\| \leq \sup \{|\frac{z_0^n}{s_{i+n-1} \cdots s_i}| : i \geq k_0 + 1\} \leq M |z_0| \), which implies \( \|(tR)^n\| = \|z_0 R^n\| \leq M |z_0| \).
This follows directly from Theorem 5, Lemma 6 and Lemma 12. 

**Proof.** Theorem 13.

If $\|Q\| = M|z_0|$, then $Q \leq M|z_0|$. Therefore, $Q = \sum_{n=1}^{\infty} tR^n$ defines a bounded linear operator on $H_1$. Given any $y \in H_1$, let $x = Qy$; then

$$(S - tI)x = (S - tI)Qy = (S - tI)\sum_{n=1}^{\infty} tR^n y$$

$$= \left( S \sum_{n=1}^{\infty} tR^n - t \sum_{n=1}^{\infty} (tR)^n \right) y$$

$$= \left( StR + S \sum_{n=2}^{\infty} tR^n - t \sum_{n=1}^{\infty} (tR)^n \right) y$$

$$= \left( tI_{H_1} + tI_{H_1} \sum_{n=1}^{\infty} tR^n - t \sum_{n=1}^{\infty} (tR)^n \right) y = ty.$$

This shows $\text{Ran}(S - tI) \supseteq H_1$. 

**Remark.** (i) (***) implies (*). This is because if (***) holds, then

$$\sum_{n=0}^{\infty} \frac{|t|^{2n}}{|s_{k_0+n+1} \cdots s_{k_0+1}|^2} = \sum_{n=0}^{\infty} \left( \frac{t}{z_0} \right)^{2n} \frac{|z_0|}{|s_{k_0+n+1} \cdots s_{k_0+1}|^2}$$

$$\leq \sum_{n=0}^{\infty} \left( \frac{t}{z_0} \right)^{2n} M^2 < \infty.$$

(ii) (***) is equivalent to the following: For any fixed $N > k_0$, there exist an $M$ and a $z_0$ with $|z_0| > |t|$ such that

$$\left| \frac{z_0^n}{s_{k+n}s_{k+n-1} \cdots s_{k}} \right| \leq M, \quad \forall k > N, \ n > 0.$$

Similarly, (*) is equivalent to: For any $N > k_0$,

$$\sum_{n=N}^{\infty} \left| \frac{1}{s_{n} \cdots s_{N}} \right|^2 < \infty.$$

(iii) In the proof of Lemma 12, we see that (***) implies $\frac{1}{s_i} \leq M \frac{\|S\|}{|z_0|}$, $\forall i > k_0$, i.e. $\lim_{i \to \infty} |s_i| \neq 0$.

If $N_0$ is finite, we define $r_1 = \inf\{|t|: t \text{ does not satisfy (***)}\}$, $r_2 = \inf\{|t|: t \text{ does not satisfy (*)}\}$. (In the case $N_0$ is infinite, we define $r_1 = r_2 = 0$.)

Since (***) implies (*), we have $r_1 \leq r_2$.

**Theorem 13.** If $S$ is a weighted backward shift and $S$ is not algebraic, then $\sigma_\Delta(S) = \{ t: r_1 \leq |t| \leq r \}$. In particular, if $S$ is the standard backward shift, then $\sigma_\Delta(S) = \{ t: |t| = 1 \}$. 

**Proof.** This follows directly from Theorem 5, Lemma 6 and Lemma 12. 

Lemma 12 together with Lemma 7 implies the next corollary which gives a more descriptive picture of the spectrum of a weighted backward shift.

**Corollary 14.** With assumptions as above, we have the following:

(i) If $|t| < r_1$, then $S - tI$ is onto but not 1-1.

(ii) If $r_1 \leq |t| < r_2$, then $S - tI$ is neither 1-1 nor onto.
(iii) If \( r_2 < |t| \leq r \), then \( S - tI \) is 1-1 but not onto.
(iv) If \( |t| < r \), then \( S - tI \) is invertible.

**Remark.** \( S - r_2I \) can be 1-1 in some cases and not 1-1 in others. In the case \( r_1 = r_2 \), we do not have (ii) of the above theorem and in the case \( r_2 = r \), we do not have (iii) of the above theorem. However, we always have if \( r_1 \leq |t| \leq r \), then \( S - tI \) is not onto, that is, \( \sigma_\Delta(S) = \{ t : r_1 \leq |t| \leq r \} \) if \( S \) is not algebraic.

The following example shows we can always choose a weighted backward shift with desired \( r_1, r_2 \) and \( r \). For simplicity, we assume \( r = 1 \).

**Example.** Suppose that \( 0 < a < b < 1 \). Define a weighted backward shift as follows: For \( k = 0, 1, 2, \ldots \), define

\[
s_i = 1 \quad \frac{1}{2}k(k+1)(n+2) < i \leq \frac{1}{2}k(k+1)(n+2) + (k+1),
\]

\[
s_i = a \quad \frac{1}{2}k(k+1)(n+2) + (k+1) < i \leq \frac{1}{2}k(k+1)(n+2) + 2(k+1),
\]

\[
s_i = d \quad \frac{1}{2}k(k+1)(n+2) + 2(k+1) < i \leq \frac{1}{2}k(k+1)(n+2) + (n+2)(k+1)
\]

where \( n \) is fixed such that \( \left( \frac{a}{b} \right)^{n+1} \leq b \) and \( b^{n+1} \leq \frac{a}{b} \), and \( d = \sqrt[2]{\frac{b^{n+2}}{a}} \). It can be verified that \( r_1 = a, r_2 = b \) and \( r = 1 \).

**Corollary 15.** For any \( L_1 < L_2 \), there exists a weighted backward shift \( S \) such that \( \sigma_\Delta(S) = \{ t : L_1 \leq |t| \leq L_2 \} \).

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