

ON THE CENTER CONDITIONS OF CERTAIN CUBIC SYSTEMS

M. A. M. ALWASH

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ABSTRACT. This paper provides a new simple proof of a recent result by C. B. Collins (Differential and Integral Equations **10** (1997), 333–356) to derive the center conditions for a class of planar cubic systems. The idea is to consider periodic solutions of a related scalar non-autonomous equation.

Consider the planar system

$$\begin{aligned} \dot{x} &= -y + x(\alpha x + \beta y + Ax^2 + Bxy + Cy^2), \\ \dot{y} &= x + y(\alpha x + \beta y + Ax^2 + Bxy + Cy^2), \end{aligned} \tag{1}$$

where $\alpha, \beta, A, B,$ and C are real constants. In this paper, we give a simple and short proof to the following recent result of [2].

Theorem. *The origin is a center for (1) if and only if*

$$A + C = 0 \text{ and } A\alpha^2 + B\alpha\beta + C\beta^2 = 0.$$

Proof. The system (1) in polar coordinates r and θ becomes

$$\begin{aligned} \dot{r} &= p(\theta)r^2 + q(\theta)r^3, \\ \dot{\theta} &= 1, \end{aligned}$$

with $p(\theta) = \alpha \cos \theta + \beta \sin \theta$ and $q(\theta) = A \cos^2 \theta + B \cos \theta \sin \theta + C \sin^2 \theta$.

The origin is a center for (1) if and only if every solution in a neighborhood of $r = 0$ is a 2π -periodic solution for the differential equation

$$\frac{dr}{d\theta} = p(\theta)r^2 + q(\theta)r^3. \tag{2}$$

Let $r(\theta, c)$ be the solution of (2) with $r(0, c) = c$. For small c , we write

$$r(\theta, c) = \sum_{n=1}^{\infty} a_n(\theta)c^n, \tag{3}$$

where $a_1(0) = 1$ and $a_n(0) = 0$ for $n > 1$. The origin is a center if and only if $a_1(2\pi) = 1$, and $a_n(2\pi) = 0$ for all $n > 1$. Substituting (3) into (2) and equating the coefficients of c yield

$$\dot{a}_n = p \sum_{i+j=n} a_i a_j + q \sum_{i+j+k=n} a_i a_j a_k. \tag{4}$$

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Solving these equations recursively gives

$$a_1 = 1, a_2 = \bar{p}, a_3 = \bar{p}^2 + \bar{q}, a_4 = \bar{p}^3 + 2\bar{p}\bar{q} + \overline{pq},$$

and

$$a_5 = \bar{p}^4 + 3\bar{p}^2\bar{q} + \overline{p^2q} + 2\bar{p}\overline{pq} + \frac{3}{2}\bar{q}^2,$$

where a bar over a function denotes its indefinite integral. The functions a_2 and a_4 are integrals of polynomials of odd degrees; hence, they are 2π -periodic. Therefore, the first two necessary conditions for a center are $a_3(2\pi) = 0$ and $a_5(2\pi) = 0$. We have

$$a_3(2\pi) = \pi(A + C).$$

If $A + C = 0$, then

$$a_5 = -\frac{\pi}{2}(A\alpha^2 + B\alpha\beta - A\beta^2).$$

Now, we prove that these conditions are also sufficient. If $\alpha\beta \neq 0$, then straightforward manipulations imply that $q = \frac{-A}{\alpha\beta}p\bar{p}$. It follows inductively from (4) that $\dot{a}_n = pF(\bar{p})$, for some continuous function F . Therefore, $a_n(2\pi) = \overline{F(\bar{p}(2\pi))} - \overline{F(\bar{p}(0))} = 0$. If $\alpha = A = 0$ and $\beta \neq 0$, then $q = -\frac{B}{\beta}p\bar{p}$; when $\beta = A = 0$ and $\alpha \neq 0$, then $q = -\frac{B}{\alpha}p\bar{p}$. In the case $\alpha = \beta = 0$, the equation has the form $\frac{dx}{d\theta} = q(\theta)r^3$. Since q has mean value zero, the solutions of this equation are 2π -periodic.

The idea of this proof is the same one used in [1] for systems with homogeneous nonlinearities.

REFERENCES

- [1] M.A.M. Alwash and N.G. Lloyd, *Non-autonomous equations related to polynomial two-dimensional systems*, Proc. Royal Soc. Edinburgh, 105A(1987), 129-152. MR **88i**:58145
- [2] C.B. Collins, *Conditions for a center in a simple class of cubic system*, Differential and Integral Equations, 10(2)(1997), 333-356. CMP 97:05

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90095
E-mail address: alwash@math.ucla.edu