

## *Qppi* GROUPS AND QUASI-EQUIVALENCE

H. P. GOETERS AND W. J. WICKLESS

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ABSTRACT. A torsion-free abelian group  $G$  is *ppi* if every map from a pure subgroup  $K$  of  $G$  into  $G$  lifts to an endomorphism of  $G$ . The class of *ppi* groups has been extensively studied, resulting in a number of nice characterizations. We obtain some characterizations for the class of homogeneous *Qppi* groups, those homogeneous groups  $G$  such that, for  $K$  pure in  $G$ , every  $\theta : K \rightarrow G$  has a lifting to a quasi-endomorphism of  $G$ . An irreducible group is *Qppi* if and only if every pure subgroup of each of its strongly indecomposable quasi-summands is strongly indecomposable. A *Qppi* group  $G$  is *ppi* if and only if every endomorphism of  $G$  is an integral multiple of an automorphism. A group  $G$  has minimal test for quasi-equivalence (*mtqe*) if whenever  $K$  and  $L$  are quasi-isomorphic pure subgroups of  $G$  then  $K$  and  $L$  are equivalent via a quasi-automorphism of  $G$ . For homogeneous groups, we show that in almost all cases the *Qppi* and *mtqe* properties coincide.

All groups considered in the paper will be torsion-free abelian of finite rank. We assume familiarity with the standard tools used in studying these groups, such as types, pure subgroups,  $p$ -rank and quasi-isomorphism. We use  $\doteq$  for quasi-equality and  $H^n$  for a direct sum of  $n$  copies of a group  $H$ . For  $W$  a (torsion-free abelian) group or ring, we write  $QW$  for the divisible hull of  $W$ . We start by defining the class of groups that will be the object of our attention.

**Definition 1.** A group  $G$  is *Qppi* if whenever  $0 \rightarrow K \rightarrow G$  is pure exact then the induced sequence  $QE(G) \rightarrow QHom(K, G) \rightarrow 0$  is exact.

What we are saying, simply, is that for any homomorphism  $\theta$  mapping a pure subgroup  $K$  into  $G$  there is a  $\phi \in QE(G)$  such that  $\phi|_K = \theta$ . Thus, there is a positive integer  $t$  such that  $t\theta$  can be lifted to an endomorphism of  $G$ .

Plainly any *ppi* group (where we require that  $\theta$  itself be lifted to an endomorphism of  $G$ ) is *Qppi*. For a complete characterization of the class of *ppi* groups see [A-O'B-R] together with [R-2]. We provide examples to show that the *Qppi* groups form a considerably larger class.

It follows directly from the definitions that a homogeneous *Qppi* group  $G$  must be irreducible, that is,  $QE(G)$  will act transitively on the rank one subspaces of  $QG$ . The precise relation between homogeneous *Qppi* groups and irreducible groups is given in our first theorem.

**Theorem 2.** Let  $G$  be irreducible and write  $G \doteq H^n$ , where  $H$  is strongly indecomposable and irreducible ([R-1], Th. 5.5). Then: (a)  $H$  is *Qppi* if and only if

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each pure subgroup of  $H$  is strongly indecomposable, and (b)  $G$  is  $Qppi$  if and only if  $H$  is  $Qppi$ .

*Proof.* (a) Since  $H$  is irreducible strongly indecomposable then  $QE(H) = \Gamma$ , a division ring ([R-1]). Thus, if  $K$  is pure in  $H$  with a nontrivial quasi-decomposition  $K \doteq U \oplus V$  then  $i \oplus 0 : K \rightarrow H$  ( $i : U \rightarrow H$  is inclusion) cannot have a lifting to an element of  $QE(H)$ . It follows that if  $H$  is  $Qppi$  then every pure subgroup of  $H$  must be strongly indecomposable. Conversely, assume that every pure subgroup of  $H$  is strongly indecomposable. Suppose that  $K \subset H$  is pure and  $\theta : K \rightarrow H$ . Let  $X$  be a rank one pure subgroup of  $K$ . Since  $H$  is irreducible,  $\theta|_X$  has a lifting to  $\phi \in QE(H)$ . If  $X = K$  we're done. Otherwise we show that  $\phi|_K = \theta$  by showing that  $\phi|_L = \theta|_L$  for each rank two pure subgroup  $L \subset K$  with  $X \subset L$ . To see this, note that, since  $L$  is homogeneous strongly indecomposable, then by Baer's Lemma ([A], Lemma 1.12)  $\text{type } L/X > \text{type } L = \text{type } H$ . Hence  $\text{Hom}(L/X, H) = 0$ . It follows that  $(\phi - \theta)|_L = 0$  and the proof of (a) is complete.

(b) It is simple to check that the class of  $Qppi$  groups is closed under taking quasi-summands. Hence, if  $G$  as above is  $Qppi$  then so is  $H$ .

Conversely, assume that  $H$  is  $Qppi$  and let  $\theta : K \rightarrow G$  with  $K$  pure in  $G$ . We show that  $\theta$  can be lifted to a quasi-endomorphism of  $G$  by induction on rank  $K$ . Since  $G$  is irreducible,  $\theta$  can be quasi-lifted when  $\text{rank } K = 1$ . Suppose that  $\text{rank } K > 1$  and that, for any pure subgroup  $L \subset G$  of smaller rank, any homomorphism from  $L$  to  $G$  can be lifted to an element of  $QE(G)$ . Let  $L \subset K$  be a pure subgroup with  $\text{rank } K/L = 1$ . By the inductive assumption there exists  $\phi \in QE(G)$  such that  $\phi|_L = \theta|_L$ . Let  $\psi = (\phi|_K - \theta) : K \rightarrow G$ . If  $\psi = 0$  we're done. If  $\psi \neq 0$ , since  $K$  is homogeneous of the same type as  $G$ , we can apply Baer's Lemma to obtain a quasi-decomposition  $K \doteq L \oplus X$  with  $X$  a rank one pure subgroup of  $G$ .

Recall that  $\Gamma = QE(H)$  can be identified with the centralizer of the simple  $QE(G)$  module  $QG$  ([R-1]). We claim that we cannot have  $\Gamma X \subset \Gamma L$  (as  $\Gamma$ -subspaces of  $QG$ ). To prove the claim suppose  $0 \neq x \in X$  can be written  $x = \gamma l$  for some  $\gamma \in \Gamma, l \in L$ . Let  $QG = \Gamma l \oplus Y$ , where  $Y$  is any complementary  $\Gamma$ -subspace of  $QG$ , and let  $\rho$  be the associated projection of  $QG$  onto  $\Gamma l$ . Then  $\rho \in \text{Hom}_\Gamma(QG, QG) = QE(G)$ , the equality by the double centralizer theorem. Using  $\rho$  we obtain a quasi-decomposition  $G \doteq G' \oplus G''$  with  $G' = \Gamma l \cap G$ . Note that  $G'$  is quasi-isomorphic to  $H$ , since  $G'$  is a quasi-summand of  $G$  with  $\text{rank } G' = \text{rank } \Gamma l \cap G = \text{rank } \Gamma = \text{rank } H$ . But, since  $L \oplus X \doteq K$ , a pure subgroup of  $G$ , the pure subgroup of  $G'$  generated by  $x$  and  $l$  will be rank two completely decomposable, contradicting part (a).

Thus  $\Gamma X \cap \Gamma L = 0$ . Arguing as in the previous paragraph, we obtain a quasi-decomposition  $G \doteq G' \oplus G''$  with  $L \subset G', X \subset G''$ . Let  $\phi' \in QE(G)$  be such that  $\phi'|_X = \theta|_X$ . Then  $\phi|_{G'} \oplus \phi'|_{G''}$  will be the desired quasi-lifting of  $\theta$ .

**Example 3.** Let  $F$  be an algebraic number field of dimension 3 over  $Q$  such that in its ring of integers  $J$  there is a decomposition of an integral prime  $p$ ,  $pJ = PP'$ , with  $\dim_{Z/pZ}(J/P) = 2$ . (Such examples are easy to construct, e.g. see [A-O'B-R].) Let  $R = J_p$ . Since  $R$  is a full subring of the field  $F$ ,  $R$  will be irreducible as an additive group. Furthermore, since  $\text{rank } R = 3$  and  $p$ -rank  $R = 2$ , in the quasi-decomposition  $R \doteq H^n$  we must have  $n = 1, R = H$ . Thus  $R$  is strongly indecomposable. Note that  $qR = R$  for all integral primes  $q \neq p$ , so that  $Z_p J \subset R$ . Denote  $J_p = Z_p J$ . Since  $R$  has  $p$ -rank two,  $R/J_p \cong Z(p^\infty)$ . Let  $K$  be a rank two pure subgroup of  $R$ . Since  $R$  is strongly indecomposable we cannot have  $K + J_p = R$ .

Hence  $(K + J_p)/J_p$  is finite. It follows that any rank two pure subgroup of  $R$  must be isomorphic to  $Z_p \oplus Z_p$ . By Theorem 2 (a),  $R$  is not *Qqpi*.

We prove for future reference that the additive group  $R$  satisfies the following property, which is weaker than the *Qqpi* property: if  $U, V$  are quasi-isomorphic pure subgroups of  $R$  (which by our above discussion just means that  $\text{rank } U = \text{rank } V$ ) then  $\phi U \doteq V$  for  $\phi$  a quasi-automorphism of  $R$  (which in this case is simply a nonzero element of  $QE(R) \cong F$ ). Since  $R$  is irreducible, if  $U, V$  are rank one pure subgroups of  $R$ , then  $\phi U \doteq V$  for  $\phi \in QE(R)$ . Write  $F = Q(r)$  for  $r \in R$  and let  $U_0 = Z_p 1 \oplus Z_p r$ . We show that for each pure subgroup  $V \subset R$  of rank two there exists  $\phi \in QE(R)$  with  $\phi U_0 \doteq V$ . The fact that arbitrary rank two pure subgroups  $U, V \subset R$  are equivalent via a quasi-automorphism of  $R$  will follow immediately. By rank considerations  $rV \cap V \neq 0$ . Take  $0 \neq v \in V$  such that  $rv \in V$ . Then, if  $\phi$  is multiplication by  $v$ , we have  $\phi U_0 \doteq V$ .

The characterization of homogeneous strongly indecomposable *qpi* groups  $G$  from [A-O'B-R] and [R-2] is that  $G \cong R \otimes A$ , where  $\text{rank } A = 1$  and  $R = E(G)$  is a strongly homogeneous ring of  $p$ -rank one for some integral prime  $p$ . A strongly homogeneous ring is a full subring  $R$  of an algebraic number field such that every element of  $R$  is an integral multiple of a unit in  $R$ . Since the class of *Qqpi* groups is closed under quasi-isomorphism and the class of groups of the form  $R \otimes A$  as above is not, we have some immediate simple examples of strongly indecomposable homogeneous *Qqpi* groups which fail to be *qpi*. We also note, in connection with Theorem 2 (b), that [A-O'B-R] has an example of a strongly indecomposable homogeneous *qpi* group  $H$  such that  $H \oplus H$  is not *qpi*.

Our  $G$  constructed in the next example shows that the endomorphism ring of a strongly indecomposable homogeneous *Qqpi* group need not be a strongly homogeneous ring and need not have  $p$ -rank one for any prime  $p$ .

**Example 4.** There is a rank 4 homogeneous *Qqpi* group  $G$  such that  $E(G)$  is the ring of integral quaternions.

**Construction of the example.** Let  $R$  denote the ring of integral quaternions and let  $S = \{0 \neq s = ai + bj + ck \in R \mid \text{the first nonzero coefficient of } s \text{ is positive and } \gcd(a, b, c) = 1\}$ . Enumerate the elements of  $S$ , say  $S = \{s_m\}$ . For  $s_m = a_m i + b_m j + c_m k$ , put  $N(s_m) = a_m^2 + b_m^2 + c_m^2$ . Let  $P_m$  be the set of integral primes  $p$  such that  $-N(s_m)$  is a nonzero square mod  $p$ . Since each  $-N(s_m) \neq 0$ , it is well known that each  $P_m$  will be an infinite set of primes. For each  $m$ , choose an infinite subset  $P'_m \subset P_m$  so that  $\{P'_m \mid 1 \leq m < \infty\}$  will be a collection of disjoint sets. For each  $p \in P'_m$  choose  $d_p \in Z$  with  $0 < d_p < p$  such that  $-N(s_m) \equiv d_p^2 \pmod p$ . Let  $G$  be the  $R$ -submodule of  $QR$  generated by  $R$  and  $\{(d_p - s_m)/p \mid 1 \leq m < \infty, p \in P'_m\}$ .

Plainly  $R \subset E(G)$ , so that  $QR \subset QE(G)$ . Since  $QR$  is a division algebra and  $QR = QG$  it follows that  $G$  is irreducible, hence homogeneous. We show that  $\text{type } G = \bar{0}$  ( $= \text{type } Z$ ) by considering the divisibility of the element  $1 \in G$ . Clearly,  $1$  is divisible in  $G$  by no prime in the complement of  $\bigcup_m P'_m$ . Suppose that  $1 = pg$  for  $g \in G, p \in P'_m$ . Writing  $g$  as a finite  $R$ -combination of the generators of  $G$  produces the equation  $1 = r(d_p - s_m) + pr'$  for some  $r, r' \in R$ . Multiplying this equation by  $(d_p + s_m)$  yields the equation  $d_p + s_m = r[d_p^2 + N(s_m)] + pr'(d_p + s_m)$ . Since  $d_p^2 \equiv -N(s_m) \pmod p$  we have  $(d_p + s_m) \in pR$ , so that  $d_p$  is divisible by  $p$ , a contradiction. Hence, for all primes  $p, 1 \notin pG$ . Thus  $\text{type } G = \text{type}_G(1) = \bar{0}$ .

Suppose that  $K \subset G$  is a pure subgroup with a nontrivial quasi-decomposition  $K \doteq U \oplus V$ . Since  $G/R$  is torsion,  $U \cap R$  and  $V \cap R$  are both nonzero. Take nonzero elements  $u = (e + ai + bj + ck) \in U \cap R$ ,  $v \in V \cap R$  and denote  $\bar{u} = e - ai - bj - ck$ . The element  $\bar{u}v \in R$  can be written in the form  $t + ls_m$  for some integers  $t, l$  and positive integer  $m$ . It follows that, for each  $p \in P'_m$ ,  $[(ld_p + t) - \bar{u}v] = l(d_p - s_m) \in pG$ . Multiplication by  $u$  yields that  $[(ld_p + t)u - N(u)v] \in pG$  for all  $p \in P'_m$ . This is impossible, since  $N(u) \neq 0$  and  $u, v$  are elements lying in different quasi-summands of a pure subgroup of  $G$ , a homogeneous group of type  $\bar{0}$ . The resulting contradiction shows that each pure subgroup of  $G$  is strongly indecomposable. By Theorem 2 (a),  $G$  is  $Qppi$ .

Since the quaternion algebra  $QR$  is contained in  $QE(G)$  and  $QR = QG$ ,  $G$  is irreducible. Moreover,  $G$  itself is strongly indecomposable, so that  $\text{rank } QE(G) = \text{rank } QG$ . Hence,  $QR = QE(G)$ . We have already noted that  $R \subset E(G)$ , so  $R \subset E(G) \subset QR$ . By considering the action of a possible endomorphism of  $G$  on the set of generators for  $G$ , it is easy to check that  $E(G)$  coincides precisely with  $R$ .

If  $G$  is strongly indecomposable, homogeneous and  $qpi$ , then  $QE(G)$  is a field. Since quasi-isomorphic groups have isomorphic quasi-endomorphism rings, our example additionally shows that the class of  $Qppi$  groups is larger than the class of groups quasi-isomorphic to a  $qpi$  group. The following simple result gives the precise connection between the  $Qppi$  and  $qpi$  properties for strongly indecomposable homogeneous groups.

**Theorem 5.** *Let  $H$  be strongly indecomposable homogeneous  $Qppi$ . Then  $H$  is  $qpi$  if and only if every element of  $R = E(H)$  is an integral multiple of a unit of  $R$ .*

*Proof.* Let  $H$  be strongly indecomposable homogeneous  $Qppi$ . By Theorem 2 (a) every pure subgroup of  $H$  is strongly indecomposable. By Theorem B of [A-O'B-R], to show that  $H$  is  $qpi$  it suffices to show that for each pair of rank one pure subgroups  $X, Y$  of  $H$  there exists a unit  $u \in R$  with  $uX = Y$ . Since  $H$  is homogeneous  $Qppi$ , there exists  $0 \neq r \in R$  with  $rX \subset Y$ . If  $r = tu, t \in Z, u$  a unit of  $R$ , it follows that  $uX = Y$ .

**Definition 6.** A group  $G$  has minimal test for quasi-equivalence ( $mtqe$ ) if whenever  $U$  and  $V$  are quasi-isomorphic pure subgroups of  $G$  then there exists  $\phi$ , a quasi-automorphism of  $G$ , with  $\phi U \doteq V$ .

We have already noted that the group in Example 3 is homogeneous with  $mtqe$  but is not  $Qppi$ . For a second example, if we eliminate the element  $s_1$  from our construction of Example 4, then we obtain a subgroup  $G' \subset G$  in which  $\langle 1 \rangle \oplus \langle s_1 \rangle$  is pure. However, it is not too hard to show that  $G'$  remains strongly indecomposable and that  $E(G')$  will coincide with  $E(G)$ , the ring of integral quaternions. Thus,  $G'$  is a strongly indecomposable homogeneous non- $Qppi$  group. It also is not too hard to show that  $G'$  has  $mtqe$ . Since the construction of this second example is not central to our work, we omit the details.

Note that the group  $G$  of Example 3 has  $QE(G)$  a number field of degree 3. The group  $G$  in the modification of Example 4 would have  $QE(G)$  a division algebra of degree 2. The next result, which we feel is somewhat surprising, shows that only division algebras of degree 2 or 3 can occur as  $QE(G)$  for a strongly indecomposable homogeneous group  $G$  for which the  $Qppi$  and  $mtqe$  properties fail to coincide.

**Theorem 7.** *Let  $G$  be irreducible and write  $G \doteq H^n$  with  $H$  strongly indecomposable and irreducible. Suppose that  $\dim_Q F > 3$ , where  $F$  is a maximal subfield of the division algebra  $\Gamma = QE(H)$ . Then  $G$  is Qqpi if and only if  $G$  has mtqe.*

*Proof.* First we assume that  $n = 1$ , that is,  $H = G$  is strongly indecomposable. Since  $\Gamma = QE(H)$ , every  $0 \neq r \in E(H)$  is a quasi-automorphism. Hence, if  $H$  is Qqpi, it follows immediately that  $H$  has mtqe. Conversely, suppose that  $H$  has mtqe. By Theorem 2 (a), to prove that  $H$  is Qqpi we need to show that any pure subgroup of  $H$  is strongly indecomposable. It will be enough to show that any rank two pure subgroup of  $H$  is strongly indecomposable.

Let  $K$  be a rank two pure subgroup of  $H$ ; say  $K = \langle x, y \rangle_*$ , the pure subgroup generated by elements  $x, y$ . Since  $H$  is irreducible there exists  $\gamma \in QE(H)$  with  $\gamma x = y$ . Extend the field  $Q(\gamma)$  to a maximal subfield  $F \subset QE(H)$ .

Suppose that for each primitive element  $\alpha \in E(H)$  with  $F = Q(\alpha)$  the pure subgroup  $\langle x, \alpha x \rangle_* \subset H$  is completely decomposable. By the proof of the existence of a primitive element for algebraic number fields, for  $\alpha$  primitive we can choose a positive integer  $t$  such that  $\beta = \alpha + t\alpha^2$  will also be primitive. Then the pure subgroups  $\langle x, \alpha x \rangle_*$  and  $\langle x, \beta x \rangle_*$  will be isomorphic (both being completely decomposable subgroups of the homogeneous group  $H$ ). Because  $H$  has mtqe we have  $\phi \langle x, \alpha x \rangle_* \doteq \langle x, \beta x \rangle_*$  for some  $\phi \in \Gamma$ . Thus  $\phi(x) = (q_1 + q_2\beta)x$  for some rational numbers  $q_1, q_2$ . Since  $\Gamma$  is a division algebra,  $\phi = (q_1 + q_2\beta)$ . But  $\phi(\alpha x) = (q_1 + q_2\beta)\alpha x$  cannot be a rational combination  $q_3x + q_4\beta x$ , for then  $(q_1 + q_2\beta)\alpha = q_3 + q_4\beta$  for some rationals  $q_3, q_4$ . Since  $\beta = \alpha + t\alpha^2$ , this latter equation would contradict the fact that  $\alpha$  is algebraic over  $Q$  of degree greater than three. It follows that, for some primitive element  $\alpha \in E(H)$ , the pure subgroup  $\langle x, \alpha x \rangle_*$  will be strongly indecomposable.

As before, the type of the rank one factor group  $\langle x, \alpha x \rangle_* / \langle x \rangle_*$  must be greater than the type of  $H$ . Thus, one of two possibilities must occur.

Case I: There is an infinite set of primes  $P$  such that for  $p \in P$  there exists an integer  $c_p$  with  $h_p(\alpha x - c_p x) > h_p(x)$ . Here  $h_p$  denotes the  $p$ -height of an element in  $H$ . In this case an easy calculation shows that, for every integer  $t$  with  $1 \leq t < \text{degree } \alpha$  and  $p \in P$ , we have  $h(\alpha^t x - c_p^t x) > h_p(x)$ . Let  $L$  be the pure subgroup of  $H$  generated by  $x$  and  $\{\alpha^t x \mid 1 \leq t < \text{degree } \alpha\}$ . Our set of height inequalities shows that the inner type of  $L/\langle x \rangle_*$  is greater than the type of  $H$ . Since  $y = \gamma x \in L$  and  $K = \langle x, y \rangle_*$ , then  $K/\langle x \rangle_*$  is a pure subgroup of  $L/\langle x \rangle_*$ . Hence  $\text{type}[K/\langle x \rangle_*] > \text{type } H$ , so that  $K$  cannot be homogeneous and completely decomposable. Thus  $K$  must be strongly indecomposable, as desired.

Case II: For some prime  $p$  with  $pH \neq H$  there is a set of integers  $c_n, 1 \leq n < \infty$ , with  $(\alpha x - c_n x) \in p^n H$ . Arguing as in Case I, we can conclude that  $L/\langle x \rangle_*$  is  $p$ -divisible; hence so is  $K/\langle x \rangle_*$ . Again we have that  $K$  cannot be homogeneous and completely decomposable, so  $K$  is strongly indecomposable. The proof that  $H$  is Qqpi is complete.

Now suppose that  $n > 1$  and  $G \doteq H^n$  is as in the statement of the theorem. If  $G$  has mtqe it is immediate that  $H$  has mtqe. By what we have proved already,  $H$  is Qqpi and, by Theorem 2 (b),  $G$  is Qqpi. Conversely, let  $G$  be Qqpi. Then  $H$  is also Qqpi, so, by the remark at the beginning of the proof,  $H$  has mtqe. In view of the fact that  $G \doteq H^n$  is irreducible with  $QE(G) \cong (\Gamma)_{n \times n}$ , it is not hard to see that, for  $X, Y$  any rank one pure subgroups of  $G$ , we can choose a quasi-automorphism

$\phi \in QE(G)$  with  $\phi X = Y$ . With this observation, the proof of Theorem 2 (b) goes through *mutatis mutandis* to show that  $G$  has *mtqe*.

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DEPARTMENT OF MATHEMATICS, AUBURN UNIVERSITY, AUBURN, ALABAMA 36849  
*E-mail address:* goetehp@mail.auburn.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06269  
*E-mail address:* wjwick@uconnvm.uconn.edu