BOUNDS ON THE ORDER OF CROSS CHARACTERISTIC
SUBGROUPS OF THE FINITE SIMPLE GROUPS
OF LIE TYPE

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Abstract. Let $X(r)$ and $G(q)$ be finite groups of Lie type and $r$ and $q$ be coprime. If $G(q)$ is embedded in $X(r)$, then the Landazuri-Seitz-Zalesskii theorem implies that $G(q)$ is small relative to $X(r)$. We formalize this observation and illustrate how it can be used with some applications.

1. Introduction

If $X$ is a finite group of Lie type over a field of order $r$ and $G$ is a maximal subgroup of $X$, then it is essentially known that either $G$ is a member of one of several nice families of subgroups of $X$ or $G$ is an almost simple group. If $G$ is a finite simple group of Lie type over a field of order $q$ with $(q,r) = 1$, then by the Landazuri-Seitz-Zalesskii theorem, the degree $N$ of the minimal nontrivial representation of $G$ over the nonnatural characteristic is very large compared to the degree of the minimal nontrivial representation of $G$ over the natural characteristic. This fact has been used implicitly in various places in the literature to show that $|X:G|$ is very large. In this note we formalize this observation and make it explicit in Propositions 1 and 2. Then we illustrate the strength of the observation with four applications. For the most part these applications are not new. They are intended to give the reader a sense of how Propositions 1 and 2 could be used.

Notice that the lower bound on $|X:G|$ is large when $N$ is large. On the other hand when $N$ is small, one can hope to enumerate all cross characteristic subgroups of $X$.

Liebeck and Saxl, in [L] and [LS], have obtained bounds on the orders of the almost simple maximal subgroups of a finite simple group of Lie type, which enabled them to list all maximal subgroups of large order. They proved that when $X$ is a classical group with natural projective module $V$ of dimension $n$ over $GF(r)$, then an almost simple maximal subgroup of $X$ is of order less then $r^{3n}$, with some known exceptions. The bound $n^{12 \log_2 n}$, that we obtain in Proposition 2 for cross characteristic subgroups is stronger, in fact this bound is independent of $r$. The situation for the exceptional groups is similar.
2. Main result

The following lemma is Lemma 2.1 in [TZ].

**Lemma 1.** If \(2 \leq r, 2 \leq a_1 < a_2 < \ldots < a_k\) are integers and \(\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in \{1, -1\}\), then
\[
\frac{1}{2} \leq \frac{(r^{a_1} + \epsilon_1)(r^{a_2} + \epsilon_2) \ldots (r^{a_k} + \epsilon_k)}{r^{a_1 + a_2 + \ldots + a_k}} \leq 2.
\]

Let \(G\) be a simple algebraic group over an algebraically closed field of characteristic \(p\). Let \(\sigma\) be a surjective homomorphism \(G \to G\) such that \(G_\sigma\) is finite. The finite groups \(O^d(rG_\sigma)\) obtained this way are the finite groups of Lie type in characteristic \(p\).

**Lemma 2.** If \(X(r)\) is a finite simple group of Lie type, then
\[
r^{d-2} \leq |X(r)| \leq r^{d+1},
\]
where \(d\) is the dimension of the corresponding algebraic simple group except for \(2B_3(r), 2G_2(r)\) and \(2F_4(r)\), where \(d = \frac{d}{2}\) (see the table, \(r\) is a power of a prime \(p\)). The lemma holds for \(PSp(2)(r), G_2(2)\) and \(2F_4(2)\) too.

<table>
<thead>
<tr>
<th>(X(r))</th>
<th>(d)</th>
<th>(N)</th>
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<tr>
<td>(PSL_n(r))</td>
<td>(n^2 - 1)</td>
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<tr>
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<td>(2n^d + n)</td>
<td>(2n)</td>
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<td>(PSU_n(r), n \geq 3)</td>
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<td>(PO_{2n}^-(r), n \geq 4)</td>
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<td>(PO_{2n+1}(r), n \geq 3, r - \text{odd})</td>
<td>(2n^2 + n)</td>
<td>(2n + 1)</td>
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<tr>
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<td>(E_8(r))</td>
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<td>(S_2(2^{2k+1}))</td>
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<td>(^3D_4(r))</td>
<td>52</td>
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<td>(F_4(r))</td>
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<td>(G_2(r))</td>
<td>14</td>
<td>7 - (\delta_{p,2})</td>
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**Proof.** If \(X(r) \neq ^3D_4(q)\) we know that
\[
|X(r)| = \frac{r^a}{s}(r^{a_1} + \epsilon_1)(r^{a_2} + \epsilon_2) \ldots (r^{a_k} + \epsilon_k),
\]
for some integers \(1 \leq a_1 < a_2 < \ldots < a_k\), such that \(a + a_1 + \ldots + a_k = d\). The number \(s\) is the order of the center of the simply connected group of type \(G(r)\). If \(a_1 > 1\), then by Lemma 1 we get
\[
\frac{r^d}{2s} \leq |X(r)| \leq \frac{2r^d}{s} \leq r^{d+1}.
\]
If \(a_1 = 1\), then \(G(r)\) is \(2B_2(r), 2G_2(r)\) or \(2F_4(r)\) and \(r = r_1^l\) for \(l > 1\). Now we can apply Lemma 1 for \(r_1^l\) instead of \(r\) to get the same result. If \(X(r) \neq ^2E_6(2)\) and
X(r) \neq PSU_n(2)$ we can easily check that $r^2 \geq 2s$ and this proves the lemma. If $X(r) = PSU_n(2)$, then $s \leq 3$ and therefore

$$|PSU_n(2)| \geq 2^{\frac{n(n-1)}{2}} \prod_{i=3}^{n} (2^i - (-1)^i) \geq 2^{n^2-3} = 2^{d-2},$$

because $(2^i + 1)(2^i + 1 - 1) \geq 2^{2i+1}$. If $X(r)$ is $2E_6(2)$ or $3D_4(q)$ the Lemma is easily verified.

**Proposition 1.** Let $G(q)$ be a finite simple group of Lie type embedded in the group $PGL_n(K)$, where $K$ is a field of characteristic coprime to $q$. Then

$$|G(q)| \leq n^{10 + 2\log_q n}.$$

**Proof.** This is a corollary of the Landazuri-Seitz-Zalesskii theorem [SZ] that gives a lower bound for $n$. Let $G(q)$ be $PSL_m(q)$ or $PSU_m(q)$. Then $n \geq q^{m/2}$ and therefore $m \leq \log_q n + 2$. Also $d = m^2 - 1$ and by Lemma 2

$$|G| \leq q^{m+1} = q^{\frac{m^2}{2}} \leq q^{\left(\log_q n\right)^2 + 2\log_q n + 4} = n^{\log_q n + 2} q^4 \leq n^{\log_q n + 10}.$$ 

The last step is true since $m \geq 3$ implies $n \geq q$, and if $m = 2$, then $n \geq \frac{(q-1)}{2}$ and $q \leq 2n + 1 \leq n^2$ except for $n = 2$ and $q = 5$, when the proposition is obvious. Let $G(q)$ be $PO_m(r)$ (m and r odd), $PSp_m(r)$ (m even) or $PO_m^n(r)$ (m even). Then $n \geq q^{m/2-1} \geq q$ and $d \leq (m^2 + m)/2$. Now we get that $m \leq 2\log_q n + 2$ and

$$|G| \leq q^{(m^2+m)/2+1} \leq q^{(2\log_q n + 2) + 2\log_q n + 2} = n^{2\log_q n + 5} q^4 \leq n^{2\log_q n + 9}.$$ 

Finally if $G(q)$ is an exceptional finite simple group of Lie type, then $n \geq q^{(d+1)/10}$. For example if $G(q) = E_6(q)$, then the LSZ theorem says that $n \geq q^{27}(q^2 - 1) \geq q^{26} \geq q^{24.9}$. Similarly for all the exceptional groups. We get that $|G(q)| \leq q^{d+1} \leq n^{10}$. Now we can easily check that the claim of the proposition holds for all the exceptions of the LSZ theorem and the proof is complete.

**Proposition 2.** Let $G(q) \leq X(r)$ be two finite simple groups of Lie type and $(r, q) = 1$. Then

$$|G(q)| \leq N^{12\log_q N}$$

and

$$|X(r) : G(q)| \geq r^{(d-2 - 12\log_q N + \log_q N)}$$

where $d$ and $N$ are from the table. In fact, $N$ is the minimal degree of the representations of $X(r)$ over a field of the natural characteristic.

**Proof.** From the LSZ theorem if $G(q) \neq PSL_2(q)$, then $N \geq q$ and by Proposition 1

$$|G(q)| \leq N^{10 + 2\log_q N} \leq N^{12\log_q N}.$$

If $G = PSL_2(q)$, then $N^2 \geq q$ and $|G(q)| \leq N^6 \leq N^{12\log_q N}$. The second part follows from the first part and Lemma 2.
3. Applications

A factorization of a finite group $G$ is an expression $G = AB$ for some proper subgroups $A$ and $B$ of $G$. The factorization is maximal if both $A$ and $B$ are maximal. The maximal factorizations of the finite classical groups are found in [LPS]. Here we consider factorizations of a finite classical group by a group of Lie type in a different characteristic. We obtain restrictions for such factorizations.

**Proposition 3.** Let $G(q)$ be a finite simple group of Lie type and $X_n(r)$ be one of the finite classical groups $PSL_n(r), PSU_n(r)$ ($n \geq 3$), $PO_n(r)$ ($n \geq 7$ and $r$ odd), $PSp_n(r)$ ($n$ even $\geq 4$), or $PO_n^+(r)$ ($n$ even $\geq 8$) with $(q,r) = 1$. If

$$X_n(r) = H \cdot G(q)$$

is a factorization of $X_n(r)$, then $G(q)$ is one of the following groups:

- $PSL_2(q), q \leq 31$;
- $PSL_3(q), q \leq 4$;
- $PSp_4(q), q = 3, 5, 7$;
- $PSp_6(q), q = 2, 3$;
- $PSp_8(3)$;
- $PSp_8(2)$;
- $PSU_3(q), q = 3, 4$;
- $PSU_4(q), q = 2, 3$;
- $PSU_5(2)$;
- $PO_7(3)$;
- $G_2(q), q = 3, 4$;
- $S_8(8)$.

In addition, $r$ and $n$ must satisfy one of the following:

$\text{r} = 2$ and $n \leq 57$;

$2 < r \leq 16$ and $r + n \leq 23$;

$16 < r \leq 23$ and $n \leq 8$;

$23 < r \leq 29$ and $n \leq 4$;

or $29 < r \leq 59$ and $n = 2$.

**Proof.** The smallest index of a subgroup of $X_n(r)$ is found in [Co] and is at least $r^{n-2}$. Therefore

$$n^{10 + 2 \log_2 n} \geq |G(q)| \geq |X_n(r) : H| \geq r^{n-2}.$$  

The above inequality is not true if $n \geq 137$ for any $r$ and $q$, $(r,q) = 1$. Therefore $n \leq 136$. Let $l(G(q))$ be the minimal degree of the representations of $G(q)$ over a field of cross characteristic. Then the group $G(q)$ is restricted to the groups that satisfy the following condition: $l(G(q)) \leq 136$ and $|G(q)| \geq r^{n-2} \geq r^{l(G(q))-2}$ or if $q$ is even then $|G(q)| \geq 3^{l(G(q))-2}$ and if $q$ is odd then $|G(q)| \geq 2^{l(G(q))-2}$. The above list of groups consists of the finite simple groups of Lie type satisfying these two conditions. Now for a fixed $r$ the possibilities for $n$ are restricted by the following

(1) 

$$l(G(q)) \leq n \leq \log_r |G(q)| + 2$$

for at least one of the groups $G(q), (r,q) = 1$ in the above list. Let $d(G(q))$ be the minimal degree of a nontrivial projective representation of $G(q)$ over the field of complex numbers. For the finite classical groups $d(G(q))$ can be found in [TZ]. If $(r, |G(q)|) = 1$, then additional restriction for $n$ is $d(G(q)) \leq n$. Now if $r = 2$, the inequalities in (1) amount to $n \leq 57$. Similarly if $r = 3$ we get $n \leq 20$ and so on for $r = 4, ..., 16$. If $r > 16$ we get $n \leq 8$. If $r > 23$ we get $n \leq 6$. If $n \leq 6$ and $r > 11$, then by [Co] the smallest index of a subgroup of $X_n(r)$ is at least $(r^n - 1)/(r - 1)$ and therefore $n$ is restricted to

(2) 

$$(r^n - 1)/(r - 1) \leq |G(q)|.$$ 

Now by (1) and (2), if $r > 23$ then $n \leq 4$. Finally if $r > 29$ then $n$ must be 2. There are two possibilities: $G = PSL_2(4)$ and $r + 1 \leq |G| = 60$, or $G = PSL_3(2)$, $r + 1 \leq |G| = 168$ and $(r, |G|) \neq 1$ (since $d(G) > 2$). These imply the final statement.

**Lemma 3.** Let $G = X_n(r)$ be a finite classical group as in Proposition 3. If $G$ acts transitively on a set $\Omega$ and the stabilizer of a point is isomorphic to a finite simple group of Lie type $H = G(q)$ for $(q,r) = 1$, then
(1) for any element \( 1 \neq g \in G \), the fixed point ratio
\[
\frac{f(g)}{|\Omega|} \leq \frac{n^{10 + 2\log q \cdot n}}{r^{n - 2}},
\]
where \( f(g) \) is the number of elements of \( \Omega \) fixed by \( g \);
(2) the permutation rank of \( G \) on \( \Omega \) satisfies
\[
\text{rank}(G) \geq r^{n^2 / 4 - 24 \log n \log q \cdot n}.
\]

Proof. By Proposition 1, the fixed point ratio
\[
\frac{f(g)}{|\Omega|} = \frac{|g^G \cap H|}{|g^G|} \leq \frac{|H|}{|G : C_G(g)|} \leq \frac{n^{10 + 2\log q \cdot n}}{r^{n - 2}}.
\]

Also \( |\Omega| \leq \text{rank}(G)|H| \) and therefore
\[
\text{rank}(G) \geq \frac{|G|}{|H|^2} \geq r^{d - 2 - 24 \log n \log q \cdot n} \geq r^{n^2 / 4 - 24 \log n \log q \cdot n}.
\]

Let \( G \) be a transitive permutation group on the set \( \Omega \), \( |\Omega| = m \). For a permutation \( g \in G \) let \( \text{orb}(g) \) be the number of orbits of \( g \) on \( \Omega \) and the index of \( g \) be \( \text{Ind}(g) = m - \text{orb}(g) \). A genus \( l \) system is a triple \( (G, \Omega, S) \), where \( S = (g_i : i = 1, \ldots, r) \) is a generating set for \( G \) such that \( g_1 g_2 \ldots g_r = 1 \) and
\[
2(m + l - 1) = \sum_{i=1}^{r} \text{Ind}(g_i).
\]

This condition is equivalent to the existence of a branched covering of the Riemann sphere by a Riemann surface of genus \( l \) with monodromy group \( G \). The study of genus 0 groups has been reduced in [A1], [GT] and [Sh] to the study of primitive almost simple groups \( G \) such that \( f(g_i)/m > 1/85 \) for some \( i \).

**Proposition 4.** If \( n > 145 \) the permutation group described in Lemma 3 is not a genus 0 group.

Proof. If \( n > 145 \), then Lemma 3 (1) implies that for any \( 1 \neq g \in G \)
\[
\frac{f(g)}{m} < \frac{1}{85}.
\]

Since \( \text{orb}(g) = \sum_{i=1}^{d} f(g^i)/d \), where \( d \) is the order of \( g \), we get that
\[
\text{Ind}(g) \geq m - (d - 1)m/(85d) - m/d = \frac{84}{85}m(1 - 1/d).
\]

Now if \( G \) is a genus 0 group with some \( S = (g_i : i = 1, \ldots, r; d_i = |g_i|) \), then
\[
2m - 2 \geq \sum_{i=1}^{r} \text{Ind}(g_i) \geq \frac{84}{85}m(r - \sum_{i=1}^{r} 1/d_i).
\]

This inequality forces \( r \) to be less than 5, and if \( r = 4 \) then \( d_1 = \ldots = d_4 = 2 \) and if \( r = 3 \) then \( \sum 1/d_i \geq 1 \). Now (9.6) in [GT] states that \( G \) is solvable or is isomorphic to \( A_5 \), which is impossible. \( \square \)
4. Bounds on the Sylow subgroups

The following application of our main proposition was suggested by R. Guralnick. We want to formalize the observation that if a simple subgroup of a finite simple group of Lie type of characteristic \( p \) has a large \( p \)-subgroup, then it has to be a group of Lie type of the same characteristic. For a prime number \( p \) define \( e_p(n) \) to be the maximal \( i \) such that \( p^i \) divides \( n \) and let \( \Phi_k(x) \) be the \( k \)-th cyclotomic polynomial. We will need the following well known number theory result; see [M] page 27.

**Lemma 4.** Let \( p \) be a prime number and \( q \) be a power of a different prime number. Then \( p \mid \Phi_k(q) \) if and only if \( k = a \cdot p^i \), where \( a \) is the minimal number \( i \) such that \( p \mid (q^i - 1) \). In this case \( e_p(\Phi_a(q)) = e_p(q^a - 1) \) and for \( j > 0 \), \( e_p(\Phi_{a^j}(q)) = 1 \), except for \( e_2(\Phi_{2a}(q)) = e_2(q + 1) \) if \( 4 \mid (q + 1) \).

**Proof.** Let \( a \) be as in Lemma 4. Then \( p \mid q^{a} - 1 \) and therefore \( p \leq q^{a} \). Since

\[
\prod_{i=1}^{n}(q^i - 1) = \prod_{i=1}^{n}\Phi_i(q)^{\lfloor \frac{a}{i} \rfloor},
\]

we can use Lemma 4 to estimate the \( p \)-part of the above product. If \( p > 2 \)

\[
p^i < (q^{a})^{\lfloor n/a \rfloor} \cdot (q^{a})^{\lfloor n/(ap) \rfloor} \cdot (q^{a})^{\lfloor n/(ap^2) \rfloor} \ldots \leq q^{n(1+\ln p^{a} + \ln p^{-2} + \ldots)} < q^{n/p^{a-1}}.
\]

When \( p = 2 \) a similar calculation yields the result.

**Lemma 6.** Let \( P \) be a cross characteristic Sylow subgroup of a finite simple classical group \( X \). If \( X = PSL_n(q) \), then \( |P| < q^{2n} \); if \( X = O_{2n+1}(q) \), \( PO_{2n}^\pm(q) \), \( PSp_{2n}(q) \), then \( |P| < q^{4n} \); and if \( X = PSU_n(q) \), then \( |P| < (q + 1)^{2n-1} \).

**Proof.** If \( X \neq PSU_n(q) \), then this Lemma follows directly from Lemma 4. If \( X = PSU_n(q) \), then we need to apply Lemma 4 for \(-q\) instead of \( q\) and repeat the proof of the Lemma using \( p \leq q^{a+1} \leq (q + 1)^{a} \).

**Lemma 7.** Let \( X(r), G(q) \) and \( N \) be as in Proposition 2. Let \( r = p^i \), \( p \) a prime number. If \( P \) is a \( p \)-Sylow subgroup of \( G(q) \), then \( |P| < N^{10} \).

**Proof.** If \( G \) is an exceptional group, then from the proof of Proposition 1 we get \( |P| < |G| < N^{10} \). If \( G = PSU_m(q) \), then by Lemma 6 \( |P| < (q + 1)^{2m-1} \) and as in the proof of Proposition 1, \( m \leq \log_q N + 2 \) and we get

\[
|P| < (q + 1)^{(2\log_q N + 3)} < (q^{2})^{5\log_q N} = N^{10}.
\]

Similar arguments apply for the rest of the classical groups.

**Proposition 5.** Let \( X(r) \) and \( N \) be as in Proposition 2, \( r = p^j \) and \( p \) a prime number. Let \( S \) be a simple subgroup of \( X \) and let \( P \) be its \( p \)-Sylow subgroup. If \( |P| \geq N^{10} \), then \( S \) is an alternating group or a group of Lie type of the same characteristic. If also \( |P| \geq p^{(N+2)/(p-1)} \), then \( S \) is a group of Lie type of the same characteristic.

**Proof.** It’s easy to observe that \( S \) is not a sporadic group. If \( S \) is the alternating group \( A_m \), then the minimal projective module is of dimension at least \( m - 2 \). Finally

\[
|P| < p^{(N+1)/(p-1)} \leq p^{(N+2)/(p-1)}.
\]
REFERENCES


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