MULTIPLIER THEOREMS FOR HERZ TYPE HARDY SPACES

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Abstract. In this paper, the authors establish a multiplier theorem for Herz type Hardy spaces.

Let $T_m$ be a multiplier operator defined in terms of Fourier transforms by

$$\hat{T}_m f(\xi) = m(\xi) \hat{f}(\xi)$$

for suitable functions $f$. It is well-known that there is a multiplier theorem for $H^1(\mathbb{R}^n)$ (see [FS]): if $\alpha > n/2$ and

$$\int_{R < |\xi| < 2R} |D^\beta m(\xi)|^2 d\xi \leq CR^{n-2|\beta|}, \ 0 < R < \infty,$$

(1)

for all $|\beta| \leq \alpha$, then $T_m$ can be extended to be a bounded operator on $H^1(\mathbb{R}^n)$. That is, $m$ is a bounded multiplier of $H^1(\mathbb{R}^n)$.

Fix a function $\eta \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq \eta \leq 1$, $\eta = 1$ on $1/2 \leq |\xi| \leq 2$ and $\text{supp} \eta \subset \{1/4 \leq |\xi| \leq 4\}$. For $\delta > 0$, let us denote $m_\delta(\xi) = m(\delta \xi) \eta(\xi)$.

It is easy to check that (1) is equivalent to

$$\sup_\delta \|\hat{m}_\delta\|_{K^{\alpha,2}_2(\mathbb{R}^n)} < \infty,$$

(2)

where $K^{\alpha,2}_2(\mathbb{R}^n)$ is a non-homogeneous Herz space (see [BS]). By using some embedding relations on Herz spaces, A. Baernstein II and E. T. Sawyer [BS] weakened (2) into

$$\sup_\delta \|\hat{m}_\delta\|_{K^{\alpha-1,1}_1(\mathbb{R}^n)} < \infty,$$

(3)

where $0 < \varepsilon < \alpha - \frac{n}{2}$. In fact, this is just a special case of their theorem. In [BS], Baernstein and Sawyer showed that $m$ is a bounded multiplier of $H^1(\mathbb{R}^n)$ under an even weaker condition than (3); see Theorem 3b in [BS, page 21].

By using the technique of Herz type Hardy spaces developed by the authors in [LY1]-[LY3] and [Y], in this paper, we shall first establish a multiplier theorem for the homogeneous Herz type Hardy space $HK^{\alpha(1-1/q),1}_q(\mathbb{R}^n)$ which is introduced by the authors of this paper in [LY1]. Then as simple consequences of this theorem, a multiplier theorem for the corresponding non-homogeneous version of the space...
$H K_q^{(1 - 1/q, 1)}(\mathbb{R}^n)$ and the special case mentioned above of the multiplier theorem of Baernstein and Sawyer for $H^1(\mathbb{R}^n)$ will be deduced.

Now, for the reader’s convenience, let us recall the definition of the Herz spaces here. For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and $A_k = B_k \setminus B_{k-1}$. We also denote by $\chi_k$ the characteristic function of the set $A_k$.

**Definition 1.** Let $\alpha \in \mathbb{R}$ and $0 < p, q \leq \infty$.

(i) The homogeneous Herz space $K_q^{\alpha, p}(\mathbb{R}^n)$ is defined in terms of

$$
\|f\|_{K_q^{\alpha, p}(\mathbb{R}^n)} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^q(\mathbb{R}^n)} \right\}^{1/p}
$$

by letting

$$
K_q^{\alpha, p}(\mathbb{R}^n) = \{ f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K_q^{\alpha, p}(\mathbb{R}^n)} < \infty \}.
$$

(ii) The non-homogeneous Herz space $K_q^{\alpha, p}(\mathbb{R}^n)$ is defined in terms of

$$
\|f\|_{K_q^{\alpha, p}(\mathbb{R}^n)} = \left\{ \|f \chi_k\|_{L^q(\mathbb{R}^n)} + \sum_{k=1}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^q(\mathbb{R}^n)} \right\}^{1/p}
$$

by letting

$$
K_q^{\alpha, p}(\mathbb{R}^n) = \{ f \in L^q_{\text{loc}}(\mathbb{R}^n) : \|f\|_{K_q^{\alpha, p}(\mathbb{R}^n)} < \infty \}.
$$

Here the usual modification was made when $p = \infty$.

In what follows, when $p = 1$, $1 < q < \infty$ and $\alpha = n(1 - 1/q)$, we shall abbreviate $K_q^{\alpha, p}(\mathbb{R}^n)$ and $K_q^{\alpha, p}(\mathbb{R}^n)$, respectively, as $K_q(\mathbb{R}^n)$ and $A^q(\mathbb{R}^n)$. The latter is also said to be the Beurling algebras; see [CL] and [GR].

**Definition 2.** Let $1 < q < \infty$. For $f \in S'(\mathbb{R}^n)$, let $Gf$ be the grand maximal function of $f$ (see [FS] for its definition).

(i) The Hardy space $H K_q(\mathbb{R}^n)$ associated with the Herz space $K_q(\mathbb{R}^n)$ is defined by

$$
H K_q(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) : Gf \in K_q(\mathbb{R}^n) \}.
$$

In this case, we also define $\|f\|_{H K_q(\mathbb{R}^n)} = \|Gf\|_{K_q(\mathbb{R}^n)}$.

(ii) The Hardy space $H A^q(\mathbb{R}^n)$ associated with the Beurling algebra $A^q(\mathbb{R}^n)$ is defined by

$$
H A^q(\mathbb{R}^n) = \{ f \in S'(\mathbb{R}^n) : Gf \in A^q(\mathbb{R}^n) \}.
$$

In this case, we also define $\|f\|_{H A^q(\mathbb{R}^n)} = \|Gf\|_{A^q(\mathbb{R}^n)}$.

We remark that $H A^q(\mathbb{R}^n)$ was first introduced by Chen and Lau in [CL] for $n = 1$, and then by García-Cuerva in [GR] for $n > 1$. Obviously, $H K_q(\mathbb{R}^n)$ is a homogeneous version of $H A^q(\mathbb{R}^n)$. Moreover, in [LY1], the authors proved that

(4) $H A^q(\mathbb{R}^n) = H K_q(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$

and

(5) $\|f\|_{H A^q(\mathbb{R}^n)} \sim \|f\|_{H K_q(\mathbb{R}^n)} + \|f\|_{L^q(\mathbb{R}^n)}$. 
It is also well-known that \( HA^q(\mathbb{R}^n) \subsetneq \mathcal{H}^q(\mathbb{R}^n) \subsetneq H^1(\mathbb{R}^n) \) for any \( q \in (1, \infty) \).

Let us now formulate our multiplier theorem for \( \mathcal{H}^q(\mathbb{R}^n) \).

**Theorem 1.** Let \( q \in (1, \infty) \) and \( m \) satisfy

\[
M \equiv \sup_{\delta} \| \hat{m}_\delta \|_{K_1^n(1-1/q,1)} < \infty.
\]

Then \( m \) is a bounded multiplier of \( \mathcal{H}^q(\mathbb{R}^n) \).

By Corollary 2 in [BS, page 22], we know that if \( m \) satisfies the condition of Theorem 1, then \( m \) is a bounded multiplier of \( L^q(\mathbb{R}^n) \) for \( 1 < q < \infty \). Therefore, from (4), (5) and Theorem 1, we have the following simple corollary.

**Corollary 1.** Let \( q \in (1, \infty) \) and \( m \) satisfy (6). Then \( m \) is a bounded multiplier of \( HA^q(\mathbb{R}^n) \).

The proof of Theorem 1 is based on the decomposition characterizations of Herz spaces and Herz type Hardy spaces in terms of central units and central atoms respectively. Let us recall that a function \( e(x) \) is said to be a central \((\alpha, q)\) unit of restrict type if it satisfies

i) \( \text{supp} e \subset B(0, r), r \geq 1 \);

ii) \( \| e \|_{L^q(\mathbb{R}^n)} \leq |B(0, r)|^{-\alpha/n} \).

**Lemma 1** ([LY2]). Let \( 0 < \alpha < \infty, \ 0 < p < \infty \) and \( 1 \leq q < \infty \). Then \( f \in K_\alpha^{\alpha,p}(\mathbb{R}^n) \) if and only if \( f \) can be expressed as

\[
f(x) = \sum_{k=0}^{\infty} \lambda_k e_k(x),
\]

where each \( e_k \) is a central \((\alpha, q)\) unit of restrict type supported on \( B_k \) and \( \sum_{k=0}^{\infty} |\lambda_k|^p < \infty \). Moreover,

\[
\inf \left\{ \left( \sum_{k} |\lambda_k|^p \right)^{1/p} \right\} \sim \| f \|_{K_\alpha^{\alpha,p}(\mathbb{R}^n)},
\]

where the infimum is taken over all of the above decompositions of \( f \).

Let us now turn to the definition of central atoms. A function \( a(x) \) is said to be a central \((1, q)\) atom if \( a \) satisfies

i) \( \text{supp} a \subset B(0, r), r > 0 \);

ii) \( \| a \|_{L^q(\mathbb{R}^n)} \leq |B(0, r)|^{1/q-1} \);

iii) \( \int_{\mathbb{R}^n} a(x)dx = 0 \).

**Lemma 2** ([LY3]). Let \( 1 < q < \infty \). Then \( f \in \mathcal{H}^q(\mathbb{R}^n) \) if and only if \( f \) can be expressed as

\[
f(x) = \sum_{k=-\infty}^{\infty} \lambda_k a_k(x),
\]

where each \( a_k \) is a central \((1, q)\) atom supported on \( B_k \) and \( \sum_{k=-\infty}^{\infty} |\lambda_k| < \infty \). Moreover,

\[
\inf \left\{ \sum_{k=-\infty}^{\infty} |\lambda_k| \right\} \sim \| f \|_{\mathcal{H}^q(\mathbb{R}^n)},
\]

where the infimum is taken over all of the above decompositions of \( f \).
To prove Theorem 1, we still need a lemma. Let \( \psi \in \mathcal{S}(\mathbb{R}^n) \), the Schwartz space of functions. In what follows, we let \( \hat{a}_\delta(\xi) \equiv \hat{a}(\delta \xi) \psi(\xi) \).

**Lemma 3.** Let \( a \) be a central \((1, q)\) atom supported on \( B(0, 1) \) and \( b_j = (\hat{a}_{2^j})' \). Then for any given \( d > 0 \), we have the following three facts:

(i) \( |b_j|_{L^q(\mathbb{R}^n)} \leq C 2^{-nj(1-1/q)} \).

(ii) \( |b_j(x)| \leq C_d 2^{-nj(1-1/d)} |x|^{-d} \), for \( |x| \geq 2^j+1 \).

(iii) \( |b_j(x)| \leq C_d 2^j (1+|x|)^{-d} \), for all \( x \) and \( j \leq 0 \).

**Proof.** Since \( 1 < q < \infty \) and
\[
 b_j(x) = 2^{-nj} \int_{|x-y|<2^j} a(2^{-j}(x-y)) \hat{\psi}(y) dy,
\]
it follows from the generalized Minkowski inequality that
\[
 \|b_j\|_{L^q(\mathbb{R}^n)} \leq 2^{-nj} \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} |a(2^{-j}(x-y))|^{q} dx \right\}^{1/q} |\hat{\psi}(y)| dy \\
\leq C 2^{-nj} 2^{nj/q} \|a\|_{L^q(\mathbb{R}^n)} \|\hat{\psi}\|_{L^q(\mathbb{R}^n)} \leq C 2^{-nj(1-1/q)}.
\]
Thus, (i) holds. Let us now assume \( |x| \geq 2^j+1 \). Note that \( |y| \geq |x|/2 \) and \( \hat{\psi} \in \mathcal{S}(\mathbb{R}^n) \). Then we have
\[
 |b_j(x)| \leq 2^{-nj} \int_{|x-y|<2^j} |a(2^{-j}(x-y))| \cdot |\hat{\psi}(y)| dy \\
\leq 2^{-nj} \left( \int_{|x-y|<2^j} |a(2^{-j}(x-y))|^{q} dy \right)^{1/q} \left( \int_{|x-y|<2^j} |\hat{\psi}(y)|^{q'} dy \right)^{1/q'} \\
= 2^{nj(1/q'-1)} \|a\|_{L^q(\mathbb{R}^n)} \left( \int_{|y|>|x|/2} |\hat{\psi}(y)|^{q'} dy \right)^{1/q'} \\
\leq C_d 2^{nj(1/q'-1)} |x|^{-d}.
\]
Thus, (ii) also holds. Finally, let us assume \( j \leq 0 \). Since \( \int_{\mathbb{R}^n} a(y) dy = 0 \), we have
\[
 b_j(x) = \int_{|y|<1} a(y) \{ \hat{\psi}(x-2^jy) - \hat{\psi}(x) \} dy.
\]
It follows from the mean value theorem that there exists a \( \theta \in (0, 1) \) such that
\[
 |b_j(x)| \leq \int_{|y|<1} |a(y)| \cdot |(\nabla_x \hat{\psi})(x-\theta 2^j y)| 2^j |y| dy,
\]
where \( \nabla_x = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right) \). Note that \( 1 + |x-\theta 2^j y| \geq (1+|x|)/2 \) and \( \hat{\psi} \in \mathcal{S}(\mathbb{R}^n) \). Then we have
\[
 |b_j(x)| \leq C_d 2^j (1+|x|)^{-d} \int_{|y|<1} |y| \cdot |a(y)| dy \leq C_d 2^j (1+|x|)^{-d}.
\]
This completes the proof of the lemma.

**Proof of Theorem 1.** By the decompositions of \( \mathcal{H}^\bullet_{K,q}(\mathbb{R}^n) \) in terms of central atoms, it suffices to prove that the inequality
\[
 \|T_m a\|_{\mathcal{H}^\bullet_{K,q}(\mathbb{R}^n)} \leq C
\]
holds for all central \((1, q)\) atoms \( a(x) \). Let \( a(x) \) be a central \((1, q)\) atom. Since \( M \) is invariant for all dilations of \( m \), we may assume \( \text{supp } a \subset B(0, 1) \). In Lemma 3,
We write
\[ m(\xi)\hat{a}(\xi) = \sum_{j=-\infty}^{\infty} m(\xi)\eta(2^{-j}\xi)\hat{a}(\xi)\psi(2^{-j}\xi) \]

where \( \hat{a}(\xi) = \hat{a}(\delta\xi)\psi(\xi) \). By letting \( N_j \equiv (m_{2^j})^\vee \), we have
\[ T_m a(x) = \sum_{j=-\infty}^{\infty} 2^{nj}(N_j * b_j)(2^j x). \]

Without loss of generality, we may assume \( M = 1 \). Thus, the inequality \( \|m\|_{L^1(\mathbb{R}^n)} \leq 1 \) holds for any \( \delta > 0 \). Therefore, it follows from the Hausdorff-Young inequality that \( \|m\|_{L^\infty(\mathbb{R}^n)} \leq 1 \) and
\[ \|N_j\|_{L^1(\mathbb{R}^n)} \leq \|N_j\|_{K^{(1-\alpha)}_1(\mathbb{R}^n)} \leq 1. \]

Let us first prove
\[ \|T_m a\|_{K_q(\mathbb{R}^n)} \leq C, \]
where \( C \) is independent of \( a \) and \( a \) is a central \((1, q)\) atom with \( \text{supp } a \subset B(0, 1) \).

We write
\[ \|T_m a\|_{K_q(\mathbb{R}^n)} = \sum_{k=-\infty}^{\infty} 2^{kn(1-1/q)}\|(T_m a)\chi_k\|_{L^q(\mathbb{R}^n)} = \sum_{k=-\infty}^{\infty} \cdots + \sum_{k=3}^{\infty} \cdots = I_1 + I_2. \]

Since \( m \) is a bounded multiplier of \( L^q(\mathbb{R}^n) \) by Corollary 2 in [BS, page 22], we have
\[ I_1 \leq C \sum_{k=-\infty}^{\infty} 2^{kn(1-1/q)}\|a\|_{L^s(\mathbb{R}^n)} \leq C \sum_{k=-\infty}^{\infty} 2^{nk(1-1/q)} \leq C. \]

On the other hand, by (8), we have
\[ I_2 = \sum_{k=3}^{\infty} 2^{kn(1-1/q)}\|(T_m a)\chi_k\|_{L^s(\mathbb{R}^n)} \leq \sum_{k=3}^{\infty} \sum_{j=-\infty}^{\infty} 2^{nj} 2^{kn(1-1/q)}\|(N_j * b_j)\chi_k(\cdot)\|_{L^s(\mathbb{R}^n)} \]
\[ = \sum_{j=-\infty}^{\infty} \sum_{l=j+3}^{\infty} 2^{n(l-1/q)}\|(N_j * b_j)\chi(\cdot)\|_{L^q(\mathbb{R}^n)} \]
\[ = \sum_{j=-\infty}^{\infty} \sum_{l=j+3}^{\infty} \cdots + \sum_{j=1}^{\infty} \sum_{l=j+3}^{\infty} \cdots \equiv I_{2,1} + I_{2,2}. \]
Let us first estimate $I_{2,1}$. By Lemma 1, $N_j$ can be expressed as

$$N_j(x) = \sum_{k=0}^{\infty} \lambda_k^j e_k^j(x),$$

where each $e_k^j$ is a central $(n(1-1/q),1)$ unit of restrict type supported on $B_k$ and

$$\inf \left( \sum_{k=0}^{\infty} |\lambda_k^j| \right) \sim \|N_j\|_{K_1^{n(1-1/q),1}(\mathbb{R}^n)}.$$

Thus,

$$I_{2,1} = \sum_{j=-\infty}^{0} \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} \|(N_j * b_j)(\cdot)\chi_l(\cdot)\|_{L^q(\mathbb{R}^n)}$$

$$\leq \sum_{j=-\infty}^{0} \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} \sum_{k=0}^{\infty} |\lambda_k^j| \cdot \|(e_k^j * b_j)\chi_l\|_{L^q(\mathbb{R}^n)}$$

$$= \sum_{j=-\infty}^{0} \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} \max\{l-2,0\} \sum_{k=0}^{\infty} |\lambda_k^j| \cdot \|(e_k^j * b_j)\chi_l\|_{L^q(\mathbb{R}^n)}$$

$$\leq \sum_{j=-\infty}^{0} \sum_{l=j+3}^{\infty} 2^{nl(1-1/q)} \sum_{k=0}^{\infty} |\lambda_k^j| \cdot \|(e_k^j * b_j)\chi_l\|_{L^q(\mathbb{R}^n)}$$

$$= I_{2,1}^1 + I_{2,1}^2.$$

By (iii) in Lemma 3 with $d = n + \varepsilon$, $0 < \varepsilon < 1$, we have

$$\|(e_k^j * b_j)\chi_l\|_{L^q(\mathbb{R}^n)} \leq C 2^j \|e_k^j\|_{L^q(\mathbb{R}^n)} \left( \int_{A_i} |x|^{-d/q} dx \right)^{1/q}$$

$$\leq C 2^j 2^{-(n+\varepsilon)} 2^{jn/q}.$$

Thus, we obtain

$$I_{2,1}^1 \leq C \sum_{j=-\infty}^{0} 2^j \sum_{l=j+3}^{\infty} 2^{-\varepsilon l} \sum_{k=0}^{\infty} |\lambda_k^j|$$

$$\leq C \sum_{j=-\infty}^{0} 2^j 2^{-\varepsilon j} \|N_j\|_{K_1^{n(1-1/q),1}(\mathbb{R}^n)}$$

$$\leq C \sum_{j=-\infty}^{0} 2^j (1-\varepsilon) \leq C.$$
Thus, we obtain

\[
I_{2,1}^2 \leq C \sum_{j=-\infty}^{0} 2^j \sum_{l=j+3}^{\infty} 2^{n_1(1-1/q)} \sum_{k=\max\{l-1,0\}}^{\infty} |\lambda_k^j| 2^{-kn_1(1-1/q)}
\]

\[
\leq C \sum_{j=-\infty}^{0} 2^j \sum_{k=0}^{\infty} |\lambda_k^j| 2^{-kn_1(1-1/q)} \sum_{l=-\infty}^{k+1} 2^{n_1(1-1/q)}
\]

\[
\leq C \sum_{j=-\infty}^{0} 2^j \|N_j\|_{K_1^{n_1(1-1/q),1}(\mathbb{R}^n)} \leq C.
\]

Hence, we obtain \(I_{2,1} \leq C\).

We now estimate \(I_{2,2}\). Let \(x \in A_l\), \(l \geq j + 3\). Then,

\[
(N_j * b_j)(x) = \int_{|y| \leq 2^{l-1}} N_j(y)b_j(x-y)dy
\]

\[
+ [(N_j\chi_{\bar{A}}_l) * b_j](x) + \int_{|y| > 2^{l+2}} N_j(y)b_j(x-y)dy,
\]

where \(\bar{A}_l = A_{l-1} \cup A_l \cup A_{l+1}\). Note that if \(|y| \leq 2^{l-2}\) and \(x \in A_l\), \(l \geq j + 3\), then \(|x-y| \geq 2^{l+1}\) and \(|x-y| \geq |x|/2\). Thus, it follows from (ii) in Lemma 3 that

\[
|b_j(x-y)| \leq C_d 2^{-n_j(1-1/q)} |x|^{-d} \leq C_d 2^{-n_j(1-1/q)} 2^{-ld}.
\]

Also, note that if \(|y| > 2^{l+3}\) and \(x \in A_l\), \(l \geq j + 3\), then \(|x-y| \geq 2^{l+1}\) and \(|x-y| \geq |y|/2\). Also, it follows from (ii) in Lemma 3 that

\[
|b_j(x-y)| \leq C_d 2^{-n_j(1-1/q)} |y|^{-d} \leq C_d 2^{-n_j(1-1/q)} 2^{-ld}.
\]

Thus, when \(x \in A_l\), \(l \geq j + 3\), we have

\[
|(N_j * b_j)(x)| \leq C_d 2^{-n_j(1-1/q)} 2^{-ld} \|N_j\|_{L^1(\mathbb{R}^n)} + |(N_j\chi_{\bar{A}}_l) * b_j(x)|
\]

\[
\leq C_d 2^{-n_j(1-1/q)-ld} + |(N_j\chi_{\bar{A}}_l) * b_j(x)|.
\]

Applying these estimates and (i) in Lemma 3 with \(d = n + \varepsilon\) to \(I_{2,2}\), we obtain

\[
I_{2,2} = \sum_{j=1}^{\infty} \sum_{l=j+3}^{\infty} 2^{n_1(1-1/q)} \|(N_j * b_j)(\cdot)\chi(\cdot)\|_{L^s(\mathbb{R}^n)}
\]

\[
\leq C \sum_{j=1}^{\infty} \sum_{l=j+3}^{\infty} 2^{n_1(1-1/q)} 2^{-n_j(1-1/q)} 2^{-ld} 2^{ln/n}
\]

\[
+ \sum_{j=1}^{\infty} \sum_{l=j+3}^{\infty} 2^{ln(1-1/q)} \|(N_j\chi_{\bar{A}}_l) * b_j(\cdot)\chi(\cdot)\|_{L^s(\mathbb{R}^n)}
\]

\[
\leq C \sum_{j=1}^{\infty} 2^{-j(d-n/q)} + \sum_{j=1}^{\infty} \sum_{l=j+3}^{\infty} 2^{ln(1-1/q)} \|N_j\chi_{\bar{A}}_l\|_{L^1(\mathbb{R}^n)} \|b_j\|_{L^s(\mathbb{R}^n)}
\]

\[
\leq C + C \sum_{j=1}^{\infty} 2^{-n_j(1-1/q)} \|N_j\|_{K_1^{n_1(1-1/q),1}(\mathbb{R}^n)} \leq C + C \sum_{j=1}^{\infty} 2^{-n_j(1-1/q)} \leq C.
\]

Now, (9) follows from the above estimates on \(I_1, I_{2,1}\) and \(I_{2,2}\).
Actually, it is easy to prove that (9) is true for any central \((1, q)\) atom. That is, the inequality
\[
\|T_m \tilde{a}\|_{K_q(\mathbb{R}^n)} \leq C
\]
holds for any central \((1, q)\) atom \(\tilde{a}\). In fact, let us assume that \(\text{supp} \tilde{a} \subset B(0, r)\). Obviously, there exists a \(k_0 \in \mathbb{Z}\) such that \(2^{k_0} < r \leq 2^{k_0+1}\). If \(I_1\) and \(I_2\) in the proof of (9) are now replaced by
\[
\sum_{k=-\infty}^{k_0+2} 2^{kn(1-1/q)} \|(T_m \tilde{a}) \chi_k\|_{L^q(\mathbb{R}^n)}
\]
and
\[
\sum_{k=k_0+3}^{\infty} 2^{kn(1-1/q)} \|(T_m \tilde{a}) \chi_k\|_{L^q(\mathbb{R}^n)}
\]
respectively, then (10) can be proved by a method similar to that of proving (9).

To prove (7), by the characterization of \(HK_q(\mathbb{R}^n)\) in terms of Riesz transforms (see [Y]), it suffices to show
\[
\sum_{j=1}^{n} \|R_j(T_m a)\|_{K_q(\mathbb{R}^n)} \leq C,
\]
where \(C\) is independent of \(a\) and \(R_j\) is the \(j\)-th Riesz transform. Since Riesz transforms are bounded on \(HK_q(\mathbb{R}^n)\) (see [Y]), we have
\[
R_j a(x) = \sum_k \lambda_k^j a_k(x)
\]
and
\[
\sum_k |\lambda_k^j| \leq C \|R_j a\|_{HK_q(\mathbb{R}^n)} \leq C,
\]
where each \(a_k^j\) is a central \((1, q)\) atom and \(C\) is independent of \(a\). Thus, it follows from (10) that
\[
\sum_{j=1}^{n} \|R_j(T_m a)\|_{K_q(\mathbb{R}^n)} = \sum_{j=1}^{n} \sum_k \|\lambda_k^j(T_m a_k^j)\|_{K_q(\mathbb{R}^n)} \leq C \sum_{j=1}^{n} \sum_k |\lambda_k^j| \leq C.
\]
Thus, (11) holds. This completes the proof of Theorem 1.

Let us now point out that if a linear operator \(T\) commutes with translations, then the boundedness of \(T\) on \(HK_q(\mathbb{R}^n)\) implies its boundedness on \(H^1(\mathbb{R}^n)\). Precisely, we have

**Theorem 2.** Let \(T\) be a linear operator that commutes with translations. If \(T\) is bounded on \(HK_q(\mathbb{R}^n)\), \(1 < q < \infty\), then \(T\) is also bounded on \(H^1(\mathbb{R}^n)\).
Proof. By the atomic decomposition of $H^1(\mathbb{R}^n)$ (see [CW]), it suffices to prove that the inequality

$$\|Ta\|_{H^1(\mathbb{R}^n)} \leq C$$

holds for any $(1, q)$ atom $a$. Let $a(x)$ be a $(1, q)$ atom supported on $B(x_0, r)$. That is, $a(x)$ satisfies the following conditions: $\text{supp} a \subset B(x_0, r), \|a\|_{L^q(\mathbb{R}^n)} \leq |B(x_0, r)|^{1/q - 1}$, and $\int_{\mathbb{R}^n} a(x) dx = 0$. Let $\tilde{a}(x) \equiv \tau_{-x_0} a(x) = a(x + x_0)$. It is easy to see that $\tilde{a}$ is a central $(1, q)$ atom supported on $B(0, r)$. Thus, from the conditions of the theorem, it follows that

$$\|T\tilde{a}\|_{H^1(\mathbb{R}^n)} \leq C,$$

where $C$ is independent of $\tilde{a}$. Since $T$ commutes with translations, we have

$$\|\tau_{-x_0} Ta\|_{H^1(\mathbb{R}^n)} \leq \|\tau_{-x_0} \tilde{a}\|_{H^1(\mathbb{R}^n)} = \|T\tau_{-x_0} a\|_{H^1(\mathbb{R}^n)} = \|T\tilde{a}\|_{H^1(\mathbb{R}^n)} \leq C.$$

Thus, $\tau_{-x_0} Ta \in H^1(\mathbb{R}^n)$ and

$$\tau_{-x_0} Ta(x) = \sum_j \lambda_j a_j(x),$$

where each $a_j$ is a $(1, q)$ atom, and $\sum_j |\lambda_j| \sim \|\tau_{-x_0} Ta\|_{H^1(\mathbb{R}^n)}$. Since $H^1(\mathbb{R}^n)$ is translation invariant, we then have $Ta \in H^1(\mathbb{R}^n)$ and

$$\|Ta\|_{H^1(\mathbb{R}^n)} \leq \sum_j |\lambda_j| \leq C.$$

This finishes the proof of Theorem 2.

Note that if $q \in (1, \infty)$, then $\alpha = n(1 - 1/q) \in (0, n)$. As a simple corollary of Theorem 1 and Theorem 2, we have

**Corollary 2.** Let $0 < \varepsilon < n$. If $m$ satisfies

$$\sup_{\delta} \|\widehat{m_\delta}\|_{K_q^{1-\varepsilon}(\mathbb{R}^n)} < \infty,$$

then $m$ is a bounded multiplier of $H^1(\mathbb{R}^n)$.

Finally, we point out that it is still an open problem whether (6) is a necessary condition for an $L^\infty(\mathbb{R}^n)$ function $m$ to be a bounded multiplier of $HK_q(\mathbb{R}^n)$ in any sense (see [BS]). And we will discuss the similar problems of multipliers on general Herz type Hardy spaces $HK_q^{\alpha, p}(\mathbb{R}^n)$ in a future paper.

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