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ON THE HYPERSPACE OF A NON-SEPARABLE METRIC SPACE

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ABSTRACT. We prove the following two results.

1) There exists a non-separable complete metric space whose Wijsman hypertopology is not Čech-complete.

2) There exist a non-separable metrizable space and two compatible metrics on it, such that the collections of the Borel sets generated by the relative Wijsman hypertopologies do not coincide.

0. INTRODUCTION

The main purpose of this paper is to underline the differences between the hyperspace of a non-separable metric space and that of a separable metric space, with special regard to the Wijsman hypertopology (for general reference, see [Be1, §§2.1 and 3.2]).

In the literature, the separable case is by far the best studied, and has proved to have interesting applications to analysis, measure theory and descriptive set theory (cf., for example, [Be2, §1] and [Ke, §12.C]). We will present here two counterexamples showing that two classical results about the hyperspace of a separable metric space do not extend to the non-separable case.

1. FIRST EXAMPLE

If (X, d) is a separable and complete metric space, then the Wijsman topology $\mathbf{w}_d$ on $c_o(X)$ (see definition below) is in turn separable and completely metrizable [Be2, Theorem 4.3]. This result can be generalized by showing that for every separable and completely metrizable space X and for every compatible metric d on X, the space $(c_o(X), \mathbf{w}_d)$ is still separable and completely metrizable [Co] (note that such a generalization is not automatic, as equivalent metrics on a set X can give rise to different Wijsman topologies on $c_o(X)$ — cf., in particular, [CLZ, Theorem 5′]).

If the metric space (X, d) is not separable, then $(c_o(X), \mathbf{w}_d)$ is neither separable nor metrizable, as each of these properties is in fact equivalent to the separability of the base space (cf. [Be1, Theorem 2.1.5] and related bibliography). However, it is worth wondering whether the complete metrizability of (X, d), or at least its actual completeness, can imply some suitable form of completeness for $(c_o(X), \mathbf{w}_d)$.

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As a natural candidate, we will consider Čech-completeness, which in the case of a metrizable space is equivalent to complete metrizability.

In the following, we give a negative answer to the above question, by exhibiting a (non-separable) complete metric space (X, d) for which $\big(c_o(X), \mathbf{w}_d\big)$ is not Čech-complete.

Let (X, d) be a metric space. For every $x \in X$ and $\varepsilon > 0$, we put:

$$\mathcal{A}_d^+(x, \varepsilon) = \big\{C \in c_o(X) \,\big|\, d(x, C) > \varepsilon\big\}$$

and

$$\mathcal{A}_d^-(x, \varepsilon) = \big\{C \in c_o(X) \,\big|\, d(x, C) < \varepsilon\big\}.$$

The Wijsman topology $\mathbf{w}_d$ on $c_o(X)$ is that having as a subbase the collection:

$$\Delta_d = \big\{\mathcal{A}_d^+(x, \varepsilon) \,\big|\, x \in X\,,\ \varepsilon > 0\big\} \cup \big\{\mathcal{A}_d^-(x, \varepsilon) \,\big|\, x \in X\,,\ \varepsilon > 0\big\}.$$

Observe that if we associate to every $A \in c_o(X)$ the continuous function $\varphi(A)$ from X to $\mathbf{R}$, defined by $\big(\varphi(A)\big)(x) = d(x, A)$, then φ is a topological embedding of $\big(c_o(X), \mathbf{w}_d\big)$ into $C(X)$, endowed with the topology of the pointwise convergence.

In the special case — which is often considered in this paper — where the metric d takes its values on the three-element set $\{0, 1, 2\}$, we will also use the notation $[x/i]_d$, for $x \in X$ and $i \in \{0, 1, 2\}$, to denote the collection $\{A \in c_o(X) \,|\, d(x, A) = i\}$. Since in this case φ is in fact an embedding of $\big(c_o(X), \mathbf{w}_d\big)$ into 3^X, we have that the collection $\big\{[x/i]_d \,\big|\, x \in X,\, i \in \{0, 1, 2\}\big\}$ is a subbase of $\big(c_o(X), \mathbf{w}_d\big)$ (consisting of clopen sets).

Now, let R be the real line endowed with the discrete topology. Let $\{A_x \,|\, x \in R\}$ be a listing of the collection of all countable subsets of R, such that for every $A \subseteq R$ with $|A| \le \aleph_0$ we have that $|\{x \in R \,|\, A_x = A\}| = \mathbf{c}$ (where $\mathbf{c}$ is the cardinality of the continuum). Also, let α be a one-to-one function from R onto $\mathbf{c}$, and define a compatible metric d on R by:

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 2, & \text{if either } \alpha(y) < \alpha(x) \text{ and } y \in A_x, \text{ or } \alpha(x) < \alpha(y) \text{ and } x \in A_y; \\ 1, & \text{otherwise.} \end{cases}$$

To show that $(c_0(R), \mathbf{w}_d)$ is not Čech-complete, it will suffice to prove that $\varphi(c_0(R))$ is not a G_δ-subset of its closure into 3^R, which is clearly compact. Let $\mathcal{K} = \varphi\big(c_0(R)\big) \cap 2^R$: for every countable (non-empty) subset A of R, we have that $\varphi(A) \notin 2^R$. Indeed, the set $M = \{\alpha(x) \,|\, x \in R,\, A_x = A\}$ has cardinality $\mathbf{c}$, thus it is cofinal in $\mathbf{c}$ and hence, choosing a $\bar{x} \in R$ with $A_{\bar{x}} = A$ and $\alpha(y) < \alpha(\bar{x})$ for every $y \in A$, we have that $\big(\varphi(A)\big)(\bar{x}) = 2$. As a consequence, $\mathcal{K} \subseteq \big\{\varphi(A) \;\big|\; |A| > \aleph_0\big\}$ and $\mathcal{K} \subseteq \big\{f \in 2^R \;\big|\; |\{x \in R \,|\, f(x) = 0\}| > \aleph_0\big\} = 2^R \setminus \Sigma(\mathbf{1})$, where $\Sigma(\mathbf{1}) = \big\{f \in 2^R \;\big|\; |\{x \in R \,|\, f(x) \neq 1\}| \le \aleph_0\big\}$ is the Σ-*product* in 2^R with base point $\mathbf{1}$ — the function having constant value 1 (for general references on Σ-products, see [En, Problem 2.7.14]).

The fact that $\Sigma(\mathbf{1}) \cap \mathcal{K} = \emptyset$ implies that $\mathcal{K}$ is not a G_δ-subset of 2^R, because $\Sigma(\mathbf{1})$ meets every non-empty G_δ-subset of 2^R. Indeed, let $\big\{\mathcal{A}_n \,\big|\, n \in \mathbf{N}\big\}$ be a family of open subsets of 2^R whose intersection contains at least an element g: for every $n \in \mathbf{N}$ we can find a finite $F_n \subseteq X$, and for every $x \in F_n$ an $i_n(x) \in 2$, such that $g \in \Big\{f \in 2^R \;\Big|\; f\big|_{F_n} = i_n\Big\} \subseteq \mathcal{A}_n$. Then $i = \bigcup_{n \in \mathbf{N}} i_n$ is still a function, and its domain D is countable; therefore if we extend i to a function f on X, taking on the value 1 on $X \setminus D$, we have that $f \in \left(\bigcap_{n\in\mathbf{N}} \mathcal{A}_n\right) \cap \Sigma(\mathbf{1})$.

As an obvious general result we have that, if a subset Y of a topological space X is a G_δ-subset of its closure in X, and Z is another subset of X such that $Y \cap Z$ is dense in Z, then $Y \cap Z$ is a G_δ-subset of Z. Thus, if we can prove that $\mathcal{K} = \varphi\big(c_0(R)\big) \cap 2^R$ is dense in 2^R, we will obtain that $\varphi\big(c_0(R)\big)$ is not a G_δ-subset of its closure in 3^R.

Consider in 2^R the Σ-product $\Sigma(\mathbf{0})$ (where $\mathbf{0}(x) = 0$ for every $x \in R$): since $\Sigma(\mathbf{0})$ is dense in 2^R, it will suffice to show that $\Sigma(\mathbf{0}) \subseteq \mathcal{K}$. Let $f \in \Sigma(\mathbf{0})$ and $A = \{x \in R \,|\, f(x) = 0\}$: we claim that $\varphi(A) = f$, that is, $d(x, A) = 1$ for every $x \in R \setminus A$. Observe that, by the definition of d, for every fixed $\bar{x} \in R \setminus A$ we have that $d(\bar{x}, A) = 2$ if and only if

(Δ)

$$A \subseteq \{y \in R \,|\, \alpha(y) < \alpha(\bar{x}) \text{ and } y \in A_{\bar{x}}\} \cup \{y \in R \,|\, \alpha(y) > \alpha(\bar{x}) \text{ and } \bar{x} \in A_y\}.$$

Let B be any countable subset of $R \setminus \{\bar{x}\}$ and $M = \{y \in R \,|\, A_y = B\}$: then $|M| = \mathbf{c}$, so that $|M \cap A| = \mathbf{c}$, too; since $M \cap A \cap \{y \in R \,|\, \alpha(y) > \alpha(\bar{x}) \text{ and } \bar{x} \in A_y\} = \emptyset$ and $|\{y \in R \,|\, \alpha(y) < \alpha(\bar{x}) \text{ and } \bar{x} \in A_y\}| < \mathbf{c}$, (Δ) does not hold, i.e., $d(\bar{x}, A) = 1$.

Let us observe that, by [Zs, Corollary 5.2], the above constructed space is Baire.

2. Second example

Let X be a separable metrizable space, and d, ρ two compatible metrics on X. As we have already recalled, the two topologies $\mathbf{w}_d$ and $\mathbf{w}_\rho$ on $c_o(X)$ may very well be different; nevertheless, if we consider the collections Ξ_d and Ξ_ρ of the Borel sets they respectively generate, then by a result of C. Hess we have coincidence [He, Proposition 3.1.1]. As a matter of fact, it turns out that the collections Ξ_d and Ξ_ρ both coincide with the *Effros sigma algebra* on $c_o(X)$ (see [Be1, Theorem 6.5.14] and related bibliography).

We will prove here that the hypothesis of separability for (X, d) cannot be dropped in the above result.

Let Q denote the set of rational numbers in the interval $]0, 2[$ and let P denote the set of irrational numbers in the interval $]1, 2[$, endowed with the Euclidean topology. For $x \in P$, put $A_x = \{q \in Q \,|\, 1/x < q < x\}$, and let $\mathcal{R} = \{A_x \,|\, x \in P\}$. Clearly, the family $\mathcal{R}$ is *incomparable*, in the sense that for $x, y \in P$ with $x \neq y$, we have that $A_x \not\subseteq A_y$ and $A_y \not\subseteq A_x$.

First, consider the 0-1 metric d on $X = Q \cup P$; we claim that $x \mapsto A_x$ is a homeomorphism between P and $\mathcal{R}$, where $\mathcal{R}$ is endowed with the topology induced by $\big(c_0(Q \cup P), \mathbf{w}_d\big)$. Indeed, for every $q \in Q$ and $x \in P$ we have that $d(q, A_x) = 0$ if and only if $x > q$ and $q > 1/x$; it follows that:

$$[q/0]_d \cap \mathcal{R} = \left\{ A_x \,\middle|\, x \in P \cap]q, 2[\cap \left]\frac{1}{q}, 2\right[\right\},$$

$$[q/1]_d \cap \mathcal{R} = \left\{ A_x \,\middle|\, x \in P \cap \left( ]1, q[\cup \left]1, \frac{1}{q}\right[\right) \right\}$$

(of course, if $q \in P$, then $[q/0]_d \cap \mathcal{R} = \emptyset$ and $[q/1]_d \cap \mathcal{R} = \mathcal{R}$). On the other hand, let $V =]\alpha, \beta[\cap P$ be a basic open subset of P (with $1 < \alpha < \beta < 2$), and let $\bar{x} \in V$: choosing $q' \in Q \cap]1/\beta, 1/\bar{x}[$ and $q'' \in Q \cap]\alpha, \bar{x}[$, we have that $A_{\bar{x}} \in [q'/1]_d \cap [q''/0]_d \cap \mathcal{R} \subseteq \{A_x \,|\, x \in V\}$.

Consequently, it is clear that the Borel sets of $\mathcal{R}$ are exactly those of the form $\{A_x \,|\, x \in B\}$, with B Borel in P. Since the Borel sets of P do not coincide with the whole of $\wp(P)$, we also have that the Borel sets of $\mathcal{R}$ do not coincide with the whole of $\wp(\mathcal{R})$.

Second, consider the metric ρ on $Q \cup P$ defined by:

$$\rho(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 2, & \text{if either } x \in P \text{ and } y \in A_x, \text{ or } y \in P \text{ and } x \in A_y; \\ 1, & \text{otherwise.} \end{cases}$$

Note that for $x, y \in P$ we have that $\rho(x, A_y) = 2$ if and only if $x = y$ (because the family $\mathcal{R}$ is incomparable), so that $[x/2]_\rho \cap \mathcal{R} = \{A_x\}$. It follows that $\left(c_0(Q \cup P), \mathbf{w}_\rho\right)$ induces the discrete topology on $\mathcal{R}$, and hence the Borel sets of $\mathcal{R}$ with respect to such a topology are the whole of $\wp(\mathcal{R})$.

Since the Borel sets of a subspace are the traces of the Borel sets of the whole space, we also have that $\Xi_d \neq \Xi_\rho$.

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