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A RATIONAL INVARIANT FOR KNOT CROSSINGS

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Abstract. A rational number-valued invariant is constructed for the crossings of knot projections. The invariant completely determines the signature and (signed) determinant of the knot obtained by changing the crossing. In particular, if the invariant is not 0, then the new knot is distinct from the old one.

1.

A knot is usually presented by a (planar) projection which contains at least 3 crossings unless the knot is trivial. Even though crossings are a critical part of the projection of a knot, it is not clear what intrinsic properties of the knot are carried by them. A reason for this is that a crossing, as we understand it, can be created or eliminated by a diffeomorphism of $\mathbb{R}^3$ (or Reidemeister moves). We assume that all diffeomorphisms are orientation-preserving unless it is said otherwise. Another reason is that a change of a crossing (the interchange of the two crossing arcs at the crossing while the rest of the projection is unchanged), before or after a Reidemeister move on the projection, may produce two distinct knots.

In this paper, we make the notion of a crossing stronger so that geometric properties of the crossing are preserved by diffeomorphisms of $\mathbb{R}^3$, and we construct a rational number-valued invariant of the crossing. The invariant can be used in an obvious way to distinguish nonequivalent crossings. The invariant of a crossing together with the signature of the knot completely determines the signature of the knot obtained by changing the crossing (Theorem 2). Another application may come from the assertion (Theorem 3) that if the invariant of a crossing is not equal to 0, then a change of the crossing produces a new knot different from the old. As for the crossings of a projection of the trivial knot, there exists (main theorem in [4]) a necessary and sufficient condition for a change of a crossing to produce a nontrivial knot. The rational invariant can be used to distinguish alternating knots; this will be studied in a subsequent paper.

To define a crossing, let E be the union of the unit circle S^1 in the x-y plane and a thin, closed, tubular neighborhood B of $W = [-1, 1] \times \{0\}$ in the unit disk (Figure 1).

Definition. A (knot) crossing c is defined to be a smooth embedding of E into $\mathbb{R}^3(S^3)$. Denote $c(S^1)$ by $K(c)$, and call c a crossing of $K(c)$ and $c(B)$ the band of c. Two crossings c and d are equivalent if there exists a diffeomorphism f of $\mathbb{R}^3$ such that $K(d) = f(K(c))$ and $d|B = fc|B$.

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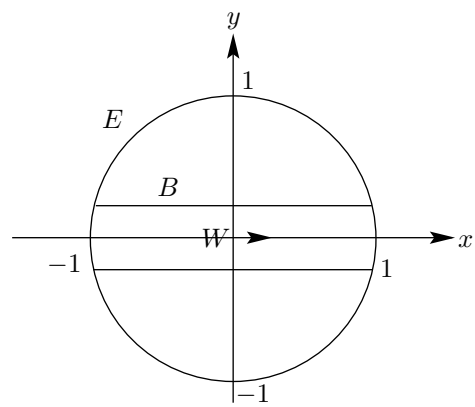


FIGURE 1

The construction of the invariant is given in Section 2 and we prove that it is well defined in Section 3. A computation of the invariant is demonstrated in Section 4. In Section 5, we relate the invariant to the change in signature of knots under the change of crossings, and we show that if the invariant of a crossing is not 0, then a change of the crossing produces a new knot.

2.

We first observe that there exists a Seifert surface of $K(c)$ containing the band of c. To see this, note that there exists a diffeomorphism f of $\mathbb{R}^3$ such that $f(K(c))$ with an orientation admits a projection (say, onto the x-y plane) where $f(c(B))$ is diffeomorphic to one of the standard forms shown in Figure 2. Notice that a crossing can be put into either one of these two standard forms by a diffeomorphism of $\mathbb{R}^3$.

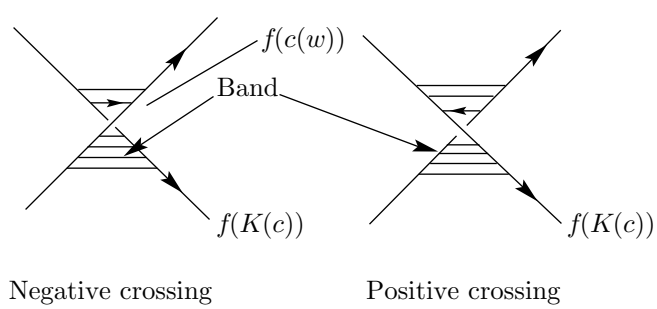


FIGURE 2

Let F be a canonical Seifert surface of $f(K(c))$. Then $f^{-1}(F)$ is a Seifert surface of $K(c)$ containing $c(B)$ as a submanifold.

Definition. For any crossing c, a Seifert surface of $K(c)$ is called a Seifert surface of the crossing if it contains $c(B)$ as a submanifold.

Given a Seifert surface F of a crossing c, choose an orientation of F and a positive normal direction of F in $\mathbb{R}^3$. Let $\alpha = (a_1, b_1, \ldots, a_g, b_g)$ be an ordered basis

of $H_1(F;\mathbb{Z})$. Then there exists a basis $\alpha^* = (a_1^*, b_1^*, \ldots, a_g^*, b_g^*)$ of $H_1(S^3 - F;\mathbb{Z})$ dual to α under the linking pairing. Let $h\colon\ H_1(F;\mathbb{Z}) \to H_1(S^3 - F;\mathbb{Z})$ be the homomorphism induced by the isotopy pushing F into $S^3 - F$ along the positive normal direction of F. Let $M_\alpha = (m_{ij})$ be the $2g \times 2g$ matrix representing h with respect to the basis α (and α^*). Then m_{ij} is equal to the linking number of $h(v)$ with u, where u and v are the ith and jth elements of α, respectively, and M_α is a Seifert matrix of $K(c)$ [3]. Let $V_\alpha = M_\alpha + M_\alpha^T$, where M_α^T denotes the transpose of M_α. Then V_α is always nonsingular [3].

Now let $\langle\ ,\ \rangle\colon\ H_1(F, \partial F;\mathbb{Z}) \otimes H_1(F;\mathbb{Z}) \to \mathbb{Z}$ be the unimodular bilinear pairing defined by $\langle x, y\rangle = u(y)$, where $u \in H^1(F;\mathbb{Z})$ is the Poincaré dual of x with respect to the orientation of F. Let $\overline{\alpha} = (\overline{a}_1, \overline{b}_1, \ldots, \overline{a}_g, \overline{b}_g)$ be the basis of $H_1(F, \partial F;\mathbb{Z})$ dual to α under this bilinear pairing. Then by definition $\langle \overline{\alpha}^T, \alpha\rangle = I$, where I is the identity matrix and the pairing is evaluated as in a multiplication of matrices.

Let w be the canonical generator of $H_1(E, S^1;\mathbb{Z}) \cong \mathbb{Z}$ determined by the standard orientation of W (see Figure 1). Let $X_\alpha(c)$ be the row coordinate vector of $c_*(w) \in H_1(F, \partial F;\mathbb{Z})$ with respect to $\overline{\alpha}$ so that $c_*(w) = X_\alpha(c)\overline{\alpha}^T$. Then the ith coordinate of $X_\alpha(c)$ is equal to the intersection number of $c(W)$ with the ith element of α.

Definition. Given a crossing c, define $\lambda(c) = X_\alpha(c)V_\alpha^{-1}X_\alpha(c)^T$.

For a crossing c, $\lambda(c)$ becomes an invariant of the equivalence class containing c, which is the main interest of this paper. The matrix V_α^{-1} has been used by Trotter in the study of the nonsingular Seifert matrices [5].

3.

We state the main theorem and then give a proof.

Theorem 1. *The rational function λ is well defined on the equivalence classes of crossings.*

The proof of the theorem is divided into four parts: we need to show that λ is invariant under change of orientation or positive normal direction of the Seifert surface, under change of basis of the 1st homology group of the Seifert surface, under a different choice of Seifert surface, and under equivalence of crossings.

3.1. Let F be a Seifert surface of a crossing c and let M_α be the Seifert matrix with respect to a basis α of $H_1(F;\mathbb{Z})$. Suppose that the orientation of F is reversed while the rest of the data is unchanged. Then in the definition of $\lambda(c)$, $X_\alpha(c)$ changes to $-X_\alpha(c)$ and V_α remains the same. Since $(-X_\alpha(c))V_\alpha^{-1}(-X_\alpha(c))^T = X_\alpha(c)V_\alpha^{-1}X_\alpha(c)$, λ is invariant.

If the positive normal direction of F is reversed, then M_α changes to M_α^T, thus leaving V_α and λ unchanged.

3.2. Suppose that β is another basis of $H_1(F;\mathbb{Z})$. Then there exists a unimodular matrix A such that $\beta = \alpha A$ with α and β regarded as row vectors. It follows easily that $M_\beta = A^T M_\alpha A$. Hence $V_\beta = A^T V_\alpha A$.

From $\langle \overline{\alpha}^T, \alpha\rangle = \langle \overline{\alpha}^T, \beta A^{-1}\rangle = \langle A^{-1}\overline{\alpha}^T, \beta\rangle = I$, $\overline{\beta} = \overline{\alpha}(A^{-1})^T$. Therefore, $c_*(w) = X_\alpha(c)\overline{\alpha}^T = X_\alpha(c)A\overline{\beta}^T$, which implies that $X_\beta(c) = X_\alpha(c)A$.

With respect to β, $\lambda(c) = X_\beta(c)V_\beta^{-1}X_\beta(c)^T = X_\alpha(c)AA^{-1}V_\alpha^{-1}(A^T)^{-1}A^T X_\alpha(c)^T$ $= X_\alpha(c)V_\alpha^{-1}X_\alpha(c)^T$. Hence λ is invariant.

3.3. Let F and F' be Seifert surfaces of a crossing c.

Let $F_0 = \text{closure}(F - c(B))$ and $F_0' = \text{closure}(F' - c(B))$. Then $\partial F_0 = \partial F_0'$ is a 2-component link. After an isotopy fixing the points of ∂F_0 and points near $c(B)$, we may assume that F_0 and F_0' are in general position such that $F_0 \cap F_0'$ is the union of ∂F_0 and circles contained in the interior of F_0 and F_0'. By applying Morse function theory [1], we see that F' is obtained from F, away from $c(B)$, by a sequence of ambient 0- or 1-surgeries on F (see below for the description of a 0-surgery). We may further assume that all the 0-surgeries are orientable ones, i.e., any surgery tube is attached on the same side of F.

To show that λ does not depend on the choice of Seifert surface, it suffices to prove that λ is invariant when F' is obtained from F by a single 0-surgery. We do not need to consider the case of a 1-surgery since a 1-surgery is complementary to a 0-surgery.

Suppose that F' is obtained from F by a single 0-surgery. Then there exists a 1-handle $H \cong D^1 \times D^2$ in $\mathbb{R}^3$ such that $H \cap F = \partial D^1 \times D^2 \subset F - c(B)$ and $H \cap F' = D^1 \times \partial D^2$. If $H \cap F$ meets two distinct components of F (note that F may not be connected), then the defining expression for λ gives the identical value for both F and F' since the surfaces have the identical 1st homology group. Therefore, we assume that $H \cap F$ is contained in a component of F. Let a and b be the elements of $H_1(F'; \mathbb{Z})$ given by the oriented circles shown in Figure 3.

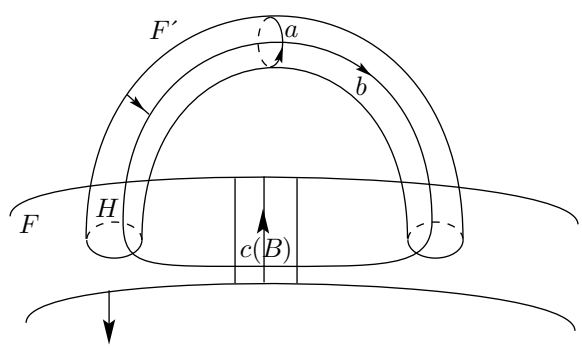


Figure 3

Let α be a basis of $h_1(F; \mathbb{Z})$. Then $\beta = (a, b) \cup \alpha$ is a basis of $h_1(F'; \mathbb{Z})$. If the positive normal directions of F and F' are chosen consistently as in the figure, then

$$M_\beta = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 0 & x & x & \dots & x \\ 0 & x & & & \\ \vdots & \vdots & & M_\alpha & \\ 0 & x & & & \end{pmatrix},$$

where the entries denoted by x are unspecified.

$$V_\beta = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ -1 & x & x & \dots & x \\ 0 & x & & & \\ \vdots & \vdots & & V_\alpha & \\ 0 & x & & & \end{pmatrix}, \qquad V_\beta^{-1} = \begin{pmatrix} x & -1 & x & \dots & x \\ -1 & 0 & 0 & \dots & 0 \\ x & 0 & & & \\ \vdots & \vdots & & V_\alpha^{-1} & \\ x & 0 & & & \end{pmatrix}.$$

Moreover, for the coordinates of $c_*(w) \in H_1(F', \partial F'; \mathbb{Z})$ with respect to $\overline{\beta}$, we have $X_\beta(c) = (0, x, X_\alpha(c))$. In terms of F' and β,

$$\lambda(c) = X_\beta(c) V_\beta^{-1} X_\beta(c)^T = (0, x, X_\alpha(c)) \begin{pmatrix} x & -1 & x & \dots & x \\ -1 & 0 & 0 & \dots & 0 \\ x & 0 & & & \\ \vdots & \vdots & & V_\alpha^{-1} & \\ x & 0 & & & \end{pmatrix} \begin{pmatrix} 0 \\ x \\ X_\alpha(c)^T \end{pmatrix}$$

$$= X_\alpha(c) V_\alpha^{-1} X_\alpha(c)^T.$$

Therefore, λ is invariant.

3.4. Suppose that a crossing c is equivalent to a crossing c' through a diffeomorphism f of $\mathbb{R}^3$. Let F be a Seifert surface of c and let α be a basis of $H_1(F; \mathbb{Z})$. Then $F' = f(F)$ is a Seifert surface of c' and $\beta = f_*(\alpha)$ is a basis of $H_1(F'; \mathbb{Z})$. Choose also orientations and positive normal directions of F and F' such that they are preserved by f. Now both data give λ the identical value at c and c'. So λ is invariant.

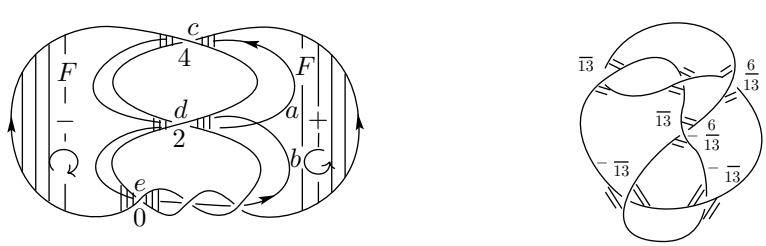
The proof of Theorem 1 is completed by combining the above four parts.

Definition. For a knot K, let M be a Seifert matrix of K associated to a Seifert surface of genus g. Define $D(K) = (-1)^g \det(M + M^T)$, where the determinant of the empty matrix is considered to be 1.

Remark. It follows immediately from the proof of Theorem 1 that $D(K)$ is an invariant of the knot K, and $D(K) \neq 0$ for any knot K. We note that $D(K) = \pm$ (the value of the Alexander polynomial of K at -1), and D(mirror image of K) $= D(K)$.

4.

The crossings c, d and e are represented by the half twisted, shaded bands in the first projection in Figure 4. They are crossings of the trivial knot.



Trivial knot $\qquad$ 6_3 knot

FIGURE 4

The shaded surface F of genus 1 in the figure is a Seifert surface for the crossings c, d and e. Choose the orientation and the positive normal direction of F as

indicated by the usual convention in the figure. Let $\alpha = (a, b)$, where a and b are given by the oriented circles shown. Then α is a basis of $H_1(F; \mathbb{Z})$ and we have

$$M_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix}, \quad V_\alpha = \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix}, \quad V_\alpha^{-1} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix},$$
$$X_\alpha(c) = (1, 0), \quad X_\alpha(d) = (-1, 1), \quad X_\alpha(e) = (0, -1).$$

The values of λ for the crossings are:

	c	d	e
λ	4	2	0

The table shows that all three crossings are not equivalent to each other, i.e., there is no diffeomorphism of $\mathbb{R}^3$ mapping one to another. Furthermore, changing the orientation of $\mathbb{R}^3$ sends V_α to $-V_\alpha$, and hence sends $\lambda(c)$ to $-\lambda(c)$. Thus these three crossings are inequivalent under any diffeomorphism of $\mathbb{R}^3$. The invariants of more crossings are given for the 6_3 knot in Figure 4.

5.

The invariant determines completely the signature of the knot obtained by a change of a crossing. It is well known [2] that the change of a positive crossing (Figure 1) increases the signature of a knot by 0 or 2, and the change of a negative crossing decreases the signature by 0 or 2. We give a proof of this assertion which implies the above claim about the invariant. The proof also implies that for any crossing $c, \lambda(c)$ is never equal to $\pm\frac{1}{2}$, and that if $\lambda(c) \neq 0$, then a change of the crossing produces a new knot.

We now clarify the notion of changing a crossing. To change a crossing, put the crossing by an equivalence into one of the two standard forms shown in Figure 1, and then interchange the arcs at the crossing. Since a crossing can be put into either one of the two standard forms, this operation is not well defined but it can be shown that no more than two distinct knots can be created by a change of a crossing.

Definition. Given a crossing c, let $K_+(c)$ or $K_-(c)$ be the knot obtained from $K(c)$ by changing c after it is put into negative or positive form, respectively.

Let $\sigma(K)$ denote the signature of a knot K.

Theorem 2. *For any crossing $c, \lambda(c) \neq \pm\frac{1}{2}$, $\sigma(K_-(c)) = \sigma(K(c)) + 2$ if and only if $\lambda(c) < -\frac{1}{2}$, and $\sigma(K_+(c)) = \sigma(K(c)) - 2$ if and only if $\lambda(c) > \frac{1}{2}$.*

Proof. Let F be a Seifert surface of c. Construct the Seifert surface F' of $K_-(c)$ or $K_+(c)$ by adding two 1-handles (half-twisted bands) to F as in Figure 5.

Let α be a basis of $H_1(F; \mathbb{Z})$. Then $\beta = (a, b) \cup \alpha$ is a basis for $H_1(F'; \mathbb{Z})$, where a and b are given by the oriented circles shown in the figure. Then

$$M_\beta = \begin{pmatrix} 0 & -1 & & X_\alpha(c) & \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & M_\alpha & \\ 0 & 0 & & & \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -1 & & -X_\alpha(c) & \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & M_\alpha & \\ 0 & 0 & & & \end{pmatrix}$$

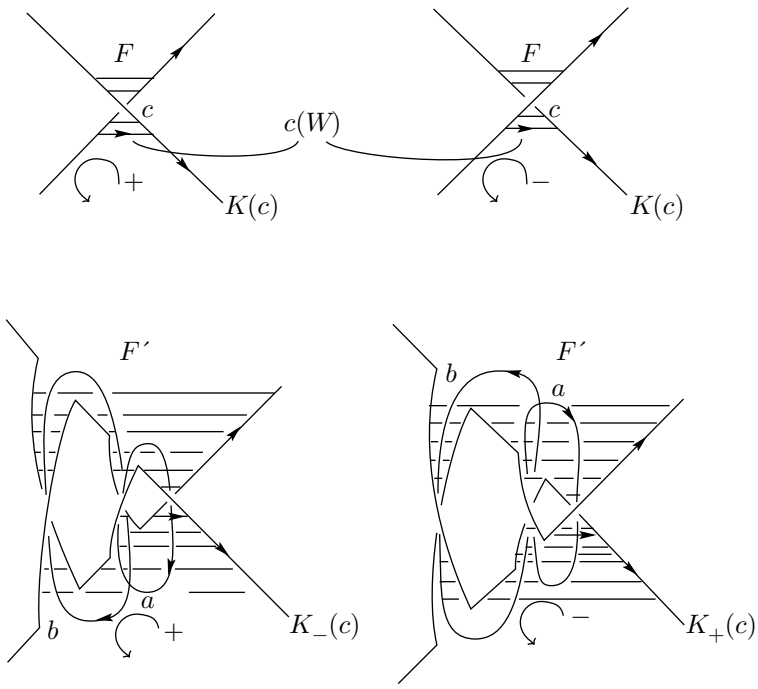


FIGURE 5

is a Seifert matrix for $K_-(c)$ or $K_+(c)$, respectively, where X_α is the coordinate vector of $c_*(w) \in H_1(F, \partial F; \mathbb{Z})$ with respect to $\overline{\alpha}$ and the orientation of F given as in the figure.

To simplify notation put $X = X_\alpha(c)$ and $V = V_\alpha$.

$$V_\beta = \begin{pmatrix} 0 & -1 & & X & \\ -1 & 2 & 0 & \dots & 0 \\ & 0 & & & \\ X^T & \vdots & & V & \\ & 0 & & & \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -1 & & -X & \\ -1 & -2 & 0 & \dots & 0 \\ & 0 & & & \\ -X^T & \vdots & & V & \\ & 0 & & & \end{pmatrix}.$$

V_β is congruent to

$$\begin{pmatrix} -\frac{1}{2} & 0 & & X & \\ 0 & 2 & 0 & \dots & 0 \\ & 0 & & & \\ X^T & \vdots & & V & \\ & 0 & & & \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \frac{1}{2} & 0 & & -X & \\ 0 & -2 & 0 & \dots & 0 \\ & 0 & & & \\ -X^T & \vdots & & V & \\ & 0 & & & \end{pmatrix}.$$

Using

$$\left(\begin{array}{c|ccc} I & 0 & \dots & 0 \\ & 0 & \dots & 0 \\ \hline 0 & & & \\ -V^{-1}X^T \vdots & & I & \\ 0 & & & \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|ccc} I & 0 & \dots & 0 \\ & 0 & \dots & 0 \\ \hline 0 & & & \\ V^{-1}X^T \vdots & & I & \\ 0 & & & \end{array}\right),$$

V_β is easily seen to be congruent to

$$\begin{pmatrix} -\frac{1}{2} - \lambda(c) & 0 \\ 0 & 2 \end{pmatrix} \oplus V_\alpha \quad \text{or} \quad \begin{pmatrix} \frac{1}{2} - \lambda(c) & 0 \\ 0 & -2 \end{pmatrix} \oplus V_\alpha.$$

Since the above matrices are nonsingular [3], $\lambda(c) \neq \pm\frac{1}{2}$. Moreover,

$$\sigma(K_-(c)) = \sigma \begin{pmatrix} -\frac{1}{2} - \lambda(c) & 0 \\ 0 & 2 \end{pmatrix} + \sigma(V_\alpha)$$

and

$$\sigma(K_+(c)) = \sigma \begin{pmatrix} \frac{1}{2} - \lambda(c) & 0 \\ 0 & -2 \end{pmatrix} + \sigma(V_\alpha).$$

From $\sigma(K(c)) = \sigma(V_\alpha)$, the rest of the conclusion of the theorem follows.

Theorem 3. *Let c be a crossing with $\lambda(c) \neq 0$. Then $K(c)$, $K_-(c)$ and $K_+(c)$ are nonequivalent knots under any diffeomorphism (possibly orientation-reversing) of $\mathbb{R}^3$.*

Proof. From the remark of Section 3 and the proof of Theorem 2, we have $D(K_-(c)) = (1 + 2\lambda(c))D(K(c))$ and $D(K_+(c)) = (1 - 2\lambda(c))D(K(c))$. If any two of $K(c)$, $K_-(c)$ and $K_+(c)$ are equivalent, then $\lambda(c) = 0$.

Remark. Theorem 3 implies that a change of the crossing c or d of the trivial knot in Figure 4 produces a nontrivial knot since $\lambda(c) \neq 0$ and $\lambda(d) \neq 0$. Notice that a change of the crossing e gives the trivial knot back and that $\lambda(e) = 0$.

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