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LIPSCHITZ IMAGES WITH FRACTAL BOUNDARIES AND THEIR SMALL SURFACE WRAPPING

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ABSTRACT. Assume $E \subset H \subset \mathbf{R}^m$ and $\Phi : E \to \mathbf{R}^m$ is a Lipschitz L-mapping; $|H|$ and $||H||$ denote the volume and the surface area of H. We verify that there exists a figure $F \supset \Phi(E)$ with $||F|| \le c_L||H||$, and, of course, $|F| \le c_L|H|$, where c_L depends only on the dimension and on L. We also give an example when $E = H \subset \mathbf{R}^2$ is a square and $||\Phi(E)|| = \infty$; in fact, the boundary of $\Phi(E)$ can contain a fractal of Hausdorff dimension exceeding one.

INTRODUCTION

Related to a question of W. F. Pfeffer about bi-Lipschitz transformations of certain generalized integrals we had to deal with the following problem. Assume that the set $E \subset \mathbf{R}^m$ is a subset of a set $H \subset \mathbf{R}^m$ with m-dimensional measure $|H|$, and with surface area $||H||$, that is, the $m-1$-dimensional Hausdorff measure of the essential boundary of H equals $||H||$. One can say that E is wrapped into (the boundary of) H. The question which motivated this paper is the following: If $\Phi : E \to \mathbf{R}^m$ is a Lipschitz L-transformation, then is it true that one can wrap $\Phi(E)$ into a set which has "not much more" surface area and volume than that of H? To be more precise, is it true that there exists a constant c_L, depending only on the dimension and on the Lipschitz constant L, such that we can find a set F containing $\Phi(E)$, and satisfying $||F|| \le c_L||H||$ and $|F| \le c_L|H|$. In Theorem 1 we give an affirmative answer to this question. The first natural idea to prove Theorem 1 would be the usage of Kirszbraun's theorem [Fe, Th. 2.10.43.] to extend the Lipschitz L-mapping Φ onto $\mathbf{R}^m$ and use $F = \Phi(H)$. However this simple idea does not seem to work. In Theorem 2 we show that even in the simple case when $Q_0 \subset \mathbf{R}^2$ is a square in the plane there exists a Lipschitz mapping $\Phi_0 : \mathbf{R}^2 \to \mathbf{R}^2$, such that the perimeter of $\Phi_0(Q_0)$ is of infinite measure. In fact, the boundary of $\Phi_0(Q_0)$ can contain a fractal, a self-similar set of Hausdorff dimension bigger than one.

The proof of Theorem 1 is not difficult. Considering a suitable covering of H by open balls and the Φ image of this covering one can find the required figure. In our argument Besicovitch's covering theorem and the relative isoperimetric inequality

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are used. We thank the referee for pointing out that the less known covering theorem [Fe, Corollary 4.5.4] could provide an even simpler argument; however to keep our paper self-contained we use the few lines longer alternate argument, which also contains the nice, and not widely known, key idea of [Fe, Corollary 4.5.4].

Interpreting BV sets (sets of bounded variation) as integral currents [Fe, Section 4.5.1] and applying the area theorem ([Fe, Theorem 3.2.3] and [Fe, Corollary 4.1.26]) it is easy to see that a one-to-one Lipschitz image of a BV set is again BV, that is, one-to-one Lipschitz images of sets of finite surface area are again of finite surface area. The mapping Φ, constructed in the proof of Theorem 2 can be chosen to be conjugate (via a homeomorphism) to a folding of the square Q_0. Thus Φ is a two-to-one mapping at almost all points of Q_0, and it is one-to-one on the "folding" edge, which contains a fractal.

Preliminaries

If x, y are *points* of the Euclidean space $\mathbf{R}^m$, then the (Euclidean) distance of the points is denoted by $|x-y|$. The transformation $\Phi : \mathbf{R}^m \to \mathbf{R}^m$ is a Lipschitz L-mapping if

$$|\Phi(x) - \Phi(y)| \leq L|x-y|$$

holds for all $x, y \in \mathbf{R}^m$. The open ball centered at x and of radius r will be denoted by $B(x, r)$.

If H is a *subset* of $\mathbf{R}^m$, then $|H|$ will denote its m-dimensional Lebesgue measure. (Note that for ease of notation the norm and the measure is denoted by the same symbol, though the domains of these symbols are different.) The $m-1$-dimensional Hausdorff measure of H will be denoted by $\mathcal{H}(H)$.

A figure is a finite union of compact nondegenerate subintervals of $\mathbf{R}^m$.

Assume that $H \subset \mathbf{R}^m$ is measurable. The set of its Lebesgue density points is called the essential interior of H and is denoted by $\operatorname{int}^* H$, that is,

$$\operatorname{int}^* H = \left\{ x \in \mathbf{R}^m : \lim_{\epsilon \to 0+} \frac{|H \cap B(x,\epsilon)|}{|B(x,\epsilon)|} = 1 \right\}.$$

The set $\operatorname{cl}^* H = \mathbf{R}^m \setminus \operatorname{int}^*(\mathbf{R}^m \setminus H)$ is called the essential closure of H. Finally, $\partial^* H = \operatorname{cl}^* H \setminus \operatorname{int}^* H$ denotes the essential boundary of H. A set $H \subset \mathbf{R}^m$ is a BV set, a set of bounded variation, if it is bounded, measurable, and its perimeter $||H|| = \mathcal{H}(\partial^* H)$ is finite. The family of BV sets arises quite often in geometric integration theory as the largest class where a reasonable surface integral can be defined, see, for example, [P, Section 2] where further references can be found as well. Clearly, figures are a very simple sort of BV sets.

Recall the following theorem from [M, Th. 2.7., p.30].

Besicovitch's covering theorem. *There is a constant c_B depending only on the dimension m with the following property. Let E be a bounded subset of $\mathbf{R}^m$, and let $\mathcal{B}$ be a family of balls such that each point of E is the centre of some ball of $\mathcal{B}$. Then there are families $\mathcal{B}_1, ..., \mathcal{B}_{c_B}$ covering E such that each $\mathcal{B}_i$ is disjoint, that is,*

$$E \subset \bigcup_i \bigcup \mathcal{B}_i,$$

and

$$B \cap B' = \emptyset \text{ for } B, B' \in \mathcal{B}_i \text{ with } B \neq B'.$$

Besicovitch's covering theorem is stated in [M] for families of closed balls. Scrutinizing its proof one can easily see that the theorem holds for familes of open balls as well.

We will also use from [Z, Theorem 5.4.3., p.230] the following:

Relative isoperimetric inequality. *If $H \subset \mathbf{R}^m$ a BV-set, then for each ball $B(x,r) \subset \mathbf{R}^m$ we have*

$$\min\{|B(x,r)\cap H|, |B(x,r)\setminus H|\}^{\frac{m-1}{m}} \leq c_{iso}\mathcal{H}(B(x,r)\cap \partial^* H),$$

where the constant c_{iso} depends only on the dimension.

Next we recall some properties of self-similar sets. For the details see, for example, [Fa, Ch.9., p.113-137]. Assume that the mappings $\varphi_j : \mathbf{R}^m \to \mathbf{R}^m$, $(j = 1, ..., k)$ are similarities with similarity ratios c_j. The (measurable) set F is self-similar if

$$F = \bigcup_{j=1}^{k} \varphi_j(F).$$

The mappings φ_j satisfy the *open set condition* if there exists a non-empty bounded open set such that

$$V \supset \bigcup_{j=1}^{k} \varphi_j(V),$$

with the union disjoint. If the mappings φ_j satisfy the open set condition, then the Hausdorff dimension of the self-similar set F, denoted by s, is given by

$$\sum_{j=1}^{k} c_j^s = 1.$$

In the special case when all c_j' s have the same value, say c, then $s = \log(1/k)/\log c$.

Main results

Theorem 1. *Assume that $E \subset \mathbf{R}^m$ is compact, $\Phi : E \to \mathbf{R}^m$ is a Lipschitz L-mapping, and H is a BV set such that $E \subset \operatorname{int}^* H$. Then there exists a figure F and a constant c_L, depending only on L and on the dimension, such that $\Phi(E) \subset F$, $||F|| \leq c_L||H||$, and $|F| \leq c_L|H|$.*

Proof. For $x \in E$ put

$$g_x(r) = \frac{|B(x,r)\cap H|}{|B(x,r)|}.$$

Since $x \in \operatorname{int}^* H$ we have $\lim_{r\to 0+} g_x(r) = 1$. On the other hand H is bounded; hence $\lim_{r\to\infty} g_x(r) = 0$. Since $g_x(r)$ is continuous, we can choose r_x, such that $g_x(r_x) = 1/2$.

Using Besicovitch's covering theorem choose c_B many classes $\{\mathcal{B}_i\}$, each consisting of balls $B(x, r_x)$, such that

$$E \subset \bigcup_i \bigcup_{B(x,r_x)\in\mathcal{B}_i} B(x, r_x),$$

and balls belonging to the same class $\mathcal{B}_i$ are disjoint. Since E is compact we can also assume that each $\mathcal{B}_i$ consists of finitely many balls.

Using the relative isoperimetric inequality and $g_x(r_x) = 1/2$, we have

$$(|B(x,r_x)|/2)^{\frac{m-1}{m}} \le c_{iso}\mathcal{H}(B(x,r_x) \cap \partial^* H)$$

for each $B(x,r_x)$, that is, there exists a constant c'_{iso} such that

$$||B(x,r_x)|| \le c'_{iso}\mathcal{H}(B(x,r_x) \cap \partial^* H).$$

Assuming that balls belonging to a class $\mathcal{B}_i$ are disjoint, we have

$$\sum_{B(x,r_x)\in\mathcal{B}_i} ||B(x,r_x)|| \le c'_{iso} \sum_{B(x,r_x)\in\mathcal{B}_i} \mathcal{H}(B(x,r_x) \cap \partial^* H) \le c'_{iso}||H||,$$

and

$$\sum_{B(x,r_x)\in\mathcal{B}_i} |B(x,r_x)| = 2 \sum_{B(x,r_x)\in\mathcal{B}_i} |B(x,r_x) \cap H| \le 2|H|,$$

for each $i \le c_B$. Clearly, $\Phi(B(x,r_x)) \subset B(\Phi(x), Lr_x) \subset Q_{\Phi(x)}$, where $Q_{\Phi(x)}$ is a closed cube centered at $\Phi(x)$ and of side length $2Lr_x$. Then there exists a constant c_Q, depending only on the dimension and on L, such that $||Q_{\Phi(x)}|| \le c_Q||B(x,r_x)||$, and $|Q_{\Phi(x)}| \le c_Q|B(x,r_x)|$.

Let

$$F = \bigcup_i \bigcup_{B(x,r_x)\subset\mathcal{B}_i} Q_{\Phi(x)}.$$

Then

$$||F|| \le \sum_i \sum_{B(x,r_x)\subset\mathcal{B}_i} ||Q_{\Phi(x)}|| \le c_B c_Q c'_{iso}||H||,$$

$$|F| \le \sum_i \sum_{B(x,r_x)\subset\mathcal{B}_i} |Q_{\Phi(x)}| \le 2c_B c_Q|H|,$$

and clearly $\Phi(E) \subset F$. Therefore, one can define c_L as the maximum of the constants $c_B c_Q c'_{iso}$ and $2c_B c_Q$. This completes the proof.

Theorem 2. *There exists a Lipschitz transformation $\Phi_0 : \mathbf{R}^2 \to \mathbf{R}^2$ and a square $Q_0 \subset \mathbf{R}^2$ such that the Hausdorff dimension of $\partial^*(\Phi_0(Q_0))$ is bigger than one; hence it is of non-σ-finite linear measure. Furthermore, Φ_0 is two-to-one at almost all points of Q_0 and is one-to-one on the rest of Q_0.*

Proof. It is enough to define a suitable Lipschitz transformation Φ on a rectangle Q_1 with vertices $(0,3)$, $(48,3)$, $(48,-3)$, and $(0,-3)$, since then, by using a suitable affine transformation, we can define Φ_0 on Q_0, and using Kirszbraun's theorem we can extend Φ_0 onto $\mathbf{R}^2$.

We construct Φ by a limit of a self-similar like repetition of the same mapping.

First we define a Lipschitz folding Φ_1 of Q_1. Then we define 48 subrectangles Q_2^1,...,Q_2^{48} in Q_1, such that each Q_2^j is similar to Q_1. Our mapping Φ will coincide with Φ_1 on $Q_1 \setminus \bigcup_{j=1}^{48} Q_2^j$. On the rectangles Q_2^j we will do a folding process similar to the one in Q_1.

Put $s_1 = (0,3)$, $s_2 = (1,3)$, $s_3 = (2,3)$, $s_4 = (2,-3)$, $s_5 = (1,-3)$, and $s_6 = (0,-3)$.

First we define Φ_1 on the rectangle $S_1 = s_1s_2s_5s_6$.

Choose the points p_1, p_2, p_3, and p_4 on the line $y = x$ such that $p_1 = (0,0)$, $p_2 = (0.1, 0.1)$, $p_3 = (0.9, 0.9)$, and $p_4 = (1,1)$. Next we choose the points $q_1,\ q_2,\ q_3,\ q_4,$

and $r_1,\ r_2,\ r_3,\ r_4$ on the lines $y = x - 0.1$, and $y = x + 0.1$, respectively, such that $q_1 = (0, -0.1)$, $r_1 = (0, 0.1)$, $q_2 = (0.15, 0.05)$, $r_2 = (0.05, 0.15)$, $q_3 = (0.95, 0.85)$, $r_3 = (0.85, 0.95)$, $q_4 = (1, 0.9)$, and $r_4 = (1, 1.1)$.

Denote by Q_2^1 the rectangle $q_2q_3r_3r_2$. Observe that Q_2^1 is similar to Q_1.

Let Φ_1 equal the identity on the segments p_1s_6 and s_6s_5, and equal the reflection about the x-axis on the segments p_1s_1 and s_1s_2. On $r_2p_2p_3r_3$, that is, on the upper half of Q_2^1 we define Φ_1 to be equal to the reflection about the $y = x$ line; on the lower half of Q_2^1, that is, on $p_2q_2q_3p_3$ we define Φ_1 to be equal to the identity, that is, Φ_1 is a lengthwise folding on Q_2^1.

At points of S_1, where we have not already defined Φ_1, we extend its definition to meet the following requirements:

i) Φ_1 is Lipschitz on S_1;
ii) Φ_1 maps S_1 onto the trapezoid $p_1s_6s_5p_4$, which will be denoted by T_1;
iii) Φ_1 is one-to-one on the segment p_1p_4 and is two-to-one at any other point of S_1; hence $\Phi_1(S_1 \setminus Q_2^1) = T_1 \setminus Q_2^1$;
iv) Φ_1 maps the line segment s_2p_4 onto the segment p_4s_5;
v) Φ_1 equals the identity on the segments p_4s_5, p_1p_4.

It is an easy exercise, left to the reader, to show that the above conditions can be satisfied.

Next we define Φ_1 on the strip $S_2 = s_2s_3s_4s_5$. Denote by τ the reflection about the line $x = 1$. For $p \in S_2$ set $\Phi_1(p) = \tau(\Phi_1(\tau(p)))$. Put $Q_2^2 = \tau(Q_2^1)$. Then Q_2^2 is a rectangle similar to Q_1, and Φ_1 on Q_2^2 equals a lengthwise folding of Q_2^2. Furthermore letting $T_2 = \tau(T_1)$ we have the following properties satisfied.

i) Φ_1 is Lipschitz on $S_1 \cup S_2$;
ii) Φ_1 maps $S_1 \cup S_2$ onto $T_1 \cup T_2$;
iii) Φ_1 is one-to-one on the segments p_1p_4 and $p_4\tau(p_1)$, and is two-to-one at any other point of $S_1 \cup S_2$; hence $\Phi_1((S_1 \cup S_2) \setminus (Q_2^1 \cup Q_2^2)) = (T_1 \cup T_2) \setminus (Q_2^1 \cup Q_2^2)$;
iv) Φ_1 equals the identity on the segments p_1s_6, s_6s_4, $s_4\tau(p_1)$ and Φ_1 equals the reflection about the x-axis on the segments p_1s_1, s_1s_3, and $s_3\tau(p_1)$, that is, on the boundary of $S_1 \cup S_2$ the mapping Φ_1 coincides with a folding about the x-axis.

Denote by σ_j the translation by the vector $(0, 2j)$. The strip bounded by the points $(j-1, 3)$, $(j, 3)$, $(j, -3)$, and $(j-1, -3)$ is denoted by S_j. If $p \in S_{2j-1} \cup S_{2j}$, $(j = 2, 3, ..., 24)$, then put $\Phi_1(p) = \sigma_{j-1}(\Phi_1(\sigma_{j-1}^{-1}(p)))$, $Q_2^{2j-1} = \sigma_{j-1}(Q_2^1)$, and $Q_2^{2j} = \sigma_{j-1}(Q_2^2)$. Then one can easily see that Φ_1 satisfies the following properties:

i) Φ_1 is Lipschitz on Q_1, denote its Lipschitz constant by L;
ii) Φ_1 coincides with a lengthwise folding on the boundary of Q_1, and on the entire closed rectangles Q_2^j, $(j = 1, ..., 48)$;
iii) If $g(x)$ denotes the sawtooth function which is periodic by 2, equals $y = x$ on $[0, 1]$, and $y = -x + 2$ on $[1, 2]$, then $\Phi_1(Q_1)$ is the region bounded by the graph of g and by the lower half of the boundary of Q_1;
iv) Φ_1 is one-to-one on the part of the boundary of $\Phi_1(Q_1)$ which is on the graph of g; at any other point of Q_1 it is two-to-one.

Denote by φ_j the similarities which map Q_1 onto Q_2^j, $(j = 1, ..., 48)$, and the lower half of Q_1 is mapped onto the lower half of Q_2^j. Observe that the contraction ratio of φ_j equals $\sqrt{2}/60$.

For $p \in Q_2^j$ put $\Phi_2^j(p) = \varphi_j(\Phi_1(\varphi_j^{-1}(p)))$. Then Φ_2^j is a Lipschitz L-mapping, and it coincides with Φ_1 on the boundary of Q_2^j, since on this boundary they both equal a lengthwise folding of Q_2^j.

Put $\tilde{\Phi}_1(p) = \Phi_1(p)$ if $p \in Q_1 \setminus \bigcup_{j=1}^{48} Q_2^j$, and $\tilde{\Phi}_1(p) = \Phi_2^j(p)$ if $p \in Q_2^j$.

Then $\tilde{\Phi}_1$ is a Lipschitz L-mapping. Indeed, assume $p, q \in Q_1$. If $p, q \notin \bigcup_{j=1}^{48} Q_2^j$, or there exists j such that $p, q \in Q_2^j$, then $|\tilde{\Phi}_1(p) - \tilde{\Phi}_1(q)| \leq L|p - q|$ follows from the same property for Φ_1, or for Φ_2^j.

If $p \in Q_1$, and $q \in Q_2^j$, then the segment pq intersects the boundary of Q_2^j. Denote this intersection point by q^*. Then

$$\begin{aligned} |\tilde{\Phi}_1(p) - \tilde{\Phi}_1(q)| &\leq |\tilde{\Phi}_1(p) - \tilde{\Phi}_1(q^*)| + |\tilde{\Phi}_1(q^*) - \tilde{\Phi}_1(q)| \\ &= |\Phi_1(p) - \Phi_1(q^*)| + |\Phi_2^j(q^*) - \Phi_2^j(q)| \leq L(|p - q^*| + |q^* - q|) = L|p - q|. \end{aligned}$$

Finally if $p \in Q_2^i$ and $q \in Q_2^j$, then denoting by p^* and q^* the intersection points of pq and the boundaries of Q_2^i and Q_2^j, respectively, an argument, similar to the one in the above paragraph, can establish the Lipschitz L-property of $\tilde{\Phi}_1$.

It is also easy to see that $\tilde{\Phi}_1$ is two-to-one at almost all points of Q_1, the exceptional points are on the "folding edges" and at these points $\tilde{\Phi}_1$ is one-to-one.

Assume $k \geq 2$, $j_1, ..., j_k \in \{1, ..., 48\}$, and $p \in Q_1$. Put $\varphi_{j_1,...,j_k}(p) = \varphi_{j_1} \circ \ldots \circ \varphi_{j_k}(p)$, $Q_{k+1}^{j_1,...,j_k} = \varphi_{j_1,...,j_k}(Q_1)$, and for $p \in Q_{k+1}^{j_1,...,j_k}$, let $\Phi_{k+1}^{j_1,...,j_k}(p) = \varphi_{j_1,...,j_k}(\Phi_1(\varphi_{j_1,...,j_k}^{-1}(p)))$. Denote $Q'_{k+1} = \bigcup_{j_1,...,j_k \in \{1,...,48\}} Q_{k+1}^{j_1,...,j_k}$.

If $\tilde{\Phi}_{k-1}$, $(k = 2, ...)$ is given, then put $\tilde{\Phi}_k(p) = \tilde{\Phi}_{k-1}(p)$, for $p \in Q_1 \setminus Q'_{k+1}$, and $\tilde{\Phi}_k(p) = \Phi_{k+1}^{j_1,...,j_k}(p)$, for $p \in Q_{k+1}^{j_1,...,j_k}$. Then, it is not difficult to see, that $\tilde{\Phi}_k$ is a Lipschitz L-mapping and is a lengthwise folding on the boundaries of $Q_{k+1}^{j_1,...,j_k}$ and on the sets $Q_{k+2}^{j_1,...,j_{k+1}} = \varphi_{j_1,...,j_k,j_{k+1}}(Q_1)$.

Since $\operatorname{diam}(Q_{k+1}^{j_1,...,j_k}) \to 0$ as $k \to \infty$ it is easy to see that the sequence $\tilde{\Phi}_k$ uniformly converges on Q_1 to a Lipschitz L-mapping which we will denote by Φ.

Put $Q_\infty = \bigcap_{k=1}^{\infty} Q'_{k+1}$. Since $\Phi(Q_1)$ contains a little more than half of each $Q_{k+1}^{j_1,...,j_k}$, examining our self-similar construction and using the definition of Φ on $Q_{k+1}^{j_1,...,j_k}$, one can easily see that points of Q_∞ belong to $\partial^*(\Phi(Q_1))$.

Since $Q_\infty = \bigcup_{j=1}^{48} \varphi_j(Q_\infty)$, taking an open set V, which is slightly larger than Q_1, the open set condition also holds. Using that the φ_j's have contraction ratio $\sqrt{2}/60$ we infer that

$$\dim_H(\partial^*(\Phi(Q_1))) \geq \dim_H(Q_\infty) = \frac{\log(\frac{1}{48})}{\log(\frac{\sqrt{2}}{60})} > 1.03.$$

Finally, it is not difficult to see that Φ is two-to-one on Q_1 everywhere but the points of the "folding edge", where it is one-to-one. This completes our proof.

Remark. By modifying some constants in our construction and by using steeper sawtooth functions (for the internal folding edge in the definition of Φ_1) one can get the Hausdorff dimension of $\partial^*(\Phi(Q_1))$ arbitrarily close to 2.

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