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ABSOLUTE FIXED POINT SETS FOR MULTI-VALUED MAPS

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Abstract. The notion of a multi-valued absolute fixed point set (MAFS) will be defined and characterized in the setting of set-valued maps with images containing multiple components.

1. Introduction

The notion of an absolute fixed point set, in the setting of single-valued maps, was first studied in [1], [2], and [3]; in [3], Martin characterized the class of finite dimensional absolute fixed point sets as the class of finite dimensional absolute retracts. The notion of an absolute fixed point set was extended in [4] to the context of continuum-valued maps and a completely inherent characterization for the notion was obtained. In this paper, we relax the requirement that the maps be continuum-valued and we obtain a characterization for the resulting notion.

A *compactum* is a nonempty compact metric space, and a *continuum* is a connected compactum. If Z is a compactum, then $C(Z)$ denotes the space of all subcontinua of Z with the Hausdorff metric and 2^Z denotes the space of all subcompacta of Z with the Hausdorff metric [5]. A *map* is a continuous function, and a map into some 2^Z will be referred to as a *multi-valued map*; in particular, a map into some $C(Z)$ will be said to be *continuum-valued*.

A *fixed point of a multi-valued map* F is a point, p, such that $p \in F(p)$; the *fixed point set of* F is the set of all fixed points of F and is denoted by $\mathcal{F}(F)$. For a compactum Z, we use the phrase *multi-valued fixed point set of* Z to refer to a subset, A, of Z such that A is the fixed point set of some multi-valued map $F : Z \to 2^Z$. In case that F is continuum-valued, we call A a *continuum-valued fixed point set of* Z.

The notion that was studied in [4] is defined as follows. A *multi-valued absolute fixed point set* (MAFS) is a compactum, A, such that whenever A is embedded as a subspace, A', of any compactum, Z, then A' is a continuum-valued fixed point set of Z. It was shown that a compactum A is a MAFS if and only if A contains only finitely many components, all but at most one of which are locally connected [4, 1.1].

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Let Z be a compactum, and let $n \in \{1, 2, \cdots, \omega\}$ where ω denotes the first infinite cardinal number. We define $C_n(Z)$ for each n as follows:

$$C_n(Z) = \{A \in 2^Z : A \text{ has at most } n \text{ components}\}.$$

In particular, if $A \in C_\omega(Z)$, then A has at most countably many components.

The notion that we study in this paper may now be defined as follows. An n-MAFS is a compactum, A, such that whenever A is embedded as a subspace, A', of any compactum, Z, then there is some map $F : Z \to C_n(Z)$ for which $\mathcal{F}(F) = A'$.

We now state the characterizations that we will prove.

1.1 Theorem. *Let A be a compactum, and let $n \in \{1, 2, \dots\}$. Then, A is an n-MAFS if and only if A has only finitely many components, all but at most n of which are locally connected.*

1.2 Theorem. *Let A be a compactum. Then, A is an ω-MAFS if and only if A has at most countably many components.*

We remark that the notions of MAFS and 1-MAFS are identical, and that the characterization given in 1.1 for every finite n is analogous to that given for the class of MAFS. Therefore, it seems natural to consider an inductive argument for 1.1. However, such a proof seems to be elusive.

2. Proof of Theorem 1.1

The following lemmas will be used several times throughout this paper; the proof of 2.1 is similar that given for 1.43 in [5] and will be omitted.

2.1 Lemma. *Let Z be a compactum and let $\Lambda \subseteq 2^Z$. If Λ is connected and $\Lambda \cap C_n(Z) \neq \varnothing$, then $\bigcup \Lambda \in C_n(Z)$, where $n \in \{1, 2, \cdots, \omega\}$.*

The proof of the following lemma is similar to that given for 3.1 in [4] and will be omitted.

2.2 Lemma. *Every retract of an n-MAFS is an n-MAFS for $n \in \{1, 2, \cdots, \omega\}$.*

The result in 2.3 will be used in 2.4 to provide a necessary condition for a compactum to be an n-MAFS for any finite n. A second necessary condition will be given in 2.5.

2.3 Proposition. *A simple sequence is not an n-MAFS for any $n \in \{1, 2, \cdots\}$.*

Proof. Let $J = \{(0, 0, z) \in \mathbf{R}^3 : 0 \leq z \leq 2\}$ and $J' = \{(0, 0, z) \in \mathbf{R}^3 : 0 \leq z \leq 1\}$. Define $W_0 = \{(x, 0, 1 + \sin \frac{1}{x}) : 0 < x \leq 1\}$. For each $i = 1, 2, \cdots$, define W_i to be a $\sin \frac{1}{x}$ curve in $\mathbf{R}^3$ whose height near the z-axis is half its height elsewhere, and arrange these curves around the z-axis in such a way that they converge to W_0. Specifically, define W_i to be an isometric copy of

$$\Big\{\big(x, \frac{1}{2}(1 + \sin \frac{1}{x})\big) : 0 < x \leq \frac{2}{(4i+3)\pi}\Big\} \cup \Big\{\big(x, 1 + \sin \frac{1}{x}\big) : \frac{2}{(4i+3)\pi} \leq x \leq 1\Big\}$$

in $\mathbf{R}^3$ such that

(i) $W_i \cap W_j = \varnothing$ whenever $i \neq j$ $(i, j \in \{0, 1, 2, \cdots\})$,
(ii) $\overline{W_i} - W_i = J'$ $(i \in \{1, 2, \cdots\})$,
(iii) $\lim W_i = W_0$.

Define $Z = J \cup \bigcup_{i=0}^{\infty} W_i$. For convenience, we will refer to a point $(0, 0, a)$ as a, and we will use interval notation in the natural way to denote connected subsets of J; in particular, 2 denotes $(0, 0, 2)$, and $[0, 2] = J$.

Let p_i denote the point of W_i that corresponds to $\frac{2}{(4i-1)\pi}$ for each $i = 1, 2, \cdots$; observe that there is a subset of W_i with height 2 between p_i and the z-axis. Set $A = \{0\} \cup \{p_1, p_2, \cdots\}$. Then A is a simple sequence in Z with limit point 0. Let $F : Z \to C_n(Z)$ be any map from Z to $C_n(Z)$. We will show that A cannot be the fixed point set of F.

For the purpose of contradiction, suppose that $\mathcal{F}(F) = A$. Observe that if there was some $x \in (0, 2]$ for which some component of $F(x)$ was contained above x in J, then the continuity of F would imply that some point above x is fixed by F. However, this is impossible by the assumption that $\mathcal{F}(F) \cap J = \{0\}$. It follows that

(1) no component of $F(x)$ lies above x in J for all $x \in (0, 2]$.

Now, suppose that

$$J \subseteq F(0). \tag{$*$}$$

Then, by the continuity of F, there is some $\epsilon \in (0, 1)$ for which

(2) some subset of $F(x)$ lies near J for each $x \in [0, \epsilon]$.

Note also that since $(0, \epsilon) \cap A = \varnothing$, we have that

(3) $J' \not\subseteq F(x)$ for any $x \in (0, \epsilon)$.

Choose any $x \in (0, \epsilon)$. By (1) and (2), at least one component of $F(x)$ lies in $\bigcup_{i=0}^{\infty} W_i$. Also, since $F(x) \in C_n(Z)$, (3) gives that there are only $k \leq n$ many members of $\{W_i : i = 0, 1, 2, \cdots\}$ which meet $F(x)$, say $W_{i_1}, W_{i_2}, \cdots, W_{i_k}$. Let y be any other member of $(0, \epsilon)$. Using the continuity of F, we see that if $F(y) \cap W_j \neq \varnothing$ for some $j \notin \{i_1, \cdots, i_k\}$, then $J' \subseteq F(z)$ for some $z \in [x, y]$, contradicting (3). It follows that $F(x) \subseteq J' \cup W_{i_1} \cup \cdots \cup W_{i_k}$ for every $x \in (0, \epsilon)$. Therefore, we have by continuity that $F(0) \subseteq J' \cup \overline{W_{i_1}} \cup \cdots \cup \overline{W_{i_k}}$. So, from $(*)$ we conclude that $i_j = 0$ for some $j \in \{1, \cdots, k\}$. Thus, (3) gives that for each $x \in (0, \epsilon)$, some component of $F(x)$ is properly contained in W_0. It follows from the continuity of F that there is some $x_0 \in W_0$ that lies between J and some component of $F(x_0)$ in W_0; we conclude that W_0 contains a fixed point of F, contrary to our supposition that $\mathcal{F}(F) = A$. Therefore, $(*)$ cannot occur.

Suppose on the other hand that

$$J \not\subseteq F(0). \tag{$**$}$$

Observe that if $2 \in F(0)$, then $(**)$ implies that some component of $F(0)$ lies strictly above 0 in J; however, it would then follow by continuity that some component of $F(x)$ lies strictly above x for some $x > 0$. Since this would contradict (1), we must have that $2 \notin F(x)$ for every $x \in J$. Therefore, the continuity of F gives that each $x \in J$ is contained in an open subset, U_x, of Z for which $2 \notin F(y)$ for every $y \in U_x$. Thus, $U = \bigcup_{x \in J} U_x$ is an open subset of Z containing J for which

(4) $2 \notin F(y)$ for every $y \in U$.

Observe that U contains a subcontinuum, Y, such that $J \subseteq Y$ and $p_i \in Y$ for all i sufficiently large. Since F is continuous, we have that $F(Y)$ is connected; thus, $\bigcup F(Y)$ belongs to $C_n(Z)$ by 2.1. Furthermore, our assumption that $\mathcal{F}(F) = A$ implies that $\bigcup F(Y)$ contains p_i for i sufficiently large. Thus, since $\bigcup F(Y)$ contains at most $n < \infty$ components, some component, K, of $\bigcup F(Y)$ must contain p_i for infinitely many i. Since K is a closed connected subset of $\bigcup F(Y)$ that contains infinitely many members of $\{p_i : i = 1, 2, \cdots\}$, it follows from our choice of p_i that $[0, 2] \subseteq K$, contrary to (4). Hence, $(**)$ cannot occur. Since we can have neither $(*)$ nor $(**)$, we must conclude that our supposition that $\mathcal{F}(F) = A$ is false. So, A cannot be the fixed point set of any $F : Z \to C_n(Z)$. This proves 2.3.

2.4 Corollary. *Every n-MAFS contains at most finitely many components for each $n \in \{1, 2, \cdots\}$.*

Proof. Let $n \in \{1, 2, \cdots\}$. If A is a compactum with infinitely many components, then A contains a simple sequence as a retract [4, 3.2]. It follows immediately from 2.2 and 2.3 that A is not an n-MAFS.

2.5 Proposition. *Let $n \in \{1, 2, \cdots\}$. If A is an n-MAFS, then all but at most n components of A must be locally connected.*

Proof. By 2.2 and 2.4, it suffices to show that

$(*)$ if A is the disjoint union of $n + 1$ nonlocally connected continua, then A is not an n-MAFS.

Let $A_1, \cdots, A_{n+1}$ denote the components of A. Since A_1 is not locally connected at some $a_\infty \in A_1$, there exists a neighborhood, V, of a_∞, and points $a_i \in V$, $i = 1, 2, \cdots$, for which $a_\infty = \lim a_i$ and such that

(1) no two of $a_\infty, a_1, a_2, \cdots$ lie in the same component of V,
(2) $C_\infty = \lim C_i$, where C_i denotes the component of a_i in V for each $i \in \{1, 2, \cdots, \infty\}$.

Let $S = \{(x, \sin \frac{1}{x}) : 0 < x \leq 1\}$. For each $i \in \{1, 2, \cdots\}$ and each $j \in \{1, 2, \cdots, i\}$, let $f_{ij} : S \to \mathbf{R}^3$ be an embedding for which

(3) $\overline{f_{ij}(S)} - f_{ij}(S) = \{0\} \times \{0\} \times [0, \frac{1}{j}] = I_j$,
(4) $f_{ij}(S) \subset \{(x, y) : x^2 + y^2 \leq \frac{1}{k}\} \times I_j$, where $k = \frac{i(i-1)}{2} + j$,
(5) $f_{ij}(S) \cap f_{i'j'}(S) \neq \varnothing$ if and only if $i = i'$ and $j = j'$.

In other words, $\{f_{ij}(S)\}$ is a sequence of $\sin \frac{1}{x}$ curves with decreasing widths and alternating heights. The first term, $f_{11}(S)$, fits inside the tube centered on the z-axis with radius 1, while the fifth term, $f_{32}(S)$ $(5 = \frac{3(3-1)}{2} + 2)$, fits inside the tube centered on the z-axis with radius $\frac{1}{5}$; also, $f_{11}(S)$ has height 1, and the heights of $f_{31}(S)$, $f_{32}(S)$, and $f_{33}(S)$ have heights 1, $\frac{1}{2}$, and $\frac{1}{3}$ respectively. Define $p = (0, 0, 0)$ and $q_j = (0, 0, \frac{1}{j})$ for each $j \in \{1, 2, \cdots\}$. For each $i \in \{1, 2, \cdots\}$ and each $j \in \{1, \cdots, i\}$, let $p_{ij} = p_k$ be a point of $f_{ij}(S)$ such that $p = \lim p_i$, where $k = \frac{i(i-1)}{2} + j$. Define $Y_1 = \bigcup_{\substack{i<\infty \\ j \leq i}} \overline{f_{ij}(S)}$. Without loss of generality, we may assume that A_1 and Y_1 are disjoint. Let $h_i : Y_1 \to \mathbf{R}^3$ be an embedding for each $i = 1, 2, \cdots, n + 1$ such that $h_1(Y_1) = Y_1$, $h_i(Y_1) \cap h_j(Y_1) = \varnothing$ whenever $i \neq j$, and $h_i(Y_1) \cap A = \varnothing$ for all $i = 1, 2, \cdots, n + 1$. Set $Y_i = h_i(Y_1)$ for each $i = 1, 2, \cdots, n+1$. Form Z_1 from A_1 and Y_1 by identifying a_∞ with p, and a_k with p_k for each $k = 1, 2, \cdots$ and define Z_i from A_i and Y_i in a similar manner. Let T be a simple $(n + 1)$-od [6, 9.8] with vertex t and endpoints $t_1, \cdots t_{n+1}$ such that $T \cap \bigcup_{i=1}^{n+1} Z_i = \varnothing$. Define Z to be the continuum obtained by identifying each t_i with the image of $h_i(p) \in Z_i$. Then Z is a continuum that contains a homeomorphic copy of A. We will use A, V, C_i, I_i, Y_i, and T to denote the natural copies of these spaces in Z.

Suppose that there is some map $F : Z \to C_n(Z)$ with $\mathcal{F}(F) = A$. Note that since $F(t)$ has at most n components, we may assume, without loss of generality, that $F(t)$ shares no point in common with the component of Z_1 in $Z - \{t\}$. It follows from the continuity of F, and the supposition that $\mathcal{F}(F) = A$, that

(6) $F(p) \cap Z_1 = \{p\}$.

By (6) and the continuity of F, there is some open set U of Z with $p \in U$ for which

(7) $Z_1 \cap F(x) \subseteq V \cup Y_1$ for each $x \in U$.

It is clear from (3) that U contains I_{j_0} for some $j_0 \in \{1, 2, \cdots\}$; indeed, it is clear from (4) that U contains $\overline{f_{ij_0}(S)}$ for all sufficiently large i. Note that from (6), the continuity of F, and the supposition that $\mathcal{F}(F) = A$, we have that $q_{j_0} \notin F(x)$ for any $x \in I_{j_0}$. Thus, for each $x \in I_{j_0}$, there is an open set, $V_x \subseteq U$, such that $q_{j_0} \notin F(y)$ for every $y \in V_x$. So, $U' = \bigcup_{x \in I_{j_0}} V_x$ is an open subset of U for which

(8) $q_{j_0} \notin F(y)$ for every $y \in U'$.

Let E be a subcontinuum of U' that contains p_{ij_0} for all sufficiently large i. Then $F(E)$ is a subcontinuum of $C_n(Z)$, and so $\bigcup F(E)$ belongs to $C_n(Z)$ by 2.1. Since each p_{ij_0} belongs to $\mathcal{F}(F)$, it follows that some component, K, of $\bigcup F(E)$ must contain infinitely many members of $\{p_{ij_0} : i = 1, 2, \cdots\}$. Hence, K must also contain p. Observe that $K \subseteq V \cup Y_1$ by (7). So by (1), $\overline{K - V}$ is a subcontinuum of Y_1 that contains p and infinitely many members of $\{p_{ij_0} : i = 1, 2, \cdots\}$. But, by (5), any subcontinuum of Y_1 that contains both p and some p_{ij_0} must also contain I_{j_0}. Thus, $q_{j_0} \in K$, contrary to (8). So, we must conclude that our supposition that $\mathcal{F}(F) = A$ is false. This shows $(*)$, as required. Thus, we have proven 2.5.

The following proposition asserts that the properties described in 2.4 and 2.5 are, together, sufficient for a compactum to be an n-MAFS.

2.6 Proposition. *Let Z be a compactum, and let $n \in \{1, 2, \cdots\}$. If A is a compactum in Z that contains only finitely many components, all but at most n of which are locally connected, then $A = \mathcal{F}(F)$ for some map $F : Z \to C_n(Z)$.*

Proof. Let $A = A_1 \cup A_2 \cup \cdots \cup A_m$, where A_i is a component of A for each $i \in \{1, \cdots, m\}$. Note that if $m \leq n$, then $F : Z \to C_n(Z)$, given by $F(z) = A$ for every $x \in Z$, is a map with $\mathcal{F}(F) = A$. So we assume that $m > n$. Without loss of generality, we also assume that $A_{n+1}, \cdots, A_m$ are locally connected. It follows from the characterization of the class of MAFS [4, 1.1] that there is a map $G : Z \to C(Z)$ with $\mathcal{F}(G) = A_n \cup A_{n+1} \cup \cdots \cup A_m$. Define $F : Z \to C_n(Z)$ by

$$F(x) = G(x) \cup A_1 \cup \cdots \cup A_{n-1}$$

for each $x \in Z$. Then F is a map with $\mathcal{F}(F) = A$. This proves 2.6.

The theorem in 1.1 has now been proved by 2.4, 2.5, and 2.6.

3. Proof of Theorem 1.2

3.1 Proposition. *Every compactum with at most countably many components is an ω-MAFS.*

Proof. Let Z be a compactum, and let A be a compact subset of Z with at most countably many components. Then $F : Z \to C_\omega(Z)$ defined by $F(x) = A$ for all $x \in Z$ is a map with $\mathcal{F}(F) = A$. This proves 3.1.

The rest of the work in this section is directed toward demonstrating that having only countably many components is also a necessary condition for a compactum to be an ω-MAFS.

3.2 Lemma. *An ω-MAFS cannot be both totally disconnected and uncountable.*

Proof. Let A be an uncountable, totally disconnected compactum. Without loss of generality, we may assume that A is a subset of the circle in $\mathbf{R}^3$ given by

$$\{(x, y, 1 + \sin 1) : x^2 + y^2 = 1\}.$$

Let $U_1 \supset U_2 \supset \cdots$ be a strictly decreasing sequence of uncountable, simultaneously open and closed subsets of A such that $U_1 = A$ and $\bigcap_{i=1}^{\infty} U_i = \{p\}$ for some $p \in A$. Define J, J' and W_i ($i \in \{1, 2, \cdots\}$) as we did in the proof of 2.3. Now, for each $a \in A - \{p\}$, observe that there is precisely one $i_a \in \{1, 2, \cdots\}$ for which $a \in U_{i_a} - U_{i_a+1}$; let $h_a : W_{i_a} \to \mathbf{R}^3$ be the isometric embedding for which

$$h_a\big((1, 1 + \sin 1)\big) = a,$$
$$\overline{h_a(W_{i_a})} - h_a(W_{i_a}) = J'.$$

Let $S = \{(x, 1+\sin\frac{1}{x} : 0 < x \leq 1\}$, and let $h_p : S \to \mathbf{R}^3$ be the isometric embedding for which

$$h_p\big((1, 1 + \sin 1)\big) = p,$$
$$\overline{h_p(S)} - h_p(S) = J.$$

Let $Z = \overline{h_p(S)} \cup \bigcup_{a \in A - \{p\}} h_a(W_{i_a})$. Now, for each $a \in A$, define

$$x_a = \begin{cases} h_a\left(\frac{2}{(4i_a - 1)\pi}\right), & \text{if } a \neq p, \\ (0, 0, 0), & \text{if } a = p. \end{cases}$$

Observe that the map given by $a \mapsto x_a$ is an embedding of A into Z. Set $A' = \{x_a : a \in A\}$, and suppose that there exists some $F : Z \to C_\omega(Z)$ with $\mathcal{F}(F) = A'$. This supposition leads to a contradiction by an argument similar to the one given for 2.3. This proves 3.2.

The following technical lemma will facilitate the proof of 3.4.

3.3 Lemma. *Let Z be a compactum, and let V be an open subset of Z which meets uncountably many components of Z. Then, for each $n \in \{1, 2, \cdots\}$ there exist points $p_1, \cdots, p_n$ of V which lie in n distinct components of Z, and for which any open set containing p_i meets uncountably many components of Z for each $i = 1, \cdots, n$.*

Proof. Suppose that the conclusion fails for $n = 1$; that is, suppose that

(∗) for every $p \in V$, there is some open subset of V that contains p and meets at most countably many components of Z.

Let $\mathcal{C}$ denote the collection of all components of Z that meet V. For each $C \in \mathcal{C}$, let $z_C \in C \cap V$; define $A = \{z_C : C \in \mathcal{C}\}$. Then, by hypothesis, we have that

(1) A is uncountable.

Now, let $\{V_i\}_{i=1}^{\infty}$ be a sequence of open subsets of V for which $cl_Z(V_i) \subseteq V$ and for which $\bigcup_{i=1}^{\infty} V_i = V$. Clearly we have that

(2) $\bigcup_{i=1}^{\infty}(cl_Z(V_i) \cap A) = A$.

Therefore, since the union of countably many countable sets is countable, it follows immediately from (1) and (2) that $cl_Z(V_k) \cap A$ is uncountable for some $k \in \{1, 2, \cdots\}$. Setting $E = cl_Z(V_k)$ shows that

(3) E is a closed subset of Z in V that meets uncountably many components of Z.

By $(*)$ and the compactness of E, there exist finitely many open subsets, $U_1, \cdots, U_m$, of V for which $E \subseteq \bigcup_{i=1}^m U_i$ and such that each U_i meets at most countably many components of Z. But this implies that E meets at most countably many components of Z, contrary to (3). Hence, the supposition in $(*)$ is false. Therefore, the lemma holds for $n = 1$.

Now suppose that the lemma holds for $n = j$. Let $p_1, \cdots, p_j$ be points of V that satisfy the conclusion of the lemma, and let K_i denote the component of p_i in Z for each $i \in \{1, \cdots, j\}$. Set $K = \bigcup_{i=1}^j K_i$. Then $V - K$ is an open subset of Z that meets uncountably many components of Z. So, by the case for $n = 1$, there is some point $p_{j+1} \in V - K$ for which any open set containing p_{j+1} meets uncountably many components of Z. This completes the proof of 3.3.

The following notation will facilitate the proof of our next result. Let $\{0,1\}^{<\omega}$ denote the set of all finite sequences with range in $\{0,1\}$, that is,

$$\{0,1\}^{<\omega} = \{\langle a_1, \cdots, a_n\rangle : n = 1, 2, \cdots; a_i = 0, 1; \quad i = 1, \cdots, n\} \cup \{\varnothing\}.$$

For any $s \in \{0,1\}^{<\omega}$, let $|s|$ denote the length of s. Let $s0$ and $s1$ denote those elements of $\{0,1\}^{<\omega}$ that have length $|s| + 1$, that agree with s in the first $|s|$ positions, and that have 0 and 1, respectively, in the final position. If $|s| = 1$, set $s^- = \varnothing$; otherwise, let s^- denote that element of $\{0,1\}^{<\omega}$ of length $|s| - 1$ that agrees with s in the first $|s| - 1$ positions. Finally, let $\{0,1\}^\omega$ denote the set of all countably infinite sequences with range in $\{0,1\}$.

3.4 Lemma. *Let Z be a compactum with uncountably many components. Then Z has a retract which is both uncountable and totally disconnected.*

Proof. Using 3.3, there exist distinct components, $K_{\langle 0\rangle}$ and $K_{\langle 1\rangle}$, of Z, such that for some $p_{\langle i\rangle} \in K_{\langle i\rangle}$, every open set about $p_{\langle i\rangle}$ meets uncountable many components of Z. Using the fact that every component of Z is also a quasicomponent of Z [6, 5.18], let $U_{\langle i\rangle}$ be a simultaneously closed and open (clopen) subset of Z for each $i = 0, 1$ such that $K_{\langle i\rangle} \subseteq U_{\langle i\rangle}$, $U_{\langle 0\rangle} \cap U_{\langle 1\rangle} = \varnothing$, and $U_{\langle 0\rangle} \cup U_{\langle 1\rangle} \neq Z$. Choose any point, $p_\varnothing$, in $Z - (U_{\langle 0\rangle} \cup U_{\langle 1\rangle})$, and define $f_\varnothing : Z - (U_{\langle 0\rangle} \cup U_{\langle 1\rangle}) \to Z$ by

$$f_\varnothing(x) = p_\varnothing \text{ for all } x \in Z - (U_{\langle 0\rangle} \cup U_{\langle 1\rangle}).$$

We will now construct U_s, V_s, K_s, p_s, and f_s for each $s \in \{0,1\}^{<\omega}$ by induction on $\{0,1\}^{<\omega}$. Set $V_\varnothing = U_\varnothing = Z$, and let $s \in \{0,1\}^{<\omega}$. Let V_s be an open set about p_s such that

(1) the diameter of V_s is $\leq 2^{-|s|}$,
(2) $V_s \subseteq U_s \cap V_{s^-}$.

By applying 3.3 to U_s, we can find distinct components, K_{s0} and K_{s1}, of U_s (hence, of Z), for which

(3) $K_{si} \neq K_s$ for $i = 0, 1$,
(4) there is some $p_{si} \in K_{si} \cap V_s$ such that every open set about p_{si} meets uncountably many components of Z.

Let U_{si} be a clopen subset of Z for each $i = 0, 1$ such that

(5) $U_{si} \subseteq U_s$,
(6) $K_{si} \subseteq U_{si}$,
(7) $U_{s0} \cap U_{s1} = \varnothing$,
(8) $U_{si} \cap K_s = \varnothing$.

Then, define $f_s : U_s - (U_{s0} \cup U_{s1}) \to Z$ by

$$f_s(x) = p_s \text{ for all } x \in U_s - (U_{s0} \cup U_{s1}).$$

Observe that since each U_{si} is open, (4) and (6) give that U_{si} contains uncountably many components of Z for each $i = 0, 1$. This completes our construction. Now, for each $r \in \{0,1\}^\omega$, let r_n denote the unique element of $\{0,1\}^{<\omega}$ of length n for which $r_n(k) = r(k)$ for all $k \in \{1, \cdots, n\}$. Let $Z' = \bigcup \{\bigcap_{n=1}^\infty U_{r_n} : r \in \{0,1\}^\omega\}$. It is easy to verify that

(9) $x \in Z - Z'$ if and only if $x \in U_s - (U_{s0} \cup U_{s1})$ for some $s \in \{0,1\}^{<\omega}$.

Since $U_s - (U_{s0} \cup U_{s1})$ is open for every $s \in \{0,1\}^{<\omega}$, (9) gives that

(10) Z' is a closed subset of Z.

Define $f' : Z - Z' \to Z$ by

$$f'(x) = f_s(x) \text{ if } x \in U_s - (U_{s0} \cup U_{s1}), \qquad s \in \{0,1\}^{<\omega}.$$

Note that f' is well defined by (5), (7), and (9). Furthermore, since f_s is constant for each $s \in \{0,1\}^{<\omega}$ and each $U_s - (U_{s0} \cup U_{s1})$ is open in Z, it is clear that

(11) f' is continuous.

For each $r \in \{0,1\}^\omega$, use (1) and (2) to define $p_r = \bigcap_{n=1}^\infty \overline{V_{r_n}}$. Then, define $f : Z \to Z$ by

$$f(x) = \begin{cases} f'(x) & \text{if } x \in Z - Z', \\ p_r, & \text{if } x \in \bigcap_{n=1}^\infty U_{r_n} \quad \text{for some } n. \end{cases}$$

Again, f is well defined by (5), (7), and (9).

By (10) and (11), we have that f is continuous at each point of $Z - Z'$. So, to see that f is continuous on Z, it is enough to show that f is continuous at each $x \in Z'$. Let $x_0 \in Z'$, and let $\{x_i\}_{i=1}^\infty$ be a sequence in Z with $\lim x_i = x_0$. By the definitions of Z' and f, there is some $r \in \{0,1\}^\omega$ for which $f(x_0) = p_r$. Let V be an open subset of Z about $f(z_0) = p_r$. Then, by (1), (2), and the definition of p_r, V contains V_{r_N} for some $N \in \{1, 2, \cdots\}$. By the definition of f, we have that $x_0 \in U_{r_{N+1}}$; hence, $x_i \in U_{r_{N+1}}$ for all i sufficiently large. But, by (2), (4), and the definitions of f_s, p_r, and f, it follows that $f(U_{r_{N+1}}) \subseteq V_{r_N}$. So $f(x_i) \in V$ for all i sufficiently large; therefore, f is continuous at x_0, as required.

Finally, observe that by (2), (3), (4), and (8), $f(Z) \cap K$ contains at most one point from each component, K, of Z; hence, we have that $f(Z)$ is totally disconnected. Furthermore, it is clear from our construction that $f(Z)$ is a retract of Z with uncountably many elements. This proves 3.4.

3.5 Proposition. *Every ω-MAFS contains at most countably many components.*

Proof. This follows immediately from 2.2, 3.2 and 3.4.

The theorem in 1.2 has now been proved by 3.1 and 3.5.

3.6 Remark. Let $\mathfrak{c}$ denote the cardinality of $\mathbf{R}$. By defining a $\mathfrak{c}$-MAFS in the natural way, it is easily seen that every compactum is a $\mathfrak{c}$-MAFS since every compactum has at most $\mathfrak{c}$ components.

3.7 Problem. Characterize the class of n-MAFS, $n \in \{1, 2, \cdots, \omega\}$, in the setting of compact Hausdorff spaces.

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