

POLISHNESS OF THE WIJSMAN TOPOLOGY REVISITED

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ABSTRACT. Let X be a completely metrizable space. Then the space of nonempty closed subsets of X endowed with the Wijsman topology is α -favorable in the strong Choquet game. As a consequence, a short proof of the Beer-Costantini Theorem on Polishness of the Wijsman topology is given.

Denote by $CL(X)$ the nonempty closed subsets of the metric space (X, d) . The *Wijsman topology* τ_W on $CL(X)$ is the weak topology generated by $\{d(x, \cdot) : x \in X\}$, where the distance functional $d(x, A) = \inf\{d(x, a) : a \in A\}$ is viewed as a function of the set argument $A \in CL(X)$. It was shown by G. Beer in [Be1] (see also [Be2]) that, given a separable complete metric space (X, d) , the corresponding hyperspace $(CL(X), \tau_W)$ is Polish. Since even uniformly equivalent metrics on X may give rise to different Wijsman topologies (cf. [LL], [CLZ]), it required a separate argument to show that Polish base spaces always generate Polish Wijsman topologies. This was established by C. Costantini in [Co1]. Thus, combining these results and the fact that X embeds in $(CL(X), \tau_W)$ as a closed subspace, one gets

Theorem 1. *The space $(CL(X), \tau_W)$ is Polish if and only if (X, d) is Polish.*

It is the purpose of this note to present a short proof of the above theorem based on the so-called *strong Choquet game* (cf. [Ch] or [Ke]). In the game, denoted by Γ , two players α and β take turns in choosing objects in the topological space X with an open base \mathcal{B} : β starts by picking (x_0, V_0) from $\mathcal{E}(X, \mathcal{B}) = \{(x, V) \in X \times \mathcal{B} : x \in V\}$ and α responds by picking $U_0 \in \mathcal{B}$ with $x_0 \in U_0 \subset V_0$. The next choice of β is some couple $(x_1, V_1) \in \mathcal{E}(X, \mathcal{B})$ with $V_1 \subset U_0$, and again α picks U_1 with $x_1 \in U_1 \subset V_1$, etc. Player α wins the run $(x_0, V_0), U_0, \dots, (x_n, V_n), U_n, \dots$ provided $\bigcap_n U_n = \bigcap_n V_n \neq \emptyset$; otherwise β wins. A *winning tactic* (abbr. w.t.) for α (cf. [Ch]) is a function $\sigma : \mathcal{E}(X, \mathcal{B}) \rightarrow \mathcal{B}$ such that α wins every run of Γ compatible with σ , i.e. such that $U_n = \sigma(x_n, V_n)$ for all n . The game Γ is α -favorable if α possesses a winning tactic; in this case X is called a *strong Choquet space* (see [Ke]). The Choquet Theorem (see [Ch], Theorem 8.7 or [Ke], Theorem 8.17) claims that a metrizable space is completely metrizable if and only if it is a strong Choquet space.

In the sequel ω will stand for the nonnegative integers, $B(x, \varepsilon)$ for the closed ball about $x \in X$ of radius ε in the metric space (X, d) , and B^c for the complement of

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$B \subset X$. For $U \subset X$ put $U^- = \{A \in CL(X) : A \cap U \neq \emptyset\}$. It is routine to show that a base \mathcal{B}_W for the Wijsman topology consists of the sets of the form

$$(V_0, \dots, V_k)_D = \bigcap_{i \leq k} V_i^- \cap \bigcap_{j \leq m} \{A \in CL(X) : d(x_j, A) > \varepsilon_j\}$$

with $V_i \subset X$ open ($i \leq k$), $x_j \in X$, $\varepsilon_j > 0$ ($j \leq m$), $D = (x_0, \dots, x_m; \varepsilon_0, \dots, \varepsilon_m)$. For every D of this kind, denote $M(D) = \bigcup_{j \leq m} B(x_j, \varepsilon_j)$.

Since metrizable of the Wijsman topology is equivalent to separability of the base space X (see [Be2], Theorem 2.1.5), in order to prove Theorem 1 it suffices by the Choquet Theorem to prove

Theorem 2. *If X is completely metrizable, then $(CL(X), \tau_W)$ is a strong Choquet space.*

Proof. Assume that (X, τ) is completely metrizable and d is a compatible metric on X . Then by the Choquet Theorem we can find a w.t. $\sigma : \mathcal{E}(X, \tau) \rightarrow \tau$ for α . Define a tactic $\sigma_W : \mathcal{E}(CL(X), \mathcal{B}_W) \rightarrow \mathcal{B}_W$ for α as follows: first, for each $V \in \tau$ and $A \in V^-$ fix a point $x_{A,V} \in A \cap V$. Then, given $(A, \mathbf{V}) \in \mathcal{E}(CL(X), \mathcal{B}_W)$ with $\mathbf{V} = (V_0, \dots, V_k)_D$ and $D = (x_0, \dots, x_m; \varepsilon_0, \dots, \varepsilon_m)$, define

$$\sigma_W(A, \mathbf{V}) = (\sigma(x_{A, V_0 \cap M(D)^c}, V_0 \cap M(D)^c), \dots, \sigma(x_{A, V_k \cap M(D)^c}, V_k \cap M(D)^c))_{\tilde{D}}$$

where $\tilde{D} = (x_0, \dots, x_m; \tilde{\varepsilon}_0, \dots, \tilde{\varepsilon}_m)$ with $\tilde{\varepsilon}_j = \frac{\varepsilon_j + d(x_j, A)}{2}$ for all $j \leq m$. Then $A \in \sigma_W(A, \mathbf{U}) \subset \mathbf{U}$. We will show that σ_W is a winning tactic for α .

Indeed, suppose that $(A_0, \mathbf{V}_0), \mathbf{U}_0, \dots, (A_n, \mathbf{V}_n), \mathbf{U}_n, \dots$ is a run of Γ in $CL(X)$ such that $\mathbf{U}_n = \sigma_W(A_n, \mathbf{V}_n)$ for all n . Denote $\mathbf{U}_n = (U_0^n, \dots, U_{l_n}^n)_{B_n}$ and $\mathbf{V}_n = (V_0^n, \dots, V_{k_n}^n)_{D_n}$ for appropriate B_n and D_n . Observe that $\mathbf{V}_{n+1} \subset \mathbf{U}_n$ and $\mathbf{V}_{n+1} \neq \emptyset$ imply that $M(D_{n+1})^c \subset M(B_n)^c$ and for all $s \leq l_n$ there exists $t \leq l_{n+1}$ such that $V_t^{n+1} \cap M(D_{n+1})^c \subset U_s^n \cap M(B_n)^c$. Hence, without loss of generality, assume that $k_{n+1} > l_n = k_n$ and $t = s$. Put $l_{-1} = -1$. Then for all $n \in \omega$ and $l_{n-1} < i \leq l_n$,

$$\begin{aligned} & (x_{A_n, V_i^n \cap M(D_n)^c}, V_i^n \cap M(D_n)^c), U_i^n \cap M(B_n)^c, \dots, \\ & (x_{A_{n+j}, V_i^{n+j} \cap M(D_{n+j})^c}, V_i^{n+j} \cap M(D_{n+j})^c), U_i^{n+j} \cap M(B_{n+j})^c, \dots \end{aligned}$$

is a run of Γ in X compatible with σ , so there exists $a_i \in \bigcap_{j \in \omega} U_i^{n+j} \cap M(B_{n+j})^c$. Denote by A the closure of $\{a_i : i \in \omega\}$ in X . Fix n and i . Then for some $N > n$, $a_i \in U_i^N \cap M(B_N)^c \subset M(B_{n+1})^c = M(\tilde{D}_n)^c$.

Consequently, if $D_n = (x_0, \dots, x_m; \varepsilon_0, \dots, \varepsilon_m)$, then $d(x_j, a_i) \geq \tilde{\varepsilon}_j$ for all $j \leq m$; thus $d(x_j, A) = \inf\{d(x_j, a_i) : i \in \omega\} \geq \tilde{\varepsilon}_j > \varepsilon_j$. It follows now that $A \in \mathbf{V}_n$ for all $n \in \omega$, whence α wins the run. \square

Remark. It is known that metrizable spaces that are not β -favorable in Γ are the hereditarily Baire spaces (see [De]), i.e. spaces, every closed subspace of which is a Baire space. In general, however, even strong Choquet spaces may be non-hereditarily Baire ([De]). It could therefore be of interest to find out if complete metrizable of X implies hereditary Baireness of $(CL(X), \tau_W)$. Note that it will certainly be a Baire space ([Zs]) and it could be non-Čech-complete, as was shown in [Co2].

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