PARTIAL ACTIONS OF GROUPS AND ACTIONS OF INVERSE SEMIGROUPS

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ABSTRACT. Given a group G, we construct, in a canonical way, an inverse semigroup $\mathcal{S}(G)$ associated to G. The actions of $\mathcal{S}(G)$ are shown to be in one-to-one correspondence with the partial actions of G, both in the case of actions on a set, and that of actions as operators on a Hilbert space. In other words, G and $\mathcal{S}(G)$ have the same representation theory.

We show that $\mathcal{S}(G)$ governs the subsemigroup of all closed linear subspaces of a G-graded C^*-algebra, generated by the grading subspaces. In the special case of finite groups, the maximum number of such subspaces is computed.

A "partial" version of the group C^*-algebra of a discrete group is introduced. While the usual group C^*-algebra of finite commutative groups forgets everything but the order of the group, we show that the partial group C^*-algebra of the two commutative groups of order four, namely $Z/4Z$ and $Z/2Z \oplus Z/2Z$, are not isomorphic.

1. INTRODUCTION

A semigroup S is said to be an inverse semigroup provided there exists, for each x in S, a unique element x^* in S such that

(i) $xx^*x = x$,

(ii) $x^*xx^* = x^*$.

According to one of the earliest results on this subject, the Vagner–Preston representation Theorem [7, V.1.10], every inverse semigroup is isomorphic to a subsemigroup of $\mathcal{I}(X)$, the inverse semigroup of partially defined bijective maps on a set X. The interested reader will also find in [7] the basic facts about inverse semigroups.

By a partially defined map on X we mean a map $\phi\colon A \to B$, where A and B are subsets of X. Given such a ϕ we will denote by $\operatorname{dom}(\phi)$ and $\operatorname{ran}(\phi)$ its domain A and range B, respectively. The multiplication rule on $\mathcal{I}(X)$ is given by composition of partial maps in the largest domain where it makes sense, that is, if ϕ and ψ are elements of $\mathcal{I}(X)$, then $\operatorname{dom}(\phi\psi) = \psi^{-1}(\operatorname{ran}(\psi) \cap \operatorname{dom}(\phi))$ and $\operatorname{ran}(\phi\psi) = \phi(\operatorname{ran}(\psi) \cap \operatorname{dom}(\phi))$.

The representation theory of inverse semigroups studies homomorphisms (i.e. semigroup homomorphisms) $\pi\colon S \to \mathcal{I}(X)$ where S is the inverse subgroup under

scrutiny and X is a set. Adopting the point of view of the theory of group actions, we make the following:

1.1. Definition. An *action* of the inverse semigroup S on the set X is a homomorphism $\pi\colon S \to \mathcal{I}(X)$.

In view of the recent interest in the concept of partial actions of groups on C^*-algebras (see [5], [9], [4], [12], [14], [11], [1]), it is interesting to study the more general concept of partial actions of groups on sets, as defined below.

1.2. Definition. Given a group G and a set X, a *partial action* of G on X is a pair

$$\Theta = \big(\{D_t\}_{t \in G}, \{\theta_t\}_{t \in G}\big),$$

where, for each t in G, D_t is a subset of X and $\theta_t\colon D_{t^{-1}} \to D_t$ is a bijective map, satisfying, for all r and s in G

(i) $D_e = X$ and $\theta_e = id_X$, the identity map on X (here, as always, e denotes the identity element of G),

(ii) $\theta_r(D_{r^{-1}} \cap D_s) = D_r \cap D_{rs}$,

(iii) $\theta_r(\theta_s(x)) = \theta_{rs}(x), \quad x \in D_{s^{-1}} \cap D_{s^{-1}r^{-1}}$.

Thus, a partial action of G on X is also a map from G into $\mathcal{I}(X)$, just like the case for actions of inverse semigroups. However, the map $t \in G \mapsto \theta_t \in \mathcal{I}(X)$ does not satisfy $\theta_r\theta_s = \theta_{rs}$ as one might imagine, but rather we have $\theta_r\theta_s \subseteq \theta_{rs}$, that is, θ_{rs} is an extension of $\theta_r\theta_s$.

One of our major goals will be to introduce, for each group G, an inverse semigroup, which we will denote by $\mathcal{S}(G)$. This turns out to be a very natural object to consider, although we have found no account of it in the literature concerning semigroups. The interest in studying $\mathcal{S}(G)$ stems from the fact that the partial actions of G on a set X are in one-to-one correspondence with the homomorphisms from $\mathcal{S}(G)$ into $\mathcal{I}(X)$.

This problem was first considered in Sieben's [14] master thesis, where a partial answer was given in the case of actions on C^*-algebras. The inverse subgroup constructed there, however, is not intrinsically obtained from G, but it depends also on the partial action under consideration.

2. The inverse semigroup associated to a group

Throughout this section G will denote a fixed group. The crucial property of partial actions which will enable us to construct our inverse semigroup is that, although θ_{rs} is only an extension of $\theta_r\theta_s$ (notation as in the previous section), we always have $\theta_r\theta_s\theta_{s^{-1}} = \theta_{rs}\theta_{s^{-1}}$. More will be said about this property later.

2.1. Definition. We let $\mathcal{S}(G)$ denote the universal semigroup defined via generators and relations as follows. To each element t in G we take a generator $[t]$ (from any fixed set having as many elements as G). For every pair of elements t, s in G we consider the relations

(i) $[s^{-1}][s][t] = [s^{-1}][st]$,

(ii) $[s][t][t^{-1}] = [st][t^{-1}]$,

(iii) $[s][e] = [s]$,

(iv) $[e][s] = [s]$.

A rule of thumb to remember the first two relations is that $[s][t]$ can be replaced by $[st]$ only if there is either an $[s^{-1}]$ hanging out on the left hand side, or a $[t^{-1}]$ on the right.

Note that, as a consequence of (i) and (iii), we have that $[t][t^{-1}][t] = [t]$, which is the first hint that $\mathcal{S}(G)$ might be shown to be an inverse semigroup. This will, in fact, be accomplished below. We should also remark that axiom (iv) is a consequence of the previous ones, since we have, for each s in G, that

$$[e][s] = [ss^{-1}][s] = [s][s^{-1}][s] = [s].$$

Therefore (iv) could be removed from the definition without any significant consequence.

By the universal property of semigroups defined via generators and relations we have:

2.2. Proposition. *Given a semigroup S and a map $f\colon G \to S$ satisfying*

(i) $f(s^{-1})f(s)f(t) = f(s^{-1})f(st),$
(ii) $f(s)f(t)f(t^{-1}) = f(st)f(t^{-1}),$
(iii) $f(s)f(e) = f(s),$

there exists a unique homomorphism $\tilde{f}\colon \mathcal{S}(G) \to S$ such that $\tilde{f}([t]) = f(t)$.

The proof is elementary but we would like to point out that, according to the remarks above, the relation $f(e)f(s) = f(s)$, corresponding to (2.1.iv), need not be included as it is a consequence of (i) – (iii).

2.3. Proposition. *There is an involutive anti-automorphism $*\colon \mathcal{S}(G) \to \mathcal{S}(G)$ such that, for every t in G, $[t]^* = [t^{-1}]$.*

Proof. Let $\mathcal{S}(G)^{op}$ be the opposite semigroup, that is $\mathcal{S}(G)^{op}$ coincides with $\mathcal{S}(G)$ in all respects, except that multiplication of the elements α and β in $\mathcal{S}(G)^{op}$ is defined by

$$\alpha \bullet \beta := \beta\alpha$$

where $\beta\alpha$, on the right hand side, corresponds to the usual multiplication in $\mathcal{S}(G)$. Define $f\colon G \to \mathcal{S}(G)$ by $f(t) = [t^{-1}]$. It is easy to see that, for s and t in G we have

(i) $f(s^{-1}) \bullet f(s) \bullet f(t) = f(s^{-1}) \bullet f(st),$
(ii) $f(s) \bullet f(t) \bullet f(t^{-1}) = f(st) \bullet f(t^{-1}),$
(iii) $f(s) \bullet f(e) = f(s).$

So, by (2.2), f extends to a homomorphism $*\colon \mathcal{S}(G) \to \mathcal{S}(G)^{op}$, which, when viewed as a map from $\mathcal{S}(G)$ to itself, rather than into the opposite semigroup, is an anti-automorphism satisfying the desired properties. $\square$

An element ε in a semigroup is said to be idempotent if $\varepsilon^2 = \varepsilon$. Idempotents in $\mathcal{S}(G)$ are studied next.

2.4. Proposition. *For every t in G let $\varepsilon_t = [t][t^{-1}]$. Then, for each t and s in G,*

(i) ε_t *is a self-adjoint idempotent in the sense that $\varepsilon_t^* = \varepsilon_t = \varepsilon_t^2$,*
(ii) $[t]\varepsilon_s = \varepsilon_{ts}[t],$
(iii) ε_t *and ε_s commute.*

Proof. The first statement is trivial. As for (ii), using (i) and (ii) of (2.1), as well as the fact that $[t][t^{-1}][t] = [t]$, we have,

$$[t]\varepsilon_s = [t][s][s^{-1}] = [ts][s^{-1}] = [ts][s^{-1}t^{-1}][ts][s^{-1}] = [ts][s^{-1}t^{-1}][t] = \varepsilon_{ts}[t].$$

Finally, using (ii), we have

$$\varepsilon_t \varepsilon_s = [t][t^{-1}]\varepsilon_s = [t]\varepsilon_{t^{-1}s}[t^{-1}] = \varepsilon_s[t][t^{-1}] = \varepsilon_s \varepsilon_t.$$

□

2.5. Proposition. *Every element α in $\mathcal{S}(G)$ admits a decomposition*

$$\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s],$$

where $n \geq 0$ and $r_1, \ldots, r_n, s$ are elements of G. In addition, one can assume that

(i) $r_i \neq r_j$ *for $i \neq j$,*
(ii) $r_i \neq s$ *and $r_i \neq e$ for all i.*

Proof. Let S be the subset of $\mathcal{S}(G)$ consisting of those α that do admit a decomposition as above. Since $n = 0$ is allowed, we see that each $[s]$ belongs to S. To prove the statement it is then enough to verify that S is a subsemigroup of $\mathcal{S}(G)$, in view of the fact that the $[s]$'s form a generating system for $\mathcal{S}(G)$.

Let $\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s]$. It suffices to demonstrate that $\alpha[t]$ belongs to S, since this will prove S to be a right ideal and hence a subsemigroup. Now, note that

$$[s][t] = [s][s^{-1}][s][t] = [s][s^{-1}][st] = \varepsilon_s[st].$$

So

$$\alpha[t] = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s][t] = \varepsilon_{r_1} \ldots \varepsilon_{r_n}\varepsilon_s[st] \in S.$$

With regard to the last part of the statement, since the idempotents ε_{r_j} commute with each other, it is easy to see how to eliminate repetitions among the ε_r's, if any, as well as any occurrence of ε_e, which is the identity element of $\mathcal{S}(G)$.

Also, if some $r_i = s$, then $\varepsilon_{r_i} = [s][s^{-1}]$ and, again by commutativity of idempotents, we have

$$\alpha = \varepsilon_{r_1} \ldots \widehat{\varepsilon_{r_i}} \ldots \varepsilon_{r_n}[s][s^{-1}][s] = \varepsilon_{r_1} \ldots \widehat{\varepsilon_{r_i}} \ldots \varepsilon_{r_n}[s].$$

So ε_{r_i} can be eliminated. □

2.6. Definition. If α is written as $\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s]$, in such a way that conditions (i) and (ii) of (2.5) are verified, we say that α is in *standard* form.

We now present some more evidence to support our earlier claim (to be proved below) that $\mathcal{S}(G)$ is an inverse semigroup.

2.7. Proposition. *For each α in $\mathcal{S}(G)$ one has $\alpha\alpha^*\alpha = \alpha$ and $\alpha^*\alpha\alpha^* = \alpha^*$.*

Proof. Let α be represented as $\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s]$. We then have

$$\alpha\alpha^*\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s][s^{-1}]\varepsilon_{r_n} \ldots \varepsilon_{r_1}\varepsilon_{r_1} \ldots \varepsilon_{r_n}[s] = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s][s^{-1}][s] = \alpha.$$

The second statement follows from the first one and our knowledge that $*$ is an anti-automorphism of $\mathcal{S}(G)$. □

The only task remaining, before we can assert that $\mathcal{S}(G)$ is in fact an inverse semigroup, is to show that α^* is the unique element satisfying the identities in the statement of our previous result. However, uniqueness results in algebraic structures originated from generators and relations are difficult to establish, unless we can find a separating family of representations. This is the purpose of our next section.

3. Representations of $\mathcal{S}(G)$

In this section we shall use the term "representation" in a very loose way, meaning any homomorphism of $\mathcal{S}(G)$ into a semigroup. These representations will often be obtained with the aid of (2.2).

The most obvious representation of $\mathcal{S}(G)$ one can think of is the map

$$\partial\colon \mathcal{S}(G) \to G$$

given by $\partial([t]) = t$. We shall refer to $\partial(\alpha)$ as the *degree* of α. Clearly $\partial(\varepsilon_{r_1} \ldots \varepsilon_{r_n}[s])$ $= s$.

We next discuss a more subtle representation of $\mathcal{S}(G)$. Let $\mathcal{P}_e(G)$ be the set of all finite subsets of G which contain the unit element e. Thus G and $\{e\}$ are, respectively, the biggest and the smallest elements of $\mathcal{P}_e(G)$. The representation we are about to consider is a homomorphism from $\mathcal{S}(G)$ into the semigroup $\mathcal{F}\big(\mathcal{P}_e(G)\big)$ consisting of all functions from $\mathcal{P}_e(G)$ into itself, under the composition rule.

Thus, for each $t \in G$ let us denote by ϕ_t the map

$$\phi_t\colon \mathcal{P}_e(G) \to \mathcal{P}_e(G)$$

given by $\phi_t(E) = tE \cup \{e\}$, for each $E \in \mathcal{P}_e(G)$. Here, as usual, tE stands for the set $\{ts : s \in E\}$.

Observe that, since $e \in E$, we can also write $\phi_t(E) = tE \cup \{t, e\}$.

3.1. Proposition. *The map $t \in G \mapsto \phi_t \in \mathcal{F}\big(\mathcal{P}_e(G)\big)$ satisfies the properties (i) – (iii) of (2.2) and hence there is a unique representation $\Lambda\colon \mathcal{S}(G) \to \mathcal{F}\big(\mathcal{P}_e(G)\big)$ such that $\Lambda([t]) = \phi_t$.*

Proof. We prove only (2.2.i). For every $t, s \in G$ and each $E \in \mathcal{P}_e(G)$, we have

$$\phi_{s^{-1}}\phi_s\phi_t(E) = \phi_{s^{-1}}\phi_s(tE \cup \{t, e\}) = \phi_{s^{-1}}(stE \cup \{st, s, e\}) = tE \cup \{t, e, s^{-1}\},$$

while

$$\phi_{s^{-1}}\phi_{st}(E) = \phi_{s^{-1}}(stE \cup \{st, e\}) = tE \cup \{t, s^{-1}, e\}. \qquad \square$$

Observe that, for any r in G and E in $\mathcal{P}_e(G)$, one has $\Lambda(\varepsilon_r)(E) = E \cup \{r\}$. In particular, if $\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s]$, we can easily prove that $\Lambda(\alpha)$, when applied to the singleton $\{e\}$, gives $\{r_1, \ldots, r_n, s, e\}$.

Based on the existence of these two representations we may prove uniqueness of our decomposition.

3.2. Proposition. *Every α in $\mathcal{S}(G)$ admits a unique standard decomposition*

$$\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s]$$

up to the order of the ε_r's.

Proof. As observed above, $\Lambda(\alpha)(\{e\}) = \{r_1, \ldots, r_n, s, e\}$ and also $\partial(\alpha) = s$. So, if one has another standard decomposition $\alpha = \varepsilon_{t_1} \ldots \varepsilon_{t_m}[u]$, then, on the one hand, we would have $s = u$ and, on the other, $\{r_1, \ldots, r_n, s, e\} = \{t_1, \ldots, t_m, u, e\}$. Hence

$$\{r_1, \ldots, r_n, s, e\}\backslash\{s, e\} = \{t_1, \ldots, t_m, u, e\}\backslash\{u, e\}$$

which implies that

$$\{r_1, \ldots, r_n\} = \{t_1, \ldots, t_m\}. \quad \square$$

With this we can compute the order of $\mathcal{S}(G)$ for finite groups.

3.3. Theorem. *If G is a finite group of order p, then $\mathcal{S}(G)$ has $2^{p-2}(p+1)$ elements.*

Proof. In fact, there are 2^{p-1} elements in $\mathcal{S}(G)$ of the form $\varepsilon_{r_1} \ldots \varepsilon_{r_n}[e]$ and for $p-1$ possible choices of $s \neq e$, there are 2^{p-2} elements of the form $\varepsilon_{r_1} \ldots \varepsilon_{r_n}[s]$. The total number is then $2^{p-1} + (p-1)2^{p-2} = 2^{p-2}(p+1)$. □

Therefore we see that the order of $\mathcal{S}(G)$ grows exponentially with the order of G. For a group of order 28, for example, the order of $\mathcal{S}(G)$ is 1,946,157,056. As for less dramatic examples, groups of order 2 through 10 cause the order of $\mathcal{S}(G)$ to be 3, 8, 20, 48, 112, 256, 576, 1280 and 2816, respectively.

3.4. Theorem. *For every group G, $\mathcal{S}(G)$ is an inverse semigroup.*

Proof. Assume $\alpha \in \mathcal{S}(G)$ admits, in addition to α^*, another "inverse", that is an element β in $\mathcal{S}(G)$ such that $\alpha\beta\alpha = \alpha$ and $\beta\alpha\beta = \beta$. Write, in standard form,

$$\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s]$$

and

$$\beta^* = \varepsilon_{t_1} \ldots \varepsilon_{t_m}[u].$$

Therefore $\beta = [u^{-1}]\varepsilon_{t_1} \ldots \varepsilon_{t_m}$ and we have $s = \partial(\alpha) = \partial(\alpha\beta\alpha) = su^{-1}s$ from which it follows that $u = s$. We also have

$$\alpha\beta\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s][s^{-1}]\varepsilon_{t_1} \ldots \varepsilon_{t_m}\varepsilon_{r_1} \ldots \varepsilon_{r_n}[s] = \varepsilon_{r_1} \ldots \varepsilon_{r_n}\varepsilon_{t_1} \ldots \varepsilon_{t_m}[s] = \alpha.$$

By the uniqueness of standard decompositions, we see that

$$\{r_1, \ldots, r_n\} \cup \{t_1, \ldots, t_m\} = \{r_1, \ldots, r_n\}$$

so

$$\{t_1, \ldots, t_m\} \subseteq \{r_1, \ldots, r_n\}.$$

The same argument, applied to the identity $\beta^*\alpha^*\beta^* = \beta^*$, yields

$$\{r_1, \ldots, r_n\} \subseteq \{t_1, \ldots, t_m\},$$

so we conclude that $\beta = \alpha^*$. □

3.5. Remark. Here are some consequences of (3.2). Note that, if e is an idempotent in $\mathcal{S}(G)$, $x \in \mathcal{S}(G)$ and xe is also idempotent, then x must necessarily be an idempotent. This says that $\mathcal{S}(G)$ is an E-unitary semigroup (see [7, p. 55]). However $\mathcal{S}(G)$ is not a Clifford semigroup (see [7, IV.2]) since idempotents are not central by (2.4). We thank the referee for pointing out these facts.

4. ACTIONS OF INVERSE SEMIGROUPS VS. PARTIAL ACTIONS OF GROUPS

Recall that a partial action of a group G on a set X is a map $\theta \colon G \to \mathcal{I}(X)$ satisfying the conditions of (1.2). Our next proposition is designed to rephrase these conditions in terms of the semigroup structure of $\mathcal{I}(X)$.

4.1. Proposition. *Let G be a group and X be a set. A map $\theta \colon G \to \mathcal{I}(X)$ gives a partial action of G on X if and only if, for all $s, t \in G$, we have*
  (i) $\theta_s\theta_t\theta_{t^{-1}} = \theta_{st}\theta_{t^{-1}}$,
  (ii) $\theta_e = id_X$.

In this case θ also satisfies

(iii) $\theta_{s^{-1}}\theta_s\theta_t = \theta_{s^{-1}}\theta_{st}$.

Proof. Taking $s = t^{-1}$ in (i), we have

$$\theta_{t^{-1}}\theta_t\theta_{t^{-1}} = \theta_e\theta_{t^{-1}} = \theta_{t^{-1}}.$$

Replacing the roles of t and t^{-1}, we also have $\theta_t\theta_{t^{-1}}\theta_t = \theta_t$. So, by the uniqueness of inverses in inverse semigroups, we conclude that $\theta_t^* = \theta_{t^{-1}}$. Define $D_t = \text{ran}(\theta_t)$; so the above conclusion tells us that

$$\text{dom}(\theta_t) = \text{ran}(\theta_t^*) = \text{ran}(\theta_{t^{-1}}) = D_{t^{-1}}.$$

That is, θ_t is a map from $D_{t^{-1}}$ to D_t, as required. Now, for any $s, t \in G$ we have

$$\theta_{t^{-1}}\theta_{s^{-1}} = \theta_{t^{-1}}\theta_{s^{-1}}\theta_s\theta_{s^{-1}} = \theta_{t^{-1}s^{-1}}\theta_s\theta_{s^{-1}}.$$

So, in particular, the domains of these maps must agree. On the one hand we have

$$\text{dom}(\theta_{t^{-1}}\theta_{s^{-1}}) = \theta_s(D_{s^{-1}} \cap D_t).$$

Noting that $\theta_s\theta_{s^{-1}}$ is the identity function on D_s, we have, on the other hand,

$$\text{dom}(\theta_{t^{-1}s^{-1}}\theta_s\theta_{s^{-1}}) = D_s \cap D_{st}.$$

Now the third condition of (1.2) can easily be checked. The converse is left to the reader.

With respect to the last part of the statement note that, since $\theta_{t^{-1}}\theta_{s^{-1}}\theta_s = \theta_{t^{-1}s^{-1}}\theta_s$ and $\theta_t^* = \theta_{t^{-1}}$, we have

$$\theta_{s^{-1}}\theta_s\theta_t = (\theta_{t^{-1}}\theta_{s^{-1}}\theta_s)^* = (\theta_{t^{-1}s^{-1}}\theta_s)^* = \theta_{s^{-1}}\theta_{st}. \quad \square$$

Our main result may now be stated.

4.2. Theorem. *For every group G and any set X, there is a one-to-one correspondence between*
(a) partial actions of G on X, and
(b) actions of $\mathcal{S}(G)$ on X.

Proof. By (2.2), homomorphisms from $\mathcal{S}(G)$ into $\mathcal{I}(X)$ are in one-to-one correspondence with maps from G to $\mathcal{I}(X)$ satisfying (i)–(iii) of (2.2). As we have seen in (4.1), these correspond to partial actions of G on X. $\quad \square$

5. Graded C^*-algebras

References for C^*-algebra theory are [10] and [6].

5.1. Definition. Let G be a group. A C^*-algebra A is said to be *graded* over G, or *G-graded*, if A is equipped with a linearly independent family of closed linear subspaces $\{A_t\}_{t \in G}$, such that for all $s, t \in G$

(i) $A_s A_t \subseteq A_{st}$,
(ii) $A_t^* = A_{t^{-1}}$,
(iii) A is the closure of $\bigoplus_{t \in G} A_t$.

In this case, each A_t is called a *grading subspace* of A.

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We would like to show that $\mathcal{S}(G)$ plays an important role in describing certain subspaces of A, related to its graduation. For example, given s and t in G, the product $A_s A_t$, although contained in A_{st}, need not coincide with it, and, in fact, need not even be dense there. We could, therefore, ask what is the maximum number of subspaces of A one can obtain by taking the closure of finite products of grading subspaces.

Before we proceed, let us agree on the following slightly unusual notational convention: If X and Y are subsets of a C^*-algebra, then XY will denote the *closed* linear span of the set of products xy with $x \in X$ and $y \in Y$.

Returning to our graded algebra, let us denote $D_t = A_t A_{t^{-1}}$. It is easy to show that A_t is a D_t-$D_{t^{-1}}$-Hilbert bimodule. A good reference for the theory of Hilbert bimodules is [8]. As is the case for all right Hilbert modules, the product $A_t D_{t^{-1}}$ must coincide with A_t [8, 1.1.4]. This translates to $A_t A_{t^{-1}} A_t = A_t$.

5.2. Definition. Given a C^*-algebra A, let $\mathcal{P}_\ell(A)$ be the set of all closed linear subspaces of A, equipped with the semigroup structure under which the product of X and Y is defined to be the closed linear span of the set of products xy with $x \in X$ and $y \in Y$, as mentioned earlier.

At this point one cannot oversee the fact that the map

$$t \in G \mapsto A_t \in \mathcal{P}_\ell(A)$$

provides a key connection between the theory of graded C^*-algebras and the theory of semigroups we have been discussing so far. As already mentioned, this map is not a homomorphism of semigroups but:

5.3. Proposition. *Given a graded C^*-algebra $A = \overline{\bigoplus_{t \in G} A_t}$, then, for every s and t in G one has*

  (i) $A_{s^{-1}} A_s A_t = A_{s^{-1}} A_{st}$,
  (ii) $A_s A_t A_{t^{-1}} = A_{st} A_{t^{-1}}$,
  (iii) $A_s A_e = A_s$.

Proof. Based on the fact that $A_t A_{t^{-1}} A_t = A_t$ and that $A_s A_t \subseteq A_{st}$ we have

$$A_{s^{-1}} A_s A_t \subseteq A_{s^{-1}} A_{st} = A_{s^{-1}} A_s A_{s^{-1}} A_{st} \subseteq A_{s^{-1}} A_s A_t,$$

which proves (i). The proof of (ii) is quite similar, while (iii) can be proved as follows:

$$A_s A_e \subseteq A_s = A_s A_{s^{-1}} A_s \subseteq A_s A_e. \quad \square$$

5.4. Theorem. *Given a graded C^*-algebra $A = \overline{\bigoplus_{t \in G} A_t}$, there is a correspondence which assigns, for each α in $\mathcal{S}(G)$, a closed subspace A^α of A satisfying, for all α and β in $\mathcal{S}(G)$ and all t in G,*

  (i) $A^{[t]} = A_t$,
  (ii) *if $\partial(\alpha) = t$, then A^α is contained in A_t,*
  (iii) *the closed linear span of the product of A^α by A^β is exactly equal to $A^{\alpha\beta}$.*

Proof. This is an immediate consequence of (5.3) and (2.2). $\quad \square$

As a consequence, we see that the collection $\{A^\alpha\}_{\alpha \in \mathcal{S}(G)}$ is closed under multiplication, i.e., it is a subsemigroup of $\mathcal{P}_\ell(A)$. Since it contains the A_t's, we have provided an answer for the question posed after (5.1), concerning the maximum number of different subspaces of A that one can obtain by taking finite products of

grading subspaces. In the specific case of finite groups, that number is at most the order of $\mathcal{S}(G)$, which we have seen to equal $2^{p-2}(p+1)$, where p is the order of G.

The question begging to be asked at this point is whether one can produce a G-graded C^*-algebra such that the A^α are all distinct. In fact this question makes sense even for infinite groups. The answer is yes but we are not yet prepared to provide an example right now. First we need to develop our theory a little further.

6. Hilbert space representations

Contrary to our previous treatment of representations of semigroups, simply as homomorphisms into another semigroup, we shall now make a more systematic study of representations of inverse semigroups on Hilbert spaces. Therefore, let H denote Hilbert's space and denote by $\mathcal{B}(H)$ the collection of all bounded linear operators on H.

6.1. Definition. Let S be an inverse semigroup. A *(Hilbert space) representation* of S on H is a map $\pi \colon S \to \mathcal{B}(H)$ such that, for all α, β in S

  (i) $\pi(\alpha\beta) = \pi(\alpha)\pi(\beta)$,
  (ii) $\pi(\alpha^*) = \pi(\alpha)^*$.

If S has a unit e, we shall also require that

 (iii) $\pi(e) = I$.

Note that, since $\alpha\alpha^*\alpha = \alpha$, the same holds for $\pi(\alpha)$, which is then a partially isometric operator on H for each $\alpha \in S$. Also, if ε is an idempotent in S, one necessarily has $\varepsilon^* = \varepsilon$ and hence $\pi(\varepsilon)$ is a selfadjoint projection on H.

The related notion for groups is that of a partial representation:

6.2. Definition. Let G be a group. A *partial representation* of G on H is a map $\pi \colon G \to \mathcal{B}(H)$ such that, for all $t, s \in G$,

  (i) $\pi(s)\pi(t)\pi(t^{-1}) = \pi(st)\pi(t^{-1})$,
  (ii) $\pi(t^{-1}) = \pi(t)^*$,
 (iii) $\pi(e) = I$.

Note that (i)–(iii) imply that $\pi(s^{-1})\pi(s)\pi(t) = \pi(s^{-1})\pi(st)$. So, regarding $\mathcal{B}(H)$ as a semigroup under multiplication (i.e., composition of operators), we see that a partial representation of G satisfies the conditions of (2.2) and hence there exists a homomorphism

$$\tilde{\pi} \colon \mathcal{S}(G) \to \mathcal{B}(H)$$

such that $\tilde{\pi}([t]) = \pi(t)$ for all t in G. It is not hard to see that $\tilde{\pi}$ also satisfies $\tilde{\pi}(\alpha^*) = \tilde{\pi}(\alpha)^*$, and hence $\tilde{\pi}$ is a representation of $\mathcal{S}(G)$ on H in the sense of (6.1).

Conversely, given a representation ρ of $\mathcal{S}(G)$ on H, it is easy to see that the map

$$t \in G \mapsto \rho([t]) \in \mathcal{B}(H)$$

is a partial representation of G. We have therefore proven:

6.3. Proposition. *There is a one-to-one correspondence between*
(a) partial representations of G on H, and
(b) representations of $\mathcal{S}(G)$ on H.

One of the important features of the theory of C^*-algebras is that several different representation theories are governed by a C^*-algebra. The theory of partial representations of groups is no exception and we shall now construct a C^*-algebra, called the *partial group C^*-algebra* of G, which will play the corresponding role with respect to partial group representations.

The representation theory of inverse semigroups has received much attention and one can also associate Banach and C^*-algebras of interest to the theory. A key result, due to Barnes [2], states that the l_1 algebra of an inverse semigroup has a separating family of irreducible representations. Also, Duncan and Paterson [3] have studied both the reduced and the full C^*-algebras of an inverse semigroup.

However, given our interest in the special case of the inverse semigroup associated to a group, with all of its characteristic features, we shall prefer to develop our theory from scratch.

Let G denote a group, which we will consider fixed for the remainder of this section.

An auxiliary C^*-algebra in our study is the universal C^*-algebra, denoted A, defined via generators and relations as follows. The set of generators consists of a symbol P_E for each finite subset $E \subseteq G$, and the relations are:

(i) $P_E = P_E^*$,
(ii) $P_E P_F = P_{E \cup F}$,

for all possible choices of E and F. Note that each P_E will then be a projection and, if $\emptyset$ denotes the empty set, $P_\emptyset$ will serve as a unit for A. Obviously, A is an abelian C^*-algebra, so by Gelfand's Theorem, $A = C(X)$ for some compact space X. For the time being the description of A as a universal C^*-algebra will be quite appropriate for our purposes, so we shall postpone the study of its spectrum until needed.

Since the collection of all P_E's is closed under multiplication and adjoints, one has that A is the closed linear span of these elements.

Let A_e be the ideal of A generated by $P_{\{e\}}$, that is, $A_e = P_{\{e\}} A$. As before, A_e is the closed linear span of the set $\{P_E \colon e \in E\}$. Also A_e is a unital C^*-algebra with unit $P_{\{e\}}$.

For each t in G, let θ_t be the automorphism of A specified by $\theta_t(P_E) = P_{tE}$. Observe that A_e is not invariant under θ_t, but if we let

$$D_t = \overline{\operatorname{span}}\{P_E \colon e, t \in E\},$$

then D_t is an ideal in A_e and the restriction of θ_t to $D_{t^{-1}}$ is a C^*-algebra isomorphism

$$\theta_t \colon D_{t^{-1}} \to D_t$$

which lives fully inside of A_e. This is an example of a partial automorphism of A_e, as defined in [5], and the map $t \mapsto \theta_t$ fits the definition of a partial action of G on A_e given in [4] and [9]. In particular, we can form the crossed product of A_e by G.

6.4. Definition. The *partial group C^*-algebra* of G is the C^*-algebra $C_p^*(G)$ given by the crossed product [4], [9] of A_e by θ, that is,

$$C_p^*(G) = A_e \rtimes_\theta G.$$

The following result improves upon (6.3) and shows the relevance of $C_p^*(G)$ in the theory of partial representations of G.

6.5. Theorem. *There is a one-to-one correspondence between*
(a) partial representations of G on H,
(b) representations of $\mathcal{S}(G)$ on H and
(c) C^*-algebra representations of $C_p^*(G)$ on H.

Proof. Let r, s be in G and let E and F be finite subsets of G containing $\{e, r\}$ and $\{e, s\}$, respectively. Then P_E is in D_r and P_F is in D_s and we have, following [4] and [9],

$$(P_E\delta_r) * (P_F\delta_s) = \theta_r(\theta_r^{-1}(P_E)P_F)\delta_{rs} = \theta_r(P_{r^{-1}E\cup F})\delta_{rs} = P_{E\cup rF}\delta_{rs}.$$

Consider, for each t in G, the element $u_t = P_{\{e,t\}}\delta_t \in C_p^*(G)$. Based on the computation above, it is easy to verify that $u_s u_t u_{t^{-1}} = u_{st}u_{t^{-1}}$, that $u_{t^{-1}} = u_t^*$ and that $u_e = 1$. This said, it is easy to see that every representation of $C_p^*(G)$, when computed on the elements u_t, gives a partial representation of G.

Conversely, suppose we are given a representation π of $\mathcal{S}(G)$ on H. Recall that the idempotents ε_r in $\mathcal{S}(G)$ are defined by $\varepsilon_r = [r][r^{-1}]$. For each finite set $E = \{r_1, \ldots, r_n\} \subseteq G$, define

$$Q_E = \pi(\varepsilon_{r_1} \ldots \varepsilon_{r_n}).$$

This is a projection in $\mathcal{B}(H)$ and we have $Q_E Q_F = Q_{E\cup F}$ for all E and F, so there exists a representation ρ of A on H such that $\rho(P_E) = Q_E$. Also let $u_t = \pi([t])$. If E contains $\{e, t^{-1}\}$, then by (2.4), we have

$$u_t\rho(P_E)u_{t^{-1}} = \pi([t]\varepsilon_{r_1} \ldots \varepsilon_{r_n}[t^{-1}]) = \pi(\varepsilon_{tr_1} \ldots \varepsilon_{tr_n}[t][t^{-1}])$$
$$= \pi(\varepsilon_{tr_1} \ldots \varepsilon_{tr_n}) = \rho(P_{tE}) = \rho(\theta_t(P_E)).$$

So we get a covariant representation of the partial action of G on A_e, in the sense of [9, 2.8], and hence a representation $\rho \times u$ of $C_p^*(G)$. In particular

$$\rho \times u(P_{\{e,t\}}\delta_t) = \rho(P_{\{e,t\}})u_t = Q_{\{e,t\}}\pi([t]) = \pi(\varepsilon_e\varepsilon_t[t]) = \pi([t]). \quad \square$$

For our following result it will be important to understand the algebra A more concretely. For this purpose let K be the power set of G. Among many different descriptions for K, one can use the model

$$K = \prod_{t\in G}\{0, 1\}.$$

Equipped with the product topology, K becomes a compact Hausdorff space. When G is an infinite countable group, for example, K is homeomorphic to Cantor's set. For each t in G let $Q_t \in C(K)$ be the tth coordinate function, i.e.,

$$Q_t: x \in K \mapsto x_t \in \mathbf{C}.$$

For each finite set $E \subseteq G$ let $Q_E = \prod_{t\in E} Q_t$. Clearly $Q_E Q_F = Q_{E\cup F}$ so we get a C^*-algebra homomorphism

$$\Phi: A \to C(K)$$

such that $\Phi(P_E) = Q_E$.

6.6. Proposition. *A and $C(K)$ are isomorphic via the homomorphism Φ above.*

Proof. Surjectivity is an immediate consequence of the Stone-Weierstrass Theorem. To prove injectivity let $a \in A$ be non-zero. Take a character f of A such that $f(a) \neq 0$. For each t in G observe that $P_{\{t\}}$ is a projection in A, so $f(P_{\{t\}})$ is a complex number in $\{0, 1\}$. Define x in K by

$$x = \left(f(P_{\{t\}}) \right)_{t \in G}.$$

For all t we have $\Phi(P_{\{t\}})(x) = Q_t(x) = f(P_{\{t\}})$ so, observing that the $P_{\{t\}}$ generate A, we see that we must also have $\Phi(a)(x) = f(a) \neq 0$, from which we conclude that $\Phi(a) \neq 0$ and hence that Φ is injective. $\qquad\square$

It is well known that a crossed product by a partial action is a graded C^*-algebra, and hence so is $C_p^*(G)$. This is also the case for the full (as opposed to partial) group C^*-algebra $C^*(G)$ (see [10], for definitions). However the grading of $C^*(G)$ is rather uninteresting from our point of view since it is saturated, that is, the product of the grading subspaces A_t and A_s always coincides with A_{ts}. In terms of the notation of (5.4), this simply says that $A^\alpha = A_{\partial(\alpha)}$. The situation for the partial group algebra is a lot more interesting.

6.7. Theorem. *Let $C_p^*(G) = \overline{\bigoplus_{t \in G} B_t}$ be the G-grading of $C_p^*(G)$ arising from its construction as a partial crossed product. Also let*

$$\alpha \in \mathcal{S}(G) \mapsto B^\alpha \in \mathcal{P}_\ell(A)$$

be the semigroup homomorphism referred to in (5.4). Then, if $\alpha \neq \beta$, one has $B^\alpha \neq B^\beta$.

Proof. We initially claim that if $\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s] \in \mathcal{S}(G)$, then

$$B^\alpha = \overline{\mathrm{span}}\{P_E \delta_s \colon e, r_1, \ldots, r_n, s \in E\}.$$

In fact, if $n = 0$, then $B^\alpha = B^{[s]} = B_s$ and the result follows from the definition of B_s as $B_s = D_s \delta_s$ and the fact that D_s is spanned by the P_E's with $e, s \in E$.

Let $n > 0$ and $\gamma = \varepsilon_{r_2} \ldots \varepsilon_{r_n}[s]$ so $\alpha = \varepsilon_{r_1}\gamma$ and $B^\alpha = B^{[r_1][r_1^{-1}]\gamma} = B_{r_1} B_{r_1^{-1}} B^\gamma$. By induction we may assume that $B^\gamma = \overline{\mathrm{span}}\{P_E \delta_s \colon e, r_2, \ldots, r_n, s \in E\}$.

On the other hand, it is easy to see that $B_{r_1} B_{r_1^{-1}} = D_{r_1} = \overline{\mathrm{span}}\{P_F \colon e, r_1 \in F\}$. We thus have

$$B_\alpha = \overline{\mathrm{span}}\{P_F P_E \delta_s \colon e, r_1 \in F; e, r_2, \ldots, r_n, s \in E\}.$$

Observing that $P_F P_E = P_{F \cup E}$ we see that the claim is proved.

Returning to the proof of our theorem, suppose

$$\alpha = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[s]$$

and

$$\beta = \varepsilon_{t_1} \ldots \varepsilon_{t_m}[u]$$

are in standard form and that $B^\alpha = B^\beta$. Then, because of (5.4.ii), that is, because $B^\alpha \subseteq B_s$ and $B^\beta \subseteq B_u$, we must have $s = u$.

On the other hand, $B^\alpha B^{\alpha^*} = B^{\alpha\alpha^*}$, but $\alpha\alpha^* = \varepsilon_{r_1} \ldots \varepsilon_{r_n}\varepsilon_s$. So, by our claim, we have $B^{\alpha\alpha^*} = \overline{\mathrm{span}}\{P_E \delta_e \colon e, r_1, \ldots, r_n, s \in E\}$, which is a subspace of B_e. Since

$B^\alpha = B^\beta$, this subspace must coincide with

$$B^{\beta\beta^*} = \overline{\text{span}}\{P_E\delta_e \colon e, t_1, \dots, t_m, u \in E\}$$

from which the final conclusion follows easily. □

In the special case of finite groups we have therefore proven:

6.8. Corollary. *Let G be a finite group of order p. Then, there is a G-graded C^*-algebra $A = \bigoplus_{t \in G} A_t$, such that the subsemigroup of $\mathcal{P}_\ell(A)$ generated by the grading spaces has exactly $2^{p-2}(p+1)$ elements; while, for any G-graded C^*-algebra the said subsemigroup cannot have more than this number of elements.*

Beyond the purpose of constructing interesting graded C^*-algebras, we believe there is a considerable amount of interesting information that can be dug out from the structure of $C^*_p(G)$.

To deal with the specific case of finite groups, one can prove that $C^*_p(G)$ is also the groupoid C^*-algebra [13] constructed from the partial action of G on $C(K)$, and compute $C^*_p(G)$ for the most elementary finite groups. In particular, for the two groups of order 4, namely $Z/4Z$ and $Z/2Z \oplus Z/2Z$, one obtains

$$C^*_p(Z/4Z) = C^7 \oplus M_2(C) \oplus M_3(C)$$

while

$$C^*_p(Z/2Z \oplus Z/2Z) = C^{11} \oplus M_3(C).$$

It is quite intriguing that, while the group C^*-algebra of finite commutative groups forgets everything but the order of the group, the partial group C^*-algebra $C^*_p(G)$ allows us to distinguish between $Z/4Z$ and $Z/2Z \oplus Z/2Z$.

References

1. B. Abadie, S. Eilers and R. Exel, "Morita equivalence and crossed products by Hilbert C^*-bimodules", preprint, Universidade de São Paulo, 1994, to appear in Trans. Amer. Math. Soc. CMP 97:17
2. B. A. Barnes, Representations of the l_1 algebra of an inverse semigroup, Trans. Amer. Math. Soc. **218**, (1976), 361–396. MR **53:**1169
3. J. Duncan and A. L. T. Paterson, "C^*-algebras of inverse semigroups", Proc. Edinburgh Math. Soc. **28** (1985), 41–58. MR **86h:**46090
4. R. Exel, "Twisted Partial Actions, A Classification of Regular C^*-Algebraic Bundles", Proc. London Math. Soc. **74** (1997), 417–443. CMP 97:05
5. R. Exel, "Circle Actions on C^*-Algebras, Partial Automorphisms and a Generalized Pimsner–Voiculescu Exact Sequence", J. Funct. Analysis **122** (1994), 361–401. MR **95g:**46122
6. J. M. G. Fell and R. S. Doran, "Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles", Pure and Applied Mathematics. vol. 125 and 126, Academic Press, 1988. MR **90c:**46001; MR **90c:**46002
7. J. M. Howie, "An introduction to Semigroup theory", Academic Press, 1976. MR **57:**6235
8. K. Jensen and K. Thomsen, "Elements of KK-Theory", Birkhäuser, 1991. MR **94b:**19008
9. K. McClanahan, "K-theory for partial crossed products by discrete groups", J. Funct. Analysis, **130** (1995), 77–117. MR **95i:**46083
10. G. K. Pedersen, "C^*-Algebras and their automorphism groups", Acad. Press, 1979. MR **81e:**46037
11. M. Pimsner, "A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by **Z**", Amer. Math. Soc., Providence, R.I., 1997. MR **97k:**46069

12. J. Quigg and I. Raeburn, "Landstad duality for partial actions", preprint, University of New-castle, 1994.
13. J. Renault, "A groupoid approach to C^*-algebras", Lecture Notes in Mathematics, vol. 793, Springer, 1980. MR **82h:**46075
14. N. Sieben, "C^*-crossed products by partial actions and actions of inverse semigroups", Masters Thesis, Arizona State University, 1994. CMP 97:14

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