A CHARACTERIZATION OF ROUND SPHERES

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Abstract. A new characterization of geodesic spheres in the simply connected space forms in terms of higher order mean curvatures is given: An immersion of an n-dimensional compact oriented manifold without boundary into n+1 dimensional Euclidean space, hyperbolic space or the open half sphere is a totally umbilic immersion if, for some r, r = 2, ..., n, the (r-1)-th mean curvature \( H_{r-1} \) does not vanish and the ratio \( H_r/H_{r-1} \) is constant.

1. Introduction

Let \( N^{n+1} \) be one of the Euclidean space \( R^{n+1} \), the hyperbolic space \( H^{n+1} \) or the open half sphere \( S^*_n \), and let \( \phi : M^n \to N^{n+1} \) be an isometric immersion of a compact oriented n-dimensional manifold without boundary \( M^n \). Let \( H_r \) denote the r-th mean curvature of \( M^n \), that is, \( H_r \) is the r-th elementary symmetric polynomial of principal curvatures of \( M^n \) divided by \( \binom{n}{r} \), and \( H_0 \) is defined to be 1. For instance, \( H_1 \) is the usual mean curvature and \( H_n \) is the Gauss-Kronecker curvature.

It was shown in [3] that if two consecutive mean curvatures \( H_{r-1}, H_r \) are constant for some r between 2 and n, then \( \phi(M^n) \) is a geodesic sphere. In this note, we generalize this result in the following way.

Theorem. Let \( N^{n+1} \) be one of \( R^{n+1}, H^{n+1} \) or \( S^*_n \), and let \( \phi : M^n \to N^{n+1} \) be an isometric immersion of a compact oriented n-dimensional manifold without boundary \( M^n \). If \( H_{r-1} \) does not vanish and the ratio \( H_r/H_{r-1} \) of two consecutive mean curvatures is a constant for some r = 2, ..., n, then \( \phi(M^n) \) is a geodesic hypersphere.

When \( N^{n+1} \) is \( R^3 \), the above theorem was proved in [1] under the convexity assumption. Note that we consider immersions, and the case \( r = 1 \) is omitted. As we defined \( H_0 \) to be 1, the ratio \( H_1/H_0 \) is equal to the usual mean curvature \( H_1 \). Because of the existence of nonspherical immersions of nonzero constant mean curvature \( H_1 \) in \( R^n \), proved by Hsiang, Teng and Yu [4] and Wente [6], we cannot expect our theorem for \( r = 1 \). However, by Alexandrov’s well-known sphere theorem

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or Montiel and Ros' theorem [5], our theorem holds even for \( r = 1 \) if we assume embeddings.

2. Proof

We use the hyperboloid model for \( \mathbb{H}^{n+1} \) and the usual embedding of \( \mathbb{S}^{n+1} \) into \( \mathbb{R}^{n+2} \). Let \( \eta \) denote a unit normal field on \( M^n \). We use the following Minkowski formula (for proof, see [5]), where \( \langle \cdot, \cdot \rangle \) denotes the usual Euclidean inner product on \( \mathbb{R}^{n+1} \) (on \( \mathbb{R}^{n+2} \) when \( N^{n+1} = \mathbb{R}^{n+1} \) (when \( N^{n+1} = \mathbb{S}^{n+1}_+ \) and the Lorentzian inner product on \( \mathbb{R}^{n+2} \) when \( N^{n+1} = \mathbb{H}^{n+1} \).

Lemma 1. For \( k = 1, \ldots, n \), the following identities hold:

1. When \( N^{n+1} = \mathbb{R}^{n+1} \),
   \[ \int_M (H_{k-1} + H_k \langle \phi, \eta \rangle) \, dM = 0. \]

2. When \( N^{n+1} = \mathbb{H}^{n+1} \),
   \[ \int_M (H_{k-1} \langle \phi, p \rangle + H_k \langle \eta, p \rangle) \, dM = 0 \text{ for any } p \in \mathbb{R}^{n+2}. \]

3. When \( N^{n+1} = \mathbb{S}^{n+1}_+ \),
   \[ \int_M (H_{k-1} \langle \phi, p \rangle - H_k \langle \eta, p \rangle) \, dM = 0 \text{ for any } p \in \mathbb{R}^{n+2}. \]

We also use the following inequalities for higher order mean curvatures (for proof, see, for example, §12 of [2]):

Lemma 2. (1) \( H_r H_{r-2} \leq H_{r-1}^2 \).

(2) If \( H_1, H_2 \) are greater than zero everywhere on \( M^n \), then
   \[ H_2 \leq H_1^2 \]

and the equality holds only at umbilical points.

Now, assume \( H_{r-1} \) does not vanish and \( H_r/H_{r-1} = a \) for a constant number \( a \) and a fixed \( r = 2, \ldots, n \).

(2.1) Proof when \( N^{n+1} = \mathbb{R}^{n+1} \). Since \( M^n \) is compact, by changing the unit normal vector on \( M \) if necessary, one can find a point in \( M^n \) where all the principal curvatures are positive. Then \( H_r, H_{r-1} \) are positive at that point. Since \( H_{r-1} \) does not vanish and \( H_r/H_{r-1} = a \) is constant on \( M^n \), \( H_r \) is positive on \( M^n \). Then, following the argument of Lemma 1 in [5], it follows that the \( H_k \) are positive on \( M^n \) for \( k = 1, \ldots, r-1 \). Then \( a > 0 \) and from the inequality (1)
   \[ H_r H_{r-2} \leq H_{r-1}^2 \]

of Lemma 2, we have

\[ (*) \quad 0 < a = H_r/H_{r-1} \leq H_{r-1}/H_{r-2}. \]

Since \( H_r = aH_{r-1} \), we have by Lemma 1,

\[ 0 = \int_M (H_{r-1} + H_r \langle \phi, \eta \rangle) \, dM = \int_M (H_{r-1} + aH_{r-1} \langle \phi, \eta \rangle) \, dM, \]

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that is,
\[
\int_M H_{r-1} \langle \phi, \eta \rangle \, dM = \int_M \left( -\frac{1}{a} H_{r-1} \right) \, dM.
\]
Since
\[
\int_M (H_{r-2} + H_{r-1} \langle \phi, \eta \rangle) \, dM = 0
\]
we then have
\[
\int_M (H_{r-2} - \frac{1}{a} H_{r-1}) \, dM = 0.
\]
Since \( H_{r-2} - \frac{1}{a} H_{r-1} \leq 0 \) from (*), it follows that
\[
H_{r-1}/H_{r-2} = a = H_r/H_{r-1},
\]
everywhere on \( M^n \). Now, proceeding inductively, we have finally
\[
H_2/H_1 = a = H_1/H_0 = H
\]
everywhere on \( M^n \), that is, the equality holds in (2) of Lemma 2. Hence every point is an umbilical point, i.e., \( \phi(M^n) \) is a geodesic hypersphere.

(2.2) Proof when \( N^{n+1} = H^{n+1} \). At a point of \( M^n \) where the distance function of \( H^{n+1} \) attains its maximum, all the principal curvatures are positive. Then (*) also holds in this case, and the \( H_k \) are positive on \( M^n \) for \( k = 1, \ldots, r-1 \). Since \( H_r = aH_{r-1} \), we have
\[
0 = \int_M (H_{r-1} \langle \phi, p \rangle + H_r \langle \eta, p \rangle) \, dM
\]
\[
= \int_M (H_{r-1} \langle \phi, p \rangle + aH_{r-1} \langle \eta, p \rangle) \, dM,
\]
that is,
\[
\int_M H_{r-1} \langle \eta, p \rangle \, dM = \int_M \left( -\frac{1}{a} H_{r-1} \langle \phi, p \rangle \right) \, dM.
\]
Since
\[
\int_M (H_{r-2} \langle \phi, p \rangle + H_{r-1} \langle \eta, p \rangle) \, dM = 0,
\]
it follows that
\[
\int_M (H_{r-2} - \frac{1}{a} H_{r-1}) \langle \phi, p \rangle \, dM = 0.
\]
Now, if we take \( p = (1, 0, \ldots, 0) \in \mathbb{R}^{n+2} \), then the sign of \( \langle \phi, p \rangle \) does not change on \( M^n \). Since \( H_{r-2} - \frac{1}{a} H_{r-1} \leq 0 \) from (*), we have \( H_{r-2} - \frac{1}{a} H_{r-1} = 0 \) everywhere on \( M^n \). Proceeding inductively as in (2.1), we can show that every point is an umbilical point. Hence \( \phi(M^n) \) is a geodesic hypersphere.
(2.3) Proof when $N^{n+1} = S^{n+1}_+$. Let $c \in S^{n+1}$ be the centre of $S^{n+1}_+$. Then at a point of $M^n$ where the height function $\langle \phi, c \rangle$ attains its maximum, all the principal curvatures are positive because $M^n$ lies in the open half sphere with the centre $c$. Then (*) holds, and the equality in (*) holds only at umbilical points. Proceeding as in (2.2), we have

$$\int_M (H - \frac{1}{a} H) \langle \phi, p \rangle \, dM = 0.$$ 

Since $M^n$ lies in the open half sphere, one can find a vector $p \in \mathbb{R}^{n+2}$ so that $\langle \phi, p \rangle$ is positive on $M^n$. Then, since $H - \frac{1}{a} H \leq 0$ by (*), it follows that $H - \frac{1}{a} H = 0$, everywhere on $M^n$. Now, arguing in the same way as above, we can show that $\phi(M^n)$ is a geodesic hypersphere.

**References**


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