A CHARACTERIZATION OF ROUND SPHERES

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Abstract. A new characterization of geodesic spheres in the simply connected space forms in terms of higher order mean curvatures is given: An immersion of an \( n \) dimensional compact oriented manifold without boundary into \( n + 1 \) dimensional Euclidean space, hyperbolic space or the open half sphere is a totally umbilic immersion if, for some \( r, r = 2, \ldots, n \), the \((r-1)\)-th mean curvature \( H_{r-1} \) does not vanish and the ratio \( H_r/H_{r-1} \) is constant.

1. Introduction

Let \( N^{n+1} \) be one of the Euclidean space \( \mathbb{R}^{n+1} \), the hyperbolic space \( \mathbb{H}^{n+1} \) or the open half sphere \( S^+_{n+1} \), and let \( \phi : M^n \rightarrow N^{n+1} \) be an isometric immersion of a compact oriented \( n \) dimensional manifold without boundary \( M^n \). Let \( H_r \) denote the \( r \)-th mean curvature of \( M^n \), that is, \( H_r \) is the \( r \)-th elementary symmetric polynomial of principal curvatures of \( M^n \) divided by \( \binom{n}{r} \), and \( H_0 \) is defined to be 1. For instance, \( H_1 \) is the usual mean curvature and \( H_n \) is the Gauss-Kronecker curvature.

It was shown in [3] that if two consecutive mean curvatures \( H_{r-1}, H_r \) are constant for some \( r \) between 2 and \( n \), then \( \phi(M^n) \) is a geodesic sphere. In this note, we generalize this result in the following way.

Theorem. Let \( N^{n+1} \) be one of \( \mathbb{R}^{n+1}, \mathbb{H}^{n+1} \) or \( S^+_{n+1} \), and let \( \phi : M^n \rightarrow N^{n+1} \) be an isometric immersion of a compact oriented \( n \)-dimensional manifold without boundary \( M^n \). If \( H_{r-1} \) does not vanish and the ratio \( H_r/H_{r-1} \) of two consecutive mean curvatures is a constant for some \( r = 2, \ldots, n \), then \( \phi(M^n) \) is a geodesic hypersphere.

When \( N^{n+1} \) is \( \mathbb{R}^3 \), the above theorem was proved in [1] under the convexity assumption. Note that we consider immersions, and the case \( r = 1 \) is omitted. As we defined \( H_0 \) to be 1, the ratio \( H_1/H_0 \) is equal to the usual mean curvature \( H_1 \). Because of the existence of nonspherical immersions of nonzero constant mean curvature \( H_1 \) in \( \mathbb{R}^n \), proved by Hsiang, Teng and Yu [4] and Wente [6], we cannot expect our theorem for \( r = 1 \). However, by Alexandrov’s well-known sphere theorem

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or Montiel and Ros’ theorem [5], our theorem holds even for \( r = 1 \) if we assume embeddings.

2. Proof

We use the hyperboloid model for \( H^{n+1} \) and the usual embedding of \( S^{n+1} \) into \( R^{n+2} \). Let \( \eta \) denote a unit normal field on \( M^n \). We use the following Minkowski formula (for proof, see [5]), where \( \langle \cdot, \cdot \rangle \) denotes the usual Euclidean inner product on \( R^{n+1} \) (on \( R^{n+2} \)) when \( N^{n+1} = R^{n+1} \) (when \( N^{n+1} = S^{n+1}_+ \)) and the Lorentzian inner product on \( R^{n+2} \) when \( N^{n+1} = H^{n+1} \).

**Lemma 1.** For \( k = 1, \ldots, n \), the following identities hold:

1. When \( N^{n+1} = R^{n+1} \),
   \[
   \int_M (H_{k-1} + H_k \langle \phi, \eta \rangle) \, dM = 0.
   \]

2. When \( N^{n+1} = H^{n+1} \),
   \[
   \int_M (H_{k-1} \langle \phi, p \rangle + H_k \langle \eta, p \rangle) \, dM = 0 \quad \text{for any } p \in R^{n+2}.
   \]

3. When \( N^{n+1} = S^{n+1}_+ \),
   \[
   \int_M (H_{k-1} \langle \phi, p \rangle - H_k \langle \eta, p \rangle) \, dM = 0 \quad \text{for any } p \in R^{n+2}.
   \]

We also use the following inequalities for higher order mean curvatures (for proof, see, for example, §12 of [2]):

**Lemma 2.** (1) \( H_r H_{r-2} \leq H_{r-1}^2 \).

(2) If \( H_1, H_2 \) are greater than zero everywhere on \( M^n \), then

\[
H_2 \leq H_1^2
\]

and the equality holds only at umbilical points.

Now, assume \( H_{r-1} \) does not vanish and \( H_r/H_{r-1} = a \) for a constant number \( a \) and a fixed \( r = 2, \ldots, n \).

(2.1) **Proof when** \( N^{n+1} = R^{n+1} \). Since \( M^n \) is compact, by changing the unit normal vector on \( M \) if necessary, one can find a point in \( M^n \) where all the principal curvatures are positive. Then \( H_r, H_{r-1} \) are positive at that point. Since \( H_{r-1} \) does not vanish and \( H_r/H_{r-1} \) is constant on \( M^n \), \( H_r \) is positive on \( M^n \). Then, following the argument of Lemma 1 in [5], it follows that the \( H_k \) are positive on \( M^n \) for \( k = 1, \ldots, r-1 \). Then \( a > 0 \) and from the inequality (1)

\[
H_r H_{r-2} \leq H_{r-1}^2
\]

of Lemma 2, we have

(*) \[ 0 < a = H_r/H_{r-1} \leq H_{r-1}/H_{r-2}. \]

Since \( H_r = aH_{r-1} \), we have by Lemma 1,

\[
0 = \int_M (H_{r-1} + H_r \langle \phi, \eta \rangle) \, dM = \int_M (H_{r-1} + aH_{r-1} \langle \phi, \eta \rangle) \, dM,
\]
that is,
\[ \int_M H_{r-1} \langle \phi, \eta \rangle \, dM = \int_M \left( -\frac{1}{a} H_{r-1} \right) \, dM. \]

Since
\[ \int_M (H_{r-2} + H_{r-1} \langle \phi, \eta \rangle) \, dM = 0 \]
we then have
\[ \int_M (H_{r-2} - \frac{1}{a} H_{r-1}) \, dM = 0. \]

Since \( H_{r-2} - \frac{1}{a} H_{r-1} \leq 0 \) from (*), it follows that
\[ H_{r-1}/H_{r-2} = a = H_r/H_{r-1}, \]
everywhere on \( M^n \). Now, proceeding inductively, we have finally
\[ H_{2}/H_1 = a = H_1/H_0 = H_1 \]
everywhere on \( M^n \), that is, the equality holds in (2) of Lemma 2. Hence every point is an umbilical point, i.e., \( \phi(M^n) \) is a geodesic hypersphere.

(2.2) Proof when \( N^{n+1} = H^{n+1} \). At a point of \( M^n \) where the distance function of \( H^{n+1} \) attains its maximum, all the principal curvatures are positive. Then (*) also holds in this case, and the \( H_k \) are positive on \( M^n \) for \( k = 1, \ldots, r-1 \). Since \( H_r = aH_{r-1} \), we have
\[ 0 = \int_M (H_{r-1} \langle \phi, p \rangle + H_r \langle \eta, p \rangle) \, dM \]
\[ = \int_M (H_{r-1} \langle \phi, p \rangle + aH_{r-1} \langle \eta, p \rangle) \, dM, \]
that is,
\[ \int_M H_{r-1} \langle \eta, p \rangle \, dM = \int_M \left( -\frac{1}{a} H_{r-1} \langle \phi, p \rangle \right) \, dM. \]

Since
\[ \int_M (H_{r-2} \langle \phi, p \rangle + H_{r-1} \langle \eta, p \rangle) \, dM = 0, \]
it follows that
\[ \int_M (H_{r-2} - \frac{1}{a} H_{r-1}) \langle \phi, p \rangle \, dM = 0. \]

Now, if we take \( p = (1, 0, \ldots, 0) \in \mathbb{R}^{n+2} \), then the sign of \( \langle \phi, p \rangle \) does not change on \( M^n \). Since \( H_{r-2} - \frac{1}{a} H_{r-1} \leq 0 \) from (*), we have \( H_{r-2} - \frac{1}{a} H_{r-1} = 0 \) everywhere on \( M^n \). Proceeding inductively as in (2.1), we can show that every point is an umbilical point. Hence \( \phi(M^n) \) is a geodesic hypersphere.
(2.3) **Proof when** \( N^{n+1} = S^{n+1}_+ \). Let \( c \in S^{n+1} \) be the centre of \( S^{n+1}_+ \). Then at a point of \( M^n \) where the height function \( \langle \phi, c \rangle \) attains its maximum, all the principal curvatures are positive because \( M^n \) lies in the open half sphere with the centre \( c \). Then (*) holds, and the equality in (*) holds only at umbilical points. Proceeding as in (2.2), we have

\[
\int_M (H_{r-2} - \frac{1}{a} H_{r-1}) \langle \phi, p \rangle \, dM = 0.
\]

Since \( M^n \) lies in the open half sphere, one can find a vector \( p \in \mathbb{R}^{n+2} \) so that \( \langle \phi, p \rangle \) is positive on \( M^n \). Then, since \( H_{r-2} - \frac{1}{a} H_{r-1} \leq 0 \) by (*), it follows that \( H_{r-2} - \frac{1}{a} H_{r-1} = 0 \), everywhere on \( M^n \). Now, arguing in the same way as above, we can show that \( \phi(M^n) \) is a geodesic hypersphere.

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