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A CHARACTERIZATION OF ROUND SPHERES

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Abstract. A new characterization of geodesic spheres in the simply connected space forms in terms of higher order mean curvatures is given: An immersion of an n dimensional compact oriented manifold without boundary into $n+1$ dimensional Euclidean space, hyperbolic space or the open half sphere is a totally umbilic immersion if, for some $r,\ r = 2, \dots, n$, the $(r-1)$-th mean curvature H_{r-1} does not vanish and the ratio H_r/H_{r-1} is constant.

1. Introduction

Let N^{n+1} be one of the Euclidean space $\mathbf{R}^{n+1}$, the hyperbolic space $\mathbf{H}^{n+1}$ or the open half sphere $\mathbf{S}_+^{n+1}$, and let $\phi : M^n \to N^{n+1}$ be an isometric immersion of a compact oriented n dimensional manifold without boundary M^n. Let H_r denote the r-th mean curvature of M^n, that is, H_r is the r-th elementary symmetric polynomial of principal curvatures of M^n divided by $\binom{n}{r}$, and H_0 is defined to be 1. For instance, H_1 is the usual mean curvature and H_n is the Gauss-Kronecker curvature.

It was shown in [3] that if two consecutive mean curvatures H_{r-1}, H_r are constant for some r between 2 and n, then $\phi(M^n)$ is a geodesic sphere. In this note, we generalize this result in the following way.

Theorem. *Let N^{n+1} be one of $\mathbf{R}^{n+1}, \mathbf{H}^{n+1}$ or $\mathbf{S}_+^{n+1}$, and let $\phi : M^n \to N^{n+1}$ be an isometric immersion of a compact oriented n-dimensional manifold without boundary M^n. If H_{r-1} does not vanish and the ratio H_r/H_{r-1} of two consecutive mean curvatures is a constant for some $r = 2, \dots, n$, then $\phi(M^n)$ is a geodesic hypersphere.*

When N^{n+1} is $\mathbf{R}^3$, the above theorem was proved in [1] under the convexity assumption. Note that we consider immersions, and the case $r = 1$ is omitted. As we defined H_0 to be 1, the ratio H_1/H_0 is equal to the usual mean curvature H_1. Because of the existence of nonspherical immersions of nonzero constant mean curvature H_1 in $\mathbf{R}^n$, proved by Hsiang, Teng and Yu [4] and Wente [6], we cannot expect our theorem for $r = 1$. However, by Alexandrov's well-known sphere theorem

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or Montiel and Ros' theorem [5], our theorem holds even for $r = 1$ if we assume embeddings.

2. Proof

We use the hyperboloid model for $\mathbf{H}^{n+1}$ and the usual embedding of $\mathbf{S}^{n+1}$ into $\mathbf{R}^{n+2}$. Let η denote a unit normal field on M^n. We use the following Minkowski formula (for proof, see [5]), where $\langle\ ,\ \rangle$ denotes the usual Euclidean inner product on $\mathbf{R}^{n+1}$ (on $\mathbf{R}^{n+2}$) when N^{n+1} is $\mathbf{R}^{n+1}$ (when N^{n+1} is $\mathbf{S}^{n+1}_+$) and the Lorentzian inner product on $\mathbf{R}^{n+2}$ when N^{n+1} is $\mathbf{H}^{n+1}$.

Lemma 1. *For $k = 1, \ldots, n$, the following identities hold:*

(1) *When N^{n+1} is $\mathbf{R}^{n+1}$,*

$$\int_M (H_{k-1} + H_k\langle\phi, \eta\rangle)\, dM = 0.$$

(2) *When N^{n+1} is $\mathbf{H}^{n+1}$,*

$$\int_M (H_{k-1}\langle\phi, p\rangle + H_k\langle\eta, p\rangle)\, dM = 0 \quad \textit{for any } p \in \mathbf{R}^{n+2}.$$

(3) *When N^{n+1} is $\mathbf{S}^{n+1}_+$,*

$$\int_M (H_{k-1}\langle\phi, p\rangle - H_k\langle\eta, p\rangle)\, dM = 0 \quad \textit{for any } p \in \mathbf{R}^{n+2}.$$

We also use the following inequalities for higher order mean curvatures (for proof, see, for example, §12 of [2]):

Lemma 2. (1) $H_r H_{r-2} \le {H_{r-1}}^2$.

(2) *If H_1, H_2 are greater than zero everywhere on M^n, then*

$$H_2 \le {H_1}^2$$

and the equality holds only at umbilical points.

Now, assume H_{r-1} does not vanish and $H_r/H_{r-1} = a$ for a constant number a and a fixed $r = 2, \ldots, n$.

(2.1) Proof when $N^{n+1} = \mathbf{R}^{n+1}$. Since M^n is compact, by changing the unit normal vector on M if necessary, one can find a point in M^n where all the principal curvatures are positive. Then H_r, H_{r-1} are positive at that point. Since H_{r-1} does not vanish and H_r/H_{r-1} is constant on M^n, H_r is positive on M^n. Then, following the argument of Lemma 1 in [5], it follows that the H_k are positive on M^n for $k = 1, \ldots, r-1$. Then $a > 0$ and from the inequality (1)

$$H_r H_{r-2} \le {H_{r-1}}^2$$

of Lemma 2, we have

$$(*) \qquad 0 < a = H_r/H_{r-1} \le H_{r-1}/H_{r-2}.$$

Since $H_r = aH_{r-1}$, we have by Lemma 1,

$$\begin{aligned} 0 &= \int_M (H_{r-1} + H_r\langle\phi, \eta\rangle)\, dM \\ &= \int_M (H_{r-1} + aH_{r-1}\langle\phi, \eta\rangle)\, dM, \end{aligned}$$

that is,

$$\int_M H_{r-1}\langle \phi, \eta\rangle \, dM = \int_M (-\frac{1}{a} H_{r-1}) \, dM.$$

Since

$$\int_M (H_{r-2} + H_{r-1}\langle \phi, \eta\rangle) \, dM = 0$$

we then have

$$\int_M (H_{r-2} - \frac{1}{a} H_{r-1}) \, dM = 0.$$

Since $H_{r-2} - \frac{1}{a}H_{r-1} \leq 0$ from (*), it follows that

$$H_{r-1}/H_{r-2} = a = H_r/H_{r-1},$$

everywhere on M^n. Now, proceeding inductively, we have finally

$$H_2/H_1 = a = H_1/H_0 = H_1$$

everywhere on M^n, that is, the equality holds in (2) of Lemma 2. Hence every point is an umbilical point, i.e., $\phi(M^n)$ is a geodesic hypersphere.

(2.2) Proof when $N^{n+1} = \mathbf{H}^{n+1}$. At a point of M^n where the distance function of $\mathbf{H}^{n+1}$ attains its maximum, all the principal curvatures are positive. Then (*) also holds in this case, and the H_k are positive on M^n for $k = 1, \ldots, r-1$. Since $H_r = aH_{r-1}$, we have

$$\begin{aligned} 0 &= \int_M (H_{r-1}\langle \phi, p\rangle + H_r\langle \eta, p\rangle) \, dM \\ &= \int_M (H_{r-1}\langle \phi, p\rangle + aH_{r-1}\langle \eta, p\rangle) \, dM, \end{aligned}$$

that is,

$$\int_M H_{r-1}\langle \eta, p\rangle \, dM = \int_M (-\frac{1}{a} H_{r-1}\langle \phi, p\rangle) \, dM.$$

Since

$$\int_M (H_{r-2}\langle \phi, p\rangle + H_{r-1}\langle \eta, p\rangle) \, dM = 0,$$

it follows that

$$\int_M (H_{r-2} - \frac{1}{a} H_{r-1})\langle \phi, p\rangle \, dM = 0.$$

Now, if we take $p = (1, 0, \ldots, 0) \in \mathbf{R}^{n+2}$, then the sign of $\langle \phi, p\rangle$ does not change on M^n. Since $H_{r-2} - \frac{1}{a}H_{r-1} \leq 0$ from (*), we have $H_{r-2} - \frac{1}{a}H_{r-1} = 0$ everywhere on M^n. Proceeding inductively as in (2.1), we can show that every point is an umbilical point. Hence $\phi(M^n)$ is a geodesic hypersphere.

(2.3) Proof when $N^{n+1} = \mathbf{S}^{n+1}_+$. Let $c \in \mathbf{S}^{n+1}$ be the centre of $\mathbf{S}^{n+1}_+$. Then at a point of M^n where the height function $\langle \phi, c \rangle$ attains its maximum, all the principal curvatures are positive because M^n lies in the open half sphere with the centre c. Then (*) holds, and the equality in (*) holds only at umbilical points. Proceeding as in (2.2), we have

$$\int_M (H_{r-2} - \frac{1}{a} H_{r-1}) \langle \phi, p \rangle \, dM = 0.$$

Since M^n lies in the open half sphere, one can find a vector $p \in \mathbf{R}^{n+2}$ so that $\langle \phi, p \rangle$ is positive on M^n. Then, since $H_{r-2} - \frac{1}{a} H_{r-1} \leq 0$ by (*), it follows that $H_{r-2} - \frac{1}{a} H_{r-1} = 0$, everywhere on M^n. Now, arguing in the same way as above, we can show that $\phi(M^n)$ is a geodesic hypersphere.

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