

STABILITY OF THE WULFF SHAPE

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ABSTRACT. We consider the functional of a hypersurface, given by a convex elliptic integrand with a volume constraint. We show that, up to homothety and translation, the only closed, oriented, stable critical point is the Wulff shape.

1. INTRODUCTION

Let $F : S^n \rightarrow \mathbf{R}^+$ be a smooth function and denote its gradient with respect to the standard metric on S^n by DF . Consider the map

$$\begin{aligned}\phi : S^n &\rightarrow \mathbf{R}^{n+1}, \\ \nu &\rightarrow F\nu + DF,\end{aligned}$$

and consider the (possibly singular) hypersurface defined by $W_F := \phi(S^n)$. The differential of ϕ is given by

$$(1) \quad d\phi_\nu = (D^2F + F1)_\nu,$$

where D^2F denotes the intrinsic Hessian of F on S^n and 1 denotes the identity on $T_\nu S^n$. If we impose the condition that

$$(2) \quad (D^2F + F1)_\nu > 0, \quad \forall \nu \in S^n,$$

where > 0 means that the matrix is positive definite, then W_F is a smooth, convex hypersurface in \mathbf{R}^{n+1} called the *Wulff shape* of F . We assume from now on that this is the case.

Now let

$$X : \Sigma \rightarrow \mathbf{R}^{n+1}$$

be a smooth immersion of a compact, oriented hypersurface, possibly with nonempty boundary. Let

$$\nu : \Sigma \rightarrow S^n$$

denote its Gauss map. We assign to each such X the functional

$$(3) \quad J_F(X) := \int_\Sigma F(\nu) dA_X.$$

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Such functionals have been treated by many authors (see for example [1], page 517, and the references therein). The algebraic $(n+1)$ -volume enclosed by Σ is given by

$$V(X) = \frac{1}{n+1} \int_{\Sigma} \langle X, \nu \rangle dA_X.$$

We are interested in those hypersurfaces which are critical points of the functional J_F restricted to those hypersurfaces enclosing a fixed volume V (and spanning a fixed boundary in the case that $\partial\Sigma \neq \emptyset$). By a standard argument involving Lagrange multipliers, this means we are considering critical points of the functional

$$(4) \quad J_{F,\Lambda}(X) := \int_{\Sigma} F(\nu) dA_X + \Lambda V(X),$$

where Λ is a constant. We will show below that such critica are characterized by the Euler-Lagrange equation

$$(5) \quad \operatorname{div}_{\Sigma} DF - nhF + \Lambda = 0$$

where h is the mean curvature of Σ . In (5) the constant $\Lambda = \Lambda_{\Sigma}$ can be determined for a critical immersion.

We will call a critical immersion X *stable* if and only if the second variation of J_F (or equivalently of $J_{F,\Lambda}$) is non-negative for all (compactly supported) variations of X preserving the enclosed $(n+1)$ -volume. We have the following result.

Theorem 1. *Let $X : \Sigma \rightarrow \mathbf{R}^{n+1}$ be a smooth immersion of an oriented, closed ($\partial\Sigma = \emptyset$), stable critical point of $J_{F,\Lambda}$. Then, up to translation, $X(\Sigma) = tW_F$, where $t = -n/\Lambda_{\Sigma}$.*

A result known as Wulff's Theorem states that W_F actually *minimizes* J_F among all hypersurfaces having the same volume, so the stability of W_F is obvious. Proofs of Wulff's Theorem can be found in [3] and [4].

Note that the case $F \equiv 1$ is the widely studied class of constant mean curvature hypersurfaces. For this special case Theorem 1 was first obtained by Barbosa and do Carmo [2]. Later, Wente [5] gave a closely related but more direct proof which avoided the use of the Jacobi operator. The proof of Theorem 1 is a modification of Wente's idea to the case of non-constant F .

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2. PRELIMINARIES

Let $X : \Sigma \rightarrow \mathbf{R}^{n+1}$ be a smooth oriented hypersurface with Gauss map $\nu : \Sigma \rightarrow S^n$. Consider a variation of X of the form

$$X_t = X + t(\psi\nu + \xi) + O(t^2)$$

where ψ is a smooth function on Σ and ξ is tangent to X . (Both ψ and ξ are assumed to have compact support if $\partial\Sigma \neq \emptyset$.) The corresponding first variation of the normal is given by

$$\delta\nu = -\nabla\psi + d\nu(\xi)$$

and the first variation of the volume element is

$$\delta dA = (\operatorname{div}\xi - nh\psi)dA.$$

We use this to compute the first variation of J_F ,

$$\begin{aligned} \delta J_F(X) &= \int_{\Sigma} (\langle DF(\nu), \delta\nu \rangle + F(\nu)\delta dA) \\ &= \int_{\Sigma} (\langle DF(\nu), -\nabla\psi \rangle - nhF\psi dA) \\ &\quad + \int_{\Sigma} (\langle DF(\nu), d\nu(\xi) \rangle + F\operatorname{div}\xi) dA. \end{aligned}$$

Note that $T_{\nu(p)}S^n = dX(T_p\Sigma)$, so that if we denote the inner product on this plane by $\langle \cdot, \cdot \rangle$, then

$$(6) \quad \langle (\nabla F \circ \nu), \cdot \rangle = d(F \circ \nu) = \nu^*dF = \langle DF(\nu), d\nu(\cdot) \rangle = \langle d\nu(DF), \cdot \rangle.$$

From this it follows that

$$(7) \quad \nabla(F \circ \nu) = d\nu(DF)$$

Integrating by parts, we then obtain

$$\begin{aligned} \delta J_F(X) &= \int_{\Sigma} (\psi(\operatorname{div}DF - nhF)) dA + \int_{\Sigma} (\langle d\nu(DF), \xi \rangle + F\operatorname{div}\xi) dA \\ &= \int_{\Sigma} (\psi(\operatorname{div}DF - nhF)) dA + \int_{\Sigma} (\langle \nabla F, \xi \rangle + F\operatorname{div}\xi) dA \\ &= \int_{\Sigma} (\psi(\operatorname{div}DF - nhF)) dA + \int_{\Sigma} (-F\operatorname{div}\xi + F\operatorname{div}\xi) dA \\ &= \int_{\Sigma} (\psi(\operatorname{div}DF - nhF)) dA. \end{aligned}$$

Finally, we recall that the first variation of the volume $V(X)$ is

$$\delta V(X) = \int_{\Sigma} \psi dA$$

Hence we obtain the Euler-Lagrange equation for the functional $J_{F,\Lambda}$:

$$(8) \quad \operatorname{div}DF - nhF + \Lambda = 0.$$

Under the convexity condition (2) this equation is elliptic. In fact its linearization is given by the elliptic, self-adjoint Jacobi operator

$$L[\psi] = \operatorname{div}((D^2F + F1)\nabla\psi) + \psi\langle d\nu, (D^2F + F1) \circ d\nu \rangle.$$

Later we will need the following lemma

Lemma 2. *Let $X : \Sigma \rightarrow \mathbf{R}^{n+1}$ be an oriented, smooth, critical immersion of $J_{F,\Lambda}$. Then*

$$(9) \quad (n + 1)\Lambda V(X) = -n \int_{\Sigma} F dA.$$

Proof. Compute

$$(10) \quad \operatorname{div}(F\nabla|X|^2/2) = \langle \nabla F, \nabla|X|^2/2 \rangle + F\Delta|X|^2/2.$$

A well-known formula gives

$$(11) \quad \Delta|X|^2/2 = nh\sigma + n,$$

where $\sigma = \langle X, \nu \rangle$ is the support function. Therefore

$$\begin{aligned} \operatorname{div}(F\nabla|X|^2/2) &= \langle \nabla F, \nabla|X|^2/2 \rangle + nh\sigma F + nF \\ &= \langle d\nu(DF), \nabla|X|^2/2 \rangle + nh\sigma F + nF \\ &= \langle DF, d\nu(\nabla|X|^2/2) \rangle + nh\sigma F + nF \\ &= \langle DF, \nabla\sigma \rangle + nh\sigma F + nF, \end{aligned}$$

using the fact that $d\nu(\nabla|X|^2/2) = \nabla\sigma$. On the other hand,

$$\operatorname{div}(\sigma DF) = \langle \nabla\sigma, DF \rangle + \sigma \operatorname{div}DF = \langle \nabla\sigma, DF \rangle + nh\sigma F - \Lambda\sigma.$$

It then follows that

$$0 = \int_{\Sigma} \operatorname{div}(F\nabla|X|^2/2) - \sigma DF dA = \int_{\Sigma} (nF + \Lambda\sigma) dA = n \int_{\Sigma} F dA + (n+1)V(X).$$

□

3. MAIN RESULT

Let $X : \Sigma \rightarrow \mathbf{R}^{n+1}$ be as above. We first consider the variation of X given by

$$(12) \quad X_t = X + t\phi(\nu) = X + t(F\nu + DF).$$

We then determine a function $s(t)$ such that

$$\tilde{X}_t := s(t)X_t$$

satisfies

$$\tilde{X}_0 = X, \quad V(\tilde{X}_t) \equiv \text{const.}$$

We then show that $\partial_{tt}^2 J_F(\tilde{X}_t) \leq 0$ holds with equality if and only if the image of X is a multiple of W_F .

Clearly,

$$J_F(\tilde{X}_t) = J_F(sX_t) = s^n J(X_t),$$

so that

$$(13) \quad \begin{aligned} \partial_{tt}^2 J_F(\tilde{X}_t)_{t=0} &= (ns''(0) + n(n-1)(s'(0))^2) J_F(X) \\ &\quad + 2ns'(0)\partial_t J_F(X)_{t=0} + \partial_{tt}^2 J_F(X)_{t=0}. \end{aligned}$$

Note that

$$(14) \quad dX_t = dX + t d(\phi \circ \nu) = dX + t((D^2F + F1) \circ d\nu) \perp \nu.$$

This implies that if ν_t is the Gauss map of X_t then $\nu_t \equiv \nu$. It follows that

$$(15) \quad \partial_t J_F(X)_{t=0} = \int_{\Sigma} F(\nu) \partial_t (dA_{X_t})_{t=0}$$

and

$$(16) \quad \partial_{tt}^2 J_F(X)_{t=0} = \int_{\Sigma} F(\nu) \partial_{tt}^2 (dA_{X_t})_{t=0}.$$

Observe that since $\partial_t(X_t)_{t=0} = DF + F\nu$, we have

$$(17) \quad \partial_t(dA_{X_t})_{t=0} = (\operatorname{div}DF - nhF)dA = -\Lambda dA.$$

In the following lemma we will compute the coefficients in (13).

Lemma 3. $s'(0) = \Lambda/n$, and $s''(0) = 2\Lambda^2/n^2$.

Proof. Recall that $s(t)$ is defined by the condition

$$(18) \quad V(\tilde{X}_t) = s^{n+1}(t)V(X_t) \equiv V(X).$$

Differentiating this gives

$$(19) \quad (n + 1)s'(t)s^n(t)V(X_t) + s^{n+1}(t)\partial_t V(X_t) \equiv 0.$$

At any time t , the first variation of the volume is

$$(20) \quad \partial_t V(X_t) = \int_{\Sigma} \langle \partial_t X, \nu_t \rangle dA_{X_t} = \int_{\Sigma} F(\nu) dA_{X_t},$$

since the normal is constant in t . In particular, when $t = 0$ this gives

$$(21) \quad \partial_t V(X_t)_{t=0} = \int_{\Sigma} F(\nu) dA_X,$$

so $s'(0) = \Lambda/n$ by Lemma 2. Also, differentiating (19) and setting $t = 0$ gives

$$(22) \quad \begin{aligned} &((n + 1)s''(0) + n(n + 1)(s'(0))^2)V(X) \\ &+ 2(n + 1)s'(0)\partial_t V(X_t)_{t=0} + \partial_{tt}^2 V(X_t)_{t=0} = 0. \end{aligned}$$

It then follows from Lemma 2, (20), and (17) that $s''(0) = 2\Lambda^2/n^2$. □

Using the lemma and (13), we obtain

$$(23) \quad \partial_{tt}^2 J_F(\tilde{X}_t)_{t=0} = \int_{\Sigma} (F(\Lambda^2(\frac{1}{n} - 1)dA + \partial_{tt}^2(dA_{X_t})_{t=0})).$$

Let $\{e_i\}_{i=1,\dots,n}$ be a local orthonormal frame for the metric induced by X defined on a neighborhood in Σ . Let $g_t = (g_{i,j}(t))$ be the matrix representation of the metric induced by X_t with respect to this frame, so that locally

$$dA_{X_t} = (\det g_t)^{1/2} dA_X.$$

Since $g_0 = id$ we have

$$(24) \quad (\det g_t)'(0) = \text{tr}(g'(0)) = -2\Lambda,$$

$$(25) \quad (\det g_t)''(0) = (\text{tr}(g'(0)))^2 + \text{tr}(g''(0)) - \text{tr}(g'(0))^2,$$

so that

$$\begin{aligned} ((\det g_t)^{1/2})''(0) &= -\frac{1}{4}(\text{tr} g')^2 + \frac{1}{2}(\text{tr} g')^2 + \frac{1}{2}(\text{tr} g'') - \frac{1}{2}\text{tr}(g')^2 \\ &= \Lambda^2 + \frac{1}{2}(\text{tr} g'') - \text{tr}(g')^2. \end{aligned}$$

Locally, we have

$$\begin{aligned} g'_{i,j} &= \langle X'_i, X_j \rangle + \langle X_i, X'_j \rangle, \\ g''_{i,j} &= 2\langle X'_i, X'_j \rangle. \end{aligned}$$

Let $A_{i,j} = \langle X'_i, X_j \rangle$. Then by using the fact that dX' is tangential, we have

$$\text{tr}(g')^2 = \text{tr}(A + A^t)^2 = |A + A^t|^2 \quad \text{tr}(g'') = 2|A|^2.$$

Therefore

$$\begin{aligned} \frac{1}{2}(\text{tr} g'') - \frac{1}{2}\text{tr}(g')^2 &= \frac{1}{2}(2 \sum_{i,j} A_{i,j}^2 - \sum_{i,j} (A_{i,j} + A_{j,i})^2) \\ &= - \sum_{i,j} (A_{i,j} A_{j,i}). \end{aligned}$$

Therefore in (23) we have, locally,

$$(26) \quad F(\Lambda^2(\frac{1}{n} - 1)dA + \partial_{tt}^2(dA_{X_t})_{t=0} = \Lambda^2/n - \sum_{i,j} (A_{i,j}A_{j,i}).$$

We now choose our frame more carefully. Fix $p \in \Sigma$ and note that $dX(T_p\Sigma) = T_{\nu(p)}S^n$. We choose an orthonormal frame so that $\{e_i(p)\}$ diagonalizes $D^2F(\nu(p))$. This is possible since the Hessian is symmetric. With respect to this basis, the matrix representing $(D^2F + F1)_{\nu(p)}$ has the form $diag(\mu_1, \dots, \mu_n)$, with $\mu_i > 0$ for all i by the convexity condition. Let $(\nu_{i,j})$ be the (symmetric) matrix representing $d\nu(p)$ with respect to the given basis. Then at p we have

$$\langle X'_i, X_j \rangle = A_{i,j} = (diag(\mu_1, \dots, \mu_n)(d\nu))_{i,j} = (\mu_i\nu_{i,j}).$$

Then

$$\begin{aligned} \Lambda^2/n - \sum_{i,j} (A_{i,j}A_{j,i}) &= 1/n(\sum_{i=1,\dots,n} \mu_i\nu_{i,i})^2 - \sum_{i,j=1,\dots,n} \mu_i\mu_j\nu_{i,j}^2 \\ &\leq (1/n)(n) \sum_{i=1,\dots,n} \mu_i^2\nu_{i,i}^2 - \sum_{i=1,\dots,n} \mu_i^2\nu_{i,i}^2 - \sum_{i \neq j} \mu_i\mu_j\nu_{i,j}^2 \\ &= - \sum_{i \neq j} \mu_i\mu_j\nu_{i,j}^2 \leq 0. \end{aligned}$$

Equality holds between the first two lines if and only if $\mu_i\nu_{i,i} \equiv t, i = 1, \dots, n$, for some number t . Equality holds in the last line if and only if $\nu_{i,j} = 0$ for all $i \neq j$. If this is the case, then

$$\sum \mu_i\nu_{i,i} = nt = -\Lambda_\Sigma,$$

so t does not depend on the point. Therefore the integrand in (26) is non-positive and vanishes identically if and only if at each point

$$(27) \quad (D^2F + F1)_\nu d\nu \equiv -(\Lambda_\Sigma/n)1.$$

However, this implies that Σ is convex and in particular that ν is a diffeomorphism of Σ onto S^n . Let σ denote the support function of Σ and let $\tilde{\sigma} := \sigma \circ \nu^{-1}$. Then the map ν^{-1} is a reparametrization of $X(\Sigma)$, and it can be expressed as

$$(28) \quad \nu^{-1} := (D\tilde{\sigma} + \tilde{\sigma}\nu) : S^n \longrightarrow \mathbf{R}^n.$$

Therefore we have

$$1 = d\nu^{-1} \circ d\nu = (D^2\tilde{\sigma} + \tilde{\sigma}1) \circ d\nu.$$

It then follows from (27) that

$$D^2(\tilde{\sigma} + F/\Lambda) + (\tilde{\sigma} + F/\Lambda)1 \equiv 0,$$

so that integrating gives

$$D(\tilde{\sigma} + F/\Lambda) + (\tilde{\sigma} + F/\Lambda)\nu \equiv \vec{c} = const.$$

The conclusion then follows.

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