ON SUBWAVELET SETS

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Abstract. In this note we give a characterization of subwavelet sets and show that any point \( x \in \mathbb{R}\setminus\{0\} \) has a neighborhood which is contained in a regularized wavelet set.

In [1] the notion of a wavelet set was introduced and in [8] subwavelet sets were considered. Wavelet sets were also introduced independently and simultaneously as the support sets of MSF (Minimally Supported Frequency) wavelets in the sequence of papers [3], [5], and [6]. (See also the recent excellent book [4].) The purpose of this note is to provide a characterization of the subwavelet sets and to use this characterization to prove that every point \( x \in \mathbb{R}\setminus\{0\} \) has a neighborhood contained in a regularized wavelet set. (Regularized wavelet sets are wavelet sets with certain nice properties; see [7].) In particular, this shows that the union of the interiors of all wavelet sets is \( \mathbb{R}\setminus\{0\} \).

We begin by introducing some preliminary terminology and notation. The measure space under consideration will always be \( \mathbb{R} \) together with its \( \sigma \)-ring \( \mathcal{L} \) of Lebesgue measurable subsets and Lebesgue measure \( \mu \). Recall (cf. [1]) that a function \( w \in L^2(\mathbb{R}) := L^2(\mathbb{R}, \mathcal{L}, \mu) \) is a wavelet if the family of (equivalence classes of) functions \( \{w_{j,k}\}_{j,k \in \mathbb{Z}} \) defined by

\[
    w_{j,k}(s) = 2^{j/2}w(2^js + k), \quad s \in \mathbb{R}, \ j, k \in \mathbb{Z},
\]

is an orthonormal basis for \( L^2(\mathbb{R}) \). A subset \( G \) of \( \mathbb{R} \) with positive measure is a wavelet set if \( \frac{1}{\sqrt{\mu(G)}} \chi_G = F(w) \), where \( w \) is a wavelet in \( L^2(\mathbb{R}) \) and \( F \) is the Fourier-Plancherel transform on \( L^2(\mathbb{R}) \). A measurable subset \( G \) of \( \mathbb{R} \) is called a regularized wavelet set if the family \( \{G + 2k\pi\}_{k \in \mathbb{Z}} \) is a partition of \( \mathbb{R} \) and the family \( \{2kG\}_{k \in \mathbb{Z}} \) is a partition of \( \mathbb{R}\setminus\{0\} \). For two measurable subsets \( F \) and \( G \) of \( \mathbb{R} \), we write \( F \sim G \) if \( \mu(F \cap G) = 0 \). It is proved in [7] that if \( W \) is any wavelet set, then there exists a regularized wavelet set \( W' \) such that \( W' \sim W \). A measurable subset \( G \) of \( \mathbb{R} \) is translation congruent modulo \( 2\pi \) to a (measurable) set \( H \subset \mathbb{R} \) if there exists a measurable bijection \( \varphi : G \to \varphi(G) \) such that \( \varphi(s) - s \) is an integral multiple of \( 2\pi \) for every \( s \) in \( G \) and \( \varphi(G) \sim H \). Analogously, \( G \subset \mathbb{R}\setminus\{0\} \) is said to be dilation congruent modulo 2 to a (measurable) set \( H \) if there exists a measurable bijection \( \psi : G \to \psi(G) \) such that \( \psi(s)/s \) is an integral power of 2 for every \( s \) in \( G \) and \( \psi(G) \sim H \). Let \( \tau : \mathbb{R} \to E := [-2\pi, -\pi) \cup [\pi, 2\pi) \) be the function defined by \( \tau(x) = x + 2j\pi \), where \( j \) is the unique integer satisfying \( x + 2j\pi \in E \), and let \( \delta : \mathbb{R}\setminus\{0\} \to E \) be the function defined by \( \delta(x) = 2^kx \), where \( k \) is the unique integer smaller than \( \frac{\log|x|}{\log 2} \).
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that are subwavelet sets.
Remark 1. In what follows we use the elementary facts that if \( G \in \mathbb{L} \cap E \), then 
\( \tau^{-1}(G), \delta^{-1}(G) \in \mathbb{L} \), and if \( H \in \mathbb{L} \) [resp., \( H \in (\mathbb{R}\setminus\{0\}) \cap \mathbb{L} \), then \( \tau(G) \in \mathbb{L} \) \[ \delta(G) \in \mathbb{L} \].

A measurable subset \( G \) of \( \mathbb{R} \) is called a subwavelet set if it is a subset of some regularized wavelet set. Our principal result characterizes measurable subsets of \( \mathbb{R} \) that are subwavelet sets.

**Theorem 2.** A set \( G \subset \mathbb{L} \) is a subwavelet set if and only if there exist sets \( G_1 \) and \( G_2 \) in \( \mathbb{L} \), each containing \( G \), such that
1. \( \tau_{G_1} \) is a measurable bijection of \( G_1 \) onto \( E \),
2. \( \tau_{G_2} \) is a measurable injection of \( G_2 \) into \( E \),
3. \( \delta_{G_2} \) is a measurable bijection of \( G_2 \) onto \( E \), and
4. \( \delta_{G_2} \) is a measurable injection of \( G_2 \) into \( E \).

**Proof:** Suppose first that \( G \) is a subset of a regularized wavelet set \( W \). Define \( G_1 = G_2 = W \), and observe that \( (a) - (d) \) follow from the definition of a regularized wavelet set and Remark 1.

For the sufficiency, suppose that there exist measurable sets \( G_1 \) and \( G_2 \) containing \( G \) such that \( (a) - (d) \) hold. We consider the maps \( h_1, h_2 : E \to E \) defined by \( h_1 := \delta_{G_1} \circ (\tau_{G_1})^{-1} \) and \( h_2 := \tau_{G_2} \circ (\delta_{G_2})^{-1} \). It is clear that \( h_1 \) and \( h_2 \) are measurable injections. We now construct a new map \( h \) from \( h_1 \) and \( h_2 \) following the idea of the proof of the Cantor-Bernstein theorem in set theory. To increase the clarity of the presentation we write \( \tilde{E} := E \) and consider \( h_1 : E \to \tilde{E} \) and \( h_2 : \tilde{E} \to E \). We denote \( f = h_2 \circ h_1 : E \to E \) and \( g := h_1 \circ h_2 : \tilde{E} \to \tilde{E} \), and note that these maps are measurable injections by Remark 1. One can check that \( E \) and \( \tilde{E} \) can be partitioned as follows:

\[
E = E_0 \cup \bigcup_{k \in \mathbb{N}} E_k \cup E'_k,
\]

\[
\tilde{E} = \tilde{E}_0 \cup \bigcup_{k \in \mathbb{N}} \tilde{E}_k \cup \tilde{E}'_k,
\]

where

\[
E_0 = \bigcap_{j \in \mathbb{N}} f^{(j)}(E),
\]

\[
\tilde{E}_0 = \bigcap_{j \in \mathbb{N}} g^{(j)}(\tilde{E}),
\]

\[
E_k = f^{(k-1)}(E) \setminus (f^{(k-1)} \circ h_2)(\tilde{E}),
\]

\[
\tilde{E}_k = (f^{(k-1)} \circ h_2)(\tilde{E}) \setminus f^{(k)}(E), \quad k \in \mathbb{N},
\]

and

\[
\tilde{E}_k = g^{(k-1)}(\tilde{E}) \setminus (g^{(k-1)} \circ h_1)(E),
\]

\[
\tilde{E}'_k = (g^{(k-1)} \circ h_1)(E) \setminus g^{(k)}(\tilde{E}), \quad k \in \mathbb{N}.
\]

We define the map \( h : E \to \tilde{E} \) to be \( h_1 \) on \( \tilde{E} = E_0 \cup \bigcup_{k \in \mathbb{N}} E_k \) and \( h_2^{-1} \) on \( \tilde{E}' = \bigcup_{k \in \mathbb{N}} E'_k \). Since \( h_1(E_0) = \tilde{E}_0 \) and, for \( k \in \mathbb{N} \), \( h_1(E_k) = \tilde{E}_k \) and \( h_2^{-1}(E'_k) = \tilde{E}_k \), it follows that \( h \) is a bijection. We define

\[
W = (\tau_{G_1})^{-1}(\tilde{E}) \cup (\tau_{G_2})^{-1}(\tilde{E}').
\]
Since $\tilde{E}'$ is a set in the range of $h_2$, it is clear from (1) that the set $W$ is translation congruent modulo $2\pi$ to $E$. Also if $x \in G$, then

$$f(\delta_{G_1}(x)) = h_2(\delta_{G_2}(x)) = \tau_{G_1}(\delta_{G_2}(\delta_{G_1}(x))) = \tau_{G_1}(x)$$

since $\delta_{G_2}(x) = \delta_{G_1}(x)$ and $\tau_{G_2}(x) = \tau_{G_1}(x)$. This shows that $\tau_{G_1}(G) \subset E_0$ and hence $G \subset W$. To complete the proof we need to check that $W$ is dilation congruent modulo 2 to $E$. This follows from the facts that $\delta_{G_1}((\tau_{G_1})^{-1}(E)) = h_1(\tilde{E})$, $\delta_{G_2}((\tau_{G_2})^{-1}(E')) = h_2^{-1}(E')$, and the function $h$ is a bijection from $E$ to $\tilde{E}(= E)$. In fact one can check that $W$ is a regularized wavelet set. \qed

**Corollary 3.** For any point $x_0 \in \mathbb{R}\setminus\{0\}$ there exists an $\varepsilon > 0$ such that the interval $I_\varepsilon := (x_0 - \varepsilon, x_0 + \varepsilon)$ is a subwavelet set.

**Proof.** It suffices to consider the case $x_0 > 0$. Choose $0 < \varepsilon < \min\{\pi/4, x_0/16\}$. We construct two sets $G_1$ and $G_2$ containing $I_\varepsilon$ and satisfying $(a) - (d)$ in Theorem 2. We write $E_+ = [\pi, 2\pi)$ and $E_- = (2\pi, -\pi]$. Note that since $\varepsilon < \min\{\pi, x_0/3\}$ the maps $\tau_{I_{\varepsilon}} : I_{\varepsilon} \to E$, $\delta_{I_{\varepsilon}} : I_{\varepsilon} \to E$ are measurable and injective. (Indeed, if $\tau(x_1) = \tau(x_2)$ with $x_1, x_2 \in I_{\varepsilon}$, then $x_1 - x_2 = 2k\pi$ for some $k \in \mathbb{Z}$, and since $|x_1 - x_2| < 2\varepsilon < 2\pi$, it follows that $k = 0$ and $x_1 = x_2$. If $\delta(x_1) = \delta(x_2)$ with $x_1, x_2 \in I_{\varepsilon}$, then $x_1/x_2 = 2^k$ for some $k \in \mathbb{Z}$. Since $\varepsilon < x_0/3$ we have

$$1/2 < (x_0 - \varepsilon)/(x_0 + \varepsilon) < x_1/x_2 < (x_0 + \varepsilon)/(x_0 - \varepsilon) < 2,$$

and so $k = 0$ and $x_1 = x_2$.) Next we show that since $\varepsilon < x_0/16$ the set $E_+ \delta(I_{\varepsilon})$ contains an interval of length greater than $3\pi/8$. To see that this is true, we observe that the set $\delta(I_{\varepsilon})$ is either an interval of length $2^k(2\varepsilon)$, where the integer $k$ is uniquely determined by the inequalities $\pi \leq 2^k x_0 < 2\pi$, or it is a union of two intervals of combined lengths no more than $2^k + 1(2\varepsilon)$. In the first case, the set $E_+ \delta(I_{\varepsilon})$ is either an interval or the union of two intervals, and if we assume that each such interval has length no greater than $3\pi/8$, we get the following contradiction:

$$\pi = \mu(E_+) = \mu(E_+ \delta(I_{\varepsilon})) + \mu(\delta(I_{\varepsilon}))$$

$$\leq 2(3\pi/8) + 2^k(2\varepsilon) < 3\pi/4 + 2^k(2x_0/16) < \pi.$$ 

In the second case (i.e., $\delta(I_{\varepsilon})$ is a union of intervals), the set $E_+ \delta(I_{\varepsilon})$ is an interval, and if we assume it has length no larger than $3\pi/8$, we get a similar contradiction:

$$\pi = \mu(E_+) = \mu(E_+ \delta(I_{\varepsilon})) + \mu(\delta(I_{\varepsilon}))$$

$$\leq (3\pi/8) + 2^k + 1(2\varepsilon) < 3\pi/4 + 2^k + 1(2x_0/16) < \pi.$$ 

Thus $2^k(2x_0 + \delta(I_{\varepsilon}))$ contains an interval of length greater than $3\pi$. Hence there exists $\ell$ in $\mathbb{N}$ such that $E_+ + 2\ell \pi \subset 2^k(E_+ \delta(I_{\varepsilon}))$. We define $G_1 = (E_\varepsilon \setminus \tau(I_{\varepsilon})) \cup I_{\varepsilon} \cup (E_\varepsilon \setminus \tau(I_{\varepsilon}))$. It is clear that $\tau(G_1) = E$. Since the maps $\tau((E_\varepsilon \setminus \tau(I_{\varepsilon})))$, $\delta_{I_{\varepsilon}}$, and $\tau((E_\varepsilon \setminus \tau(I_{\varepsilon})))$ are all injective and the sets $\tau(E_\varepsilon \setminus \tau(I_{\varepsilon}))$, $\tau(I_{\varepsilon})$, and $\tau(E_\varepsilon \setminus \tau(I_{\varepsilon}))$ are pairwise disjoint, it follows that $\tau(G_1)$ is injective and hence is a measurable bijective map. From the choice of $\ell$ we conclude that $\delta(E_\varepsilon \setminus \tau(I_{\varepsilon})) + 2(\ell\pi) \subset E_\varepsilon \setminus \tau(I_{\varepsilon})$. Hence the sets $\delta(E_\varepsilon \setminus \tau(I_{\varepsilon}))$, $\delta(I_{\varepsilon})$, and $\delta(E_\varepsilon \setminus \tau(I_{\varepsilon}))$ are pairwise disjoint, and since the maps $\delta((E_\varepsilon \setminus \tau(I_{\varepsilon})))$, $\delta(I_{\varepsilon})$, and $\delta(E_\varepsilon \setminus \tau(I_{\varepsilon}))$ are injective, it follows that $\delta(G_1)$ is a measurable injective map. Thus $G_1$ has the desired properties.

To construct $G_2$, we observe first that the collection $\{2^{-n}E_\varepsilon + 2\pi\} \cup \{2^{-n}E_\varepsilon - 2\pi\}$ is an interval partition of the set $E \setminus \{2\pi\}$. Moreover $\tau(I_{\varepsilon})$ is
either an interval of length $2\varepsilon$ or the union of two intervals of combined lengths $2\varepsilon$. Since $\varepsilon < \pi/4$, there exists an $n_0 \in \mathbb{N}$ such that $\tau(2^{-n_0}E) \cap \tau(I_\varepsilon) = \emptyset$. In other words, $\tau(2^{-n_0}E) \subset E \setminus \tau(I_\varepsilon)$. We define $G_2 = I_\varepsilon \cup 2^{-n_0}(E \setminus \tau(I_\varepsilon))$. Using arguments similar to those above, one shows that $\delta|_{G_2} : G_2 \rightarrow E$ is a measurable bijective map, and using the fact that

$$\tau(2^{-n_0}(E \setminus \delta(I_\varepsilon))) = (2^{-n_0}(E \setminus \delta(I_\varepsilon)) + 2\pi) \cup (2^{-n_0}(E \setminus \delta(I_\varepsilon)) - 2\pi) \subset \tau(2^{-n_0}E) \subset E \setminus \tau(I_\varepsilon),$$

we obtain that $\tau|_{G_2} : G_2 \rightarrow E$ is a measurable injective map. Thus $G_2$ has the desired properties, and the proof is complete.

A regularized wavelet set $W$ is called a regularized MRA-wavelet set [2] if the family $\{\tilde{W} + 2k\pi\}_{k \in \mathbb{Z}}$ is a partition of $\mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\}$, where $\tilde{W} = \bigcup_{n \in \mathbb{N}} 2^{-n}(W)$. A set is called an MRA-subwavelet set if it is a subset of a regularized MRA-wavelet set.

**Question 4.** Is there a characterization of MRA-subwavelet sets similar to that given in Theorem 2 for subwavelet sets?

**References**


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