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ON SUBWAVELET SETS

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ABSTRACT. In this note we give a characterization of subwavelet sets and show that any point $x \in \mathbb{R}\backslash 0$ has a neighborhood which is contained in a regularized wavelet set.

In [1] the notion of a wavelet set was introduced and in [8] subwavelet sets were considered. Wavelet sets were also introduced independently and simultaneously as the support sets of MSF (Minimally Supported Frequency) wavelets in the sequence of papers [3], [5], and [6]. (See also the recent excellent book [4].) The purpose of this note is to provide a characterization of the subwavelet sets and to use this characterization to prove that every point $x \in \mathbb{R}\backslash\{0\}$ has a neighborhood contained in a regularized wavelet set. (Regularized wavelet sets are wavelet sets with certain nice properties; see [7].) In particular, this shows that the union of the interiors of all wavelet sets is $\mathbb{R}\backslash\{0\}$.

We begin by introducing some preliminary terminology and notation. The measure space under consideration will always be $\mathbb{R}$ together with its σ-ring $\mathbb{L}$ of Lebesgue measurable subsets and Lebesgue measure μ. Recall (cf. [1]) that a function $w \in L^2(\mathbb{R}) := L^2(\mathbb{R}, \mathbb{L}, \mu)$ is a wavelet if the family of (equivalence classes of) functions $\{w_{j,k}\}_{j,k\in\mathbb{Z}}$ defined by

$$w_{j,k}(s) = 2^{j/2} w(2^j s + k), \quad s \in \mathbb{R},\ j, k \in \mathbb{Z},$$

is an orthonormal basis for $L^2(\mathbb{R})$. A subset G of $\mathbb{R}$ with positive measure is a *wavelet set* if $\frac{1}{\sqrt{\mu(G)}}\chi_G = \mathcal{F}(w)$, where w is a wavelet in $L^2(\mathbb{R})$ and $\mathcal{F}$ is the Fourier-Plancherel transform on $L^2(\mathbb{R})$. A measurable subset G of $\mathbb{R}$ is called a *regularized wavelet set* if the family $\{G + 2k\pi\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}$ and the family $\{2^k G\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}\backslash\{0\}$. For two measurable subsets F and G of $\mathbb{R}$, we write $F \sim G$ if $\mu(F \bigtriangledown G) = 0$. It is proved in [7] that if W is any wavelet set, then there exists a regularized wavelet set W' such that $W' \sim W$. A measurable subset G of $\mathbb{R}$ is *translation congruent modulo* 2π to a (measurable) set $H \subset \mathbb{R}$ if there exists a measurable bijection $\varphi : G \to \varphi(G)$ such that $\varphi(s) - s$ is an integral multiple of 2π for every s in G and $\varphi(G) \sim H$. Analogously, $G \subset \mathbb{R}\backslash\{0\}$ is said to be *dilation congruent modulo* 2 to a (measurable) set H if there exists a measurable bijection $\psi : G \to \psi(G)$ such that $\psi(s)/s$ is an integral power of 2 for every s in G and $\psi(G) \sim H$. Let $\tau : \mathbb{R} \to E := [-2\pi, -\pi) \cup [\pi, 2\pi)$ be the function defined by $\tau(x) = x + 2j\pi$, where j is the unique integer satisfying $x + 2j\pi \in E$, and let $\delta : \mathbb{R}\backslash\{0\} \to E$ be the function defined by $\delta(x) = 2^k x$, where k is the unique integer

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for which $2^k x \in E$. For a function $f : X \to X$ and $k \in \mathbb{Z}$ we write $f^{(0)}$ for the map $x \to x$ on X and $f^{(k)}$ for the composition of f [resp. f^{-1}] with itself $|k|$ times if $k > 0$ [resp. $k < 0$].

Remark 1. In what follows we use the elementary facts that if $G \in \mathbb{L} \cap E$, then $\tau^{-1}(G)$, $\delta^{-1}(G) \in \mathbb{L}$, and if $H \in \mathbb{L}$ [resp., $H \in (\mathbb{R}\backslash\{0\}) \cap \mathbb{L}$], then $\tau(G) \in \mathbb{L}$ [$\delta(G) \in \mathbb{L}$].

A measurable subset G of $\mathbb{R}$ is called a *subwavelet* set if it is a subset of some regularized wavelet set. Our principal result characterizes measurable subsets of $\mathbb{R}$ that are subwavelet sets.

Theorem 2. *A set $G \subset \mathbb{L}$ is a subwavelet set if and only if there exist sets G_1 and G_2 in $\mathbb{L}$, each containing G, such that*

(*a*) $\tau_{|G_1}$ *is a measurable bijection of G_1 onto E,*
(*b*) $\tau_{|G_2}$ *is a measurable injection of G_2 into E,*
(*c*) $\delta_{|G_2}$ *is a measurable bijection of G_2 onto E, and*
(*d*) $\delta_{|G_1}$ *is a measurable injection of G_2 into E.*

Proof. Suppose first that G is a subset of a regularized wavelet set W. Define $G_1 = G_2 = W$, and observe that $(a) - (d)$ follow from the definition of a regularized wavelet set and Remark 1.

For the sufficiency, suppose that there exist measurable sets G_1 and G_2 containing G such that $(a) - (d)$ hold. We consider the maps $h_1,\ h_2 : E \to E$ defined by $h_1 := \delta_{|G_1} \circ (\tau_{|G_1})^{-1}$ and $h_2 := \tau_{|G_2} \circ (\delta_{|G_2})^{-1}$. It is clear that h_1 and h_2 are measurable injections. We now construct a new map h from h_1 and h_2 following the idea of the proof of the Cantor-Bernstein theorem in set theory. To increase the clarity of the presentation we write $\widetilde{E} := E$ and consider $h_1 : E \to \widetilde{E}$ and $h_2 : \widetilde{E} \to E$. We denote $f = h_2 \circ h_1 : E \to E$ and $g := h_1 \circ h_2 : \widetilde{E} \to \widetilde{E}$, and note that these maps are measurable injections by Remark 1. One can check that E and $\widetilde{E}$ can be partitioned as follows:

$$E = E_0 \dot{\cup} \left(\dot{\bigcup_{k\in\mathbb{N}}} E_k \dot{\cup} E_k' \right),$$

$$\widetilde{E} = \widetilde{E}_0 \dot{\cup} \left(\dot{\bigcup_{k\in\mathbb{N}}} \widetilde{E}_k \dot{\cup} \widetilde{E}_k' \right),$$

where

$$E_0 = \bigcap_{j\in\mathbb{N}} f^{(j)}(E), \qquad \widetilde{E}_0 = \bigcap_{j\in\mathbb{N}} g^{(j)}(\widetilde{E}),$$

$$E_k = f^{(k-1)}(E)\backslash(f^{(k-1)} \circ h_2)(\widetilde{E}), \quad E_k' = (f^{(k-1)} \circ h_2)(\widetilde{E})\backslash f^{(k)}(E), \quad k \in \mathbb{N},$$

and

$$\widetilde{E}_k = g^{(k-1)}(\widetilde{E})\backslash(g^{(k-1)} \circ h_1)(E), \quad \widetilde{E}_k' = (g^{(k-1)} \circ h_1)(E)\backslash g^{(k)}(\widetilde{E}), \quad k \in \mathbb{N}.$$

We define the map $h : E \to \widetilde{E}$ to be h_1 on $\hat{E} = E_0 \dot{\cup} \left(\dot{\bigcup}_{k\in\mathbb{N}} E_k \right)$ h_2^{-1} on $\hat{E}' = \dot{\bigcup}_{k\in\mathbb{N}} E_k'$. Since $h_1(E_0) = \widetilde{E}_0$ and, for $k \in \mathbb{N}$, $h_1(E_k) = \widetilde{E}_k'$ and $h_2^{-1}(E_k') = \widetilde{E}_k$, it follows that h is a bijection. We define

$$W = (\tau_{|G_1})^{-1}(\hat{E}) \cup (\tau_{|G_2})^{-1}(\hat{E}'). \tag{1}$$

Since $\hat{E}'$ is a set in the range of h_2, it is clear from (1) that the set W is translation congruent modulo 2π to E. Also if $x \in G$, then

$$f(\tau_{|G_1}(x)) = h_2\left(\delta_{|G_1}(x)\right) = \tau_{|G_2}\left(\delta_{|G_2}^{-1}(\delta_{|G_1}(x))\right) = \tau_{|G_2}(x) = \tau_{|G_1}(x)$$

since $\delta_{|G_2}(x) = \delta_{|G_1}(x)$ and $\tau_{|G_2}(x) = \tau_{|G_1}(x)$. This shows that $\tau_{|G_1}(G) \subset E_0$ and hence $G \subset W$. To complete the proof we need to check that W is dilation congruent modulo 2 to E. This follows from the facts that $\delta_{|G_1}((\tau_{|G_1})^{-1}(\hat{E})) = h_1(\hat{E})$, $\delta_{|G_2}((\tau_{|G_2})^{-1}(\hat{E}')) = h_2^{-1}(\hat{E}')$, and the function h is a bijection from E to $\widetilde{E}(= E)$. In fact one can check that W is a regularized wavelet set. □

Corollary 3. *For any point $x_0 \in \mathbb{R}\backslash\{0\}$ there exists an $\varepsilon > 0$ such that the interval $I_\varepsilon := (x_0 - \varepsilon, x_0 + \varepsilon)$ is a subwavelet set.*

Proof. It suffices to consider the case $x_0 > 0$. Choose $0 < \varepsilon < \min\{\pi/4, x_0/16\}$. We construct two sets G_1 and G_2 containing I_ε and satisfying $(a) - (d)$ in Theorem 2. We write $E_+ = [\pi, 2\pi)$ and $E_- = [-2\pi, -\pi)$. Note that since $\varepsilon < \min\{\pi, x_0/3\}$ the maps $\tau_{|I_\varepsilon} : I_\varepsilon \to E$, $\delta_{|I_\varepsilon} : I_\varepsilon \to E$ are measurable and injective. (Indeed, if $\tau(x_1) = \tau(x_2)$ with $x_1, x_2 \in I_\varepsilon$, then $x_1 - x_2 = 2k\pi$ for some $k \in \mathbb{Z}$, and since $|x_1 - x_2| < 2\varepsilon < 2\pi$, it follows that $k = 0$ and so $x_1 = x_2$. If $\delta(x_1) = \delta(x_2)$ with $x_1, x_2 \in I_\varepsilon$, then $x_1/x_2 = 2^k$ for some $k \in \mathbb{Z}$. Since $\varepsilon < x_0/3$ we have

$$1/2 < (x_0 - \varepsilon)/(x_0 + \varepsilon) < x_1/x_2 < (x_0 + \varepsilon)/(x_0 - \varepsilon) < 2,$$

and so $k = 0$ and $x_1 = x_2$.) Next we show that since $\varepsilon < x_0/16$ the set $E_+\backslash\delta(I_\varepsilon)$ contains an interval of length greater than $3\pi/8$. To see that this is true, we observe that the set $\delta(I_\varepsilon)$ is either an interval of length $2^k(2\varepsilon)$, where the integer k is uniquely determined by the inequalities $\pi \leq 2^k x_0 < 2\pi$, or it is a union of two intervals of combined lengths no more than $2^{k+1}(2\varepsilon)$. In the first case, the set $E_+\backslash\delta(I_\varepsilon)$ is either an interval or the union of two intervals, and if we assume that each such interval has length no greater than $3\pi/8$, we get the following contradiction:

$$\begin{aligned}\pi = \mu(E_+) &= \mu(E_+\backslash\delta(I_\varepsilon)) + \mu(\delta(I_\varepsilon)) \\ &\leq 2(3\pi/8) + 2^k(2\varepsilon) < 3\pi/4 + 2^k(2x_0/16) < \pi.\end{aligned}$$

In the second case (i.e., $\delta(I_\varepsilon)$ is a union of intervals), the set $E_+\backslash\delta(I_\varepsilon)$ is an interval, and if we assume it has length no larger than $3\pi/8$, we get a similar contradiction:

$$\begin{aligned}\pi = \mu(E_+) &= \mu(E_+\backslash\delta(I_\varepsilon)) + \mu(\delta(I_\varepsilon)) \\ &\leq (3\pi/8) + 2^{k+1}(2\varepsilon) < 3\pi/8 + 2^{k+1}(2x_0/16) < \pi.\end{aligned}$$

Thus $2^3\,(E_+\backslash\delta(I_\varepsilon))$ contains an interval of length greater than 3π. Hence there exists ℓ in $\mathbb{N}$ such that $E_+ + 2\ell\pi \subset 2^3\,(E_+\backslash\delta(I_\varepsilon))$. We define $G_1 = (E_-\backslash\tau(I_\varepsilon)) \cup I_\varepsilon \cup ((E_+\backslash\tau(I_\varepsilon)) + 2\ell\pi)$. It is clear that $\tau(G_1) = E$. Since the maps $\tau_{|(E_-\backslash\tau(I_\varepsilon))}$, $\tau_{|I_\varepsilon}$, and $\tau_{|(E_+\backslash\tau(I_\varepsilon))}$ are all injective and the sets $\tau(E_-\backslash\tau(I_\varepsilon))$, $\tau(I_\varepsilon)$, and $\tau(E_+\backslash\tau(I_\varepsilon))$ are pairwise disjoint, it follows that $\tau_{|G_1}$ is injective and hence is a measurable bijective map. From the choice of ℓ we conclude that $\delta((E_+\backslash\tau(I_\varepsilon)) + 2\ell\pi) \subset E_+\backslash\tau(I_\varepsilon)$. Hence the sets $\delta(E_-\backslash\tau(I_\varepsilon))$, $\delta(I_\varepsilon)$, and $\delta(E_+\backslash\tau(I_\varepsilon))$ are pairwise disjoint, and since the maps $\delta_{(|E_+\backslash\tau(I_\varepsilon))}$, $\delta_{|I_\varepsilon}$, and $\delta_{|(E_+\backslash\tau(I_\varepsilon))}$ are injective, it follows that $\delta_{|G_1}$ is a measurable injective map. Thus G_1 has the desired properties.

To construct G_2, we observe first that the collection $\{2^{-n}E_- + 2\pi\}_{n\in\mathbb{N}} \cup \{2^{-n}E_+ - 2\pi\}_{n\in\mathbb{N}}$ is an interval partition of the set $E\backslash\{-2\pi\}$. Moreover $\tau(I_\varepsilon)$ is either an interval of length 2ε or the union of two intervals of combined lengths 2ε. Since $\varepsilon < \pi/4$, there exists an $n_0 \in \mathbb{N}$ such that $\tau(2^{-n_0}E) \cap \tau(I_\varepsilon) = \varnothing$. In other words, $\tau(2^{-n_0}E) \subset E\backslash\tau(I_\varepsilon)$. We define $G_2 = I_\varepsilon \cup 2^{-n_0}(E\backslash\delta(I_\varepsilon))$. Using arguments similar to those above, one shows that $\delta_{|G_2} : G_2 \to E$ is a measurable bijective map, and using the fact that

$$\tau(2^{-n_0}(E\backslash\delta(I_\varepsilon))) = (2^{-n_0}(E_-\backslash\delta(I_\varepsilon)) + 2\pi) \\ \cup (2^{-n_0}(E_+\backslash\delta(I_\varepsilon)) - 2\pi) \subset \tau(2^{-n_0}E) \subset E\backslash\tau(I_\varepsilon),$$

we obtain that $\tau_{|G_2} : G_2 \to E$ is a measurable injective map. Thus G_2 has the desired properties, and the proof is complete. □

A regularized wavelet set W is called a *regularized MRA-wavelet set* [2] if the family $\{\widetilde{W} + 2k\pi\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}\backslash\{2k\pi : k \in \mathbb{Z}\}$, where $\widetilde{W} = \bigcup_{n\in\mathbb{N}} 2^{-n}(W)$. A set is called an *MRA-subwavelet set* if it is a subset of a regularized MRA-wavelet set.

Question 4. Is there a characterization of MRA-subwavelet sets similar to that given in Theorem 2 for subwavelet sets?

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