ON SUBWAVELET SETS

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Abstract. In this note we give a characterization of subwavelet sets and show that any point \( x \in \mathbb{R} \setminus 0 \) has a neighborhood which is contained in a regularized wavelet set.

In [1] the notion of a wavelet set was introduced and in [8] subwavelet sets were considered. Wavelet sets were also introduced independently and simultaneously as the support sets of MSF (Minimally Supported Frequency) wavelets in the sequence of papers [3], [5], and [6]. (See also the recent excellent book [4].) The purpose of this note is to provide a characterization of the subwavelet sets and to use this characterization to prove that every point \( x \in \mathbb{R} \setminus \{0\} \) has a neighborhood contained in a regularized wavelet set. (Regularized wavelet sets are wavelet sets with certain nice properties; see [7].) In particular, this shows that the union of the interiors of all wavelet sets is \( \mathbb{R} \setminus \{0\} \).

We begin by introducing some preliminary terminology and notation. The measure space under consideration will always be \( \mathbb{R} \) together with its \( \sigma \)-ring \( \mathbb{L} \) of Lebesgue measurable subsets and Lebesgue measure \( \mu \). Recall (cf. [1]) that a function \( w \in L^2(\mathbb{R}):= L^2(\mathbb{R}, \mathbb{L}, \mu) \) is a wavelet if the family of (equivalence classes of) functions \( \{w_{j,k}\}_{j,k \in \mathbb{Z}} \) defined by

\[
w_{j,k}(s) = 2^{j/2}w(2^js + k), \quad s \in \mathbb{R}, \ j, k \in \mathbb{Z},
\]

is an orthonormal basis for \( L^2(\mathbb{R}) \). A subset \( G \) of \( \mathbb{R} \) with positive measure is a wavelet set if \( F(w) \), where \( w \) is a wavelet in \( L^2(\mathbb{R}) \) and \( F \) is the Fourier-Plancherel transform on \( L^2(\mathbb{R}) \). A measurable subset \( G \) of \( \mathbb{R} \) is called a regularized wavelet set if the family \( \{G + 2k\pi\}_{k \in \mathbb{Z}} \) is a partition of \( \mathbb{R} \) and the family \( \{2^kG\}_{k \in \mathbb{Z}} \) is a partition of \( \mathbb{R} \setminus \{0\} \). For two measurable subsets \( F \) and \( G \) of \( \mathbb{R} \), we write \( F \sim G \) if \( \mu(F \setminus G) = 0 \). It is proved in [7] that if \( W \) is any wavelet set, then there exists a regularized wavelet set \( W' \) such that \( W' \sim W \). A measurable subset \( G \) of \( \mathbb{R} \) is translation congruent modulo \( 2\pi \) to a (measurable) set \( H \subset \mathbb{R} \) if there exists a measurable bijection \( \varphi : G \to \varphi(G) \) such that \( \varphi(s) \sim s \) is an integral multiple of \( 2\pi \) for every \( s \) in \( G \) and \( \varphi(G) \sim H \). Analogously, \( G \subset \mathbb{R} \setminus \{0\} \) is said to be dilation congruent modulo \( 2 \) to a (measurable) set \( H \) if there exists a measurable bijection \( \psi : G \to \psi(G) \) such that \( \psi(s)/s \) is an integral power of \( 2 \) for every \( s \) in \( G \) and \( \psi(G) \sim H \). Let \( \tau : \mathbb{R} \to \mathbb{R} := [-2\pi, -\pi) \cup [\pi, 2\pi) \) be the function defined by \( \tau(x) = x + 2j\pi \), where \( j \) is the unique integer satisfying \( x + 2j\pi \in E \), and let \( \delta : \mathbb{R} \setminus \{0\} \to E \) be the function defined by \( \delta(x) = 2^kx \), where \( k \) is the unique integer...
for which $2^k x \in E$. For a function $f : X \to X$ and $k \in \mathbb{Z}$ we write $f^{(k)}$ for the map $x \to x$ on $X$ and $f^{(k)}$ for the composition of $f$ [resp. $f^{-1}$] with itself $|k|$ times if $k > 0$ [resp. $k < 0$].

**Remark 1.** In what follows we use the elementary facts that if $G \in \mathbb{L} \cap E$, then $\tau^{-1}(G)$, $\delta^{-1}(G) \in \mathbb{L}$, and if $H \in \mathbb{L}$ [resp., $H \in (\mathbb{R} \setminus \{0\}) \cap \mathbb{L}$], then $\tau(G) \in \mathbb{L}$ [resp. $\delta(G) \in \mathbb{L}$].

A measurable subset $G$ of $\mathbb{R}$ is called a *subwavelet* set if it is a subset of some regularized wavelet set. Our principal result characterizes measurable subsets of $\mathbb{R}$ that are subwavelet sets.

**Theorem 2.** A set $G \subset \mathbb{L}$ is a subwavelet set if and only if there exist sets $G_1$ and $G_2$ in $\mathbb{L}$, each containing $G$, such that

(a) $\tau_{G_1}$ is a measurable bijection of $G_1$ onto $E$,
(b) $\tau_{G_2}$ is a measurable injection of $G_2$ into $E$,
(c) $\delta_{G_2}$ is a measurable bijection of $G_2$ onto $E$, and
(d) $\delta_{G_1}$ is a measurable injection of $G_1$ into $E$.

**Proof.** Suppose first that $G$ is a subset of a regularized wavelet set $W$. Define $G_1 = G_2 = W$, and observe that (a) − (d) follow from the definition of a regularized wavelet set and Remark 1.

For the sufficiency, suppose that there exist measurable sets $G_1$ and $G_2$ containing $G$ such that (a) − (d) hold. We consider the maps $h_1, h_2 : E \to E$ defined by $h_1 := \delta_{G_1} \circ (\tau_{G_1})^{-1}$ and $h_2 := \tau_{G_2} \circ (\delta_{G_2})^{-1}$. It is clear that $h_1$ and $h_2$ are measurable injections. We now construct a new map $h$ from $h_1$ and $h_2$ following the idea of the proof of the Cantor-Bernstein theorem in set theory. To increase the clarity of the presentation we write $\tilde{E} := E$ and consider $h_1 : E \to \tilde{E}$ and $h_2 : \tilde{E} \to E$. We denote $f = h_2 \circ h_1 : E \to E$ and $g := h_1 \circ h_2 : \tilde{E} \to \tilde{E}$, and note that these maps are measurable injections by Remark 1. One can check that $E$ and $\tilde{E}$ can be partitioned as follows:

$$E = E_0 \cup \bigcup_{k \in \mathbb{N}} E_k \cup E'_k,$$

$$\tilde{E} = \tilde{E}_0 \cup \bigcup_{k \in \mathbb{N}} \tilde{E}_k \cup \tilde{E}'_k,$$

where

$$E_0 = \bigcap_{j \in \mathbb{N}} f^{(j)}(E), \quad \tilde{E}_0 = \bigcap_{j \in \mathbb{N}} g^{(j)}(\tilde{E}),$$

$$E_k = f^{(k-1)}(E) \setminus (f^{(k-1)} \circ h_2)(\tilde{E}), \quad E'_k = (f^{(k-1)} \circ h_2)(\tilde{E}) \setminus f^{(k)}(E), \quad k \in \mathbb{N},$$

and

$$\tilde{E}_k = g^{(k-1)}(\tilde{E}) \setminus (g^{(k-1)} \circ h_1)(E), \quad \tilde{E}'_k = (g^{(k-1)} \circ h_1)(E) \setminus g^{(k)}(\tilde{E}), \quad k \in \mathbb{N}.$$
Since $\tilde{E}'$ is a set in the range of $h_2$, it is clear from (1) that the set $W$ is translation congruent modulo $2\pi$ to $E$. Also if $x \in G$, then

$$f(\eta_{G_1}(x)) = h_2(\delta_{G_1}(x)) = \tau_{G_2}^{-1}(\delta_{G_2}(x)) = \tau_{G_2}(x) = \eta_{G_1}(x)$$

since $\delta_{G_2}(x) = \delta_{G_1}(x)$ and $\tau_{G_2}(x) = \tau_{G_1}(x)$. This shows that $\tau_{G_1}(G) \subset E_0$ and hence $G \subset W$. To complete the proof we need to check that $W$ is dilation congruent modulo 2 to $E$. This follows from the facts that $\delta_{G_1}((\tau_{G_1})^{-1}(E)) = h_1(E)$, $\delta_{G_2}((\tau_{G_2})^{-1}(E')) = h_2^{-1}(E')$, and the function $h$ is a bijection from $E$ to $\tilde{E}(= E)$. In fact one can check that $W$ is a regularized wavelet set. \hfill $\Box$

**Corollary 3.** For any point $x_0 \in \mathbb{R}\setminus\{0\}$ there exists an $\varepsilon > 0$ such that the interval $I_{\varepsilon} := (x_0 - \varepsilon, x_0 + \varepsilon)$ is a subwavelet set.

**Proof.** It suffices to consider the case $x_0 > 0$. Choose $0 < \varepsilon < \min\{\pi/4, x_0/16\}$. We construct two sets $G_1$ and $G_2$ containing $I_{\varepsilon}$ and satisfying $(a) - (d)$ in Theorem 2. We write $E_+ = [\pi, 2\pi)$ and $E_- = [-2\pi, -\pi)$. Note that since $\varepsilon < \min\{\pi, x_0/3\}$ the maps $\tau_{I_{\varepsilon}} : I_{\varepsilon} \to E$, $\delta_{I_{\varepsilon}} : I_{\varepsilon} \to E$ are measurable and injective. (Indeed, if $\tau(x_1) = \tau(x_2)$ with $x_1, x_2 \in I_{\varepsilon}$, then $x_1 - x_2 = 2k\pi$ for some $k \in \mathbb{Z}$, and since $|x_1 - x_2| < 2\varepsilon < 2\pi$, it follows that $k = 0$ and so $x_1 = x_2$.) If $\delta(x_1) = \delta(x_2)$ with $x_1, x_2 \in I_{\varepsilon}$, then $x_1/x_2 = 2^k$ for some $k \in \mathbb{Z}$. Since $\varepsilon < x_0/3$ we have

$$1/2 < (x_0 - \varepsilon)/(x_0 + \varepsilon) < x_1/x_2 < (x_0 + \varepsilon)/(x_0 - \varepsilon) < 2,$n

and so $k = 0$ and $x_1 = x_2$.) Next we show that since $\varepsilon < x_0/16$ the set $E_+ \setminus \delta(I_{\varepsilon})$ contains an interval of length greater than $3\pi/8$. To see that this is true, we observe that the set $\delta(I_{\varepsilon})$ is either an interval of length $2\varepsilon$, where the integer $k$ is uniquely determined by the inequalities $\pi \leq 2^k x_0 < 2\pi$, or it is a union of two intervals of combined lengths no more than $2^{k+1}(2\varepsilon)$. In the first case, the set $E_+ \setminus \delta(I_{\varepsilon})$ is either an interval or the union of two intervals, and if we assume that each such interval has length no greater than $3\pi/8$, we get the following contradiction:

$$\pi = \mu(E_+) = \mu(E_+ \setminus \delta(I_{\varepsilon})) + \mu(\delta(I_{\varepsilon})) < 2(3\pi/8) + 2\varepsilon < 3\pi/4 + 2\varepsilon(2x_0/16) < \pi.$$n

In the second case (i.e., $\delta(I_{\varepsilon})$ is a union of intervals), the set $E_+ \setminus \delta(I_{\varepsilon})$ is an interval, and if we assume it has length no larger than $3\pi/8$, we get a similar contradiction:

$$\pi = \mu(E_+) = \mu(E_+ \setminus \delta(I_{\varepsilon})) + \mu(\delta(I_{\varepsilon})) < 3\pi/8 + 2\varepsilon(2x_0/16) < \pi.$$n

Thus $2^3(E_+ \setminus \delta(I_{\varepsilon}))$ contains an interval of length greater than $3\pi$. Hence there exists $\ell$ in $\mathbb{N}$ such that $E_+ + 2\ell \pi < 2^3(E_+ \setminus \delta(I_{\varepsilon}))$. We define $G_1 = (E_+ \setminus \tau(I_{\varepsilon})) \cup I_{\varepsilon} \cup (E_+ \setminus \tau(I_{\varepsilon})) + 2\ell \pi)$. It is clear that $\tau(G_1) = E$. Since the maps $\tau_{\ell}(E_+ \setminus \tau(I_{\varepsilon}))$, $\tau_{I_{\varepsilon}}$, and $\tau_{(E_+ \setminus \tau(I_{\varepsilon}))}$ are all injective and the sets $\tau(E_+ \setminus \tau(I_{\varepsilon}))$, $\tau(I_{\varepsilon})$, and $\tau(E_+ \setminus \tau(I_{\varepsilon}))$ are pairwise disjoint, it follows that $\tau_{G_1}$ is injective and hence is a measurable bijective map. From the choice of $\ell$ we conclude that $\delta(E_+ \setminus \tau(I_{\varepsilon})) + 2\ell \pi) < E_+ \setminus \tau(I_{\varepsilon})$. Hence the sets $\delta(E_+ \setminus \tau(I_{\varepsilon}))$, $\delta(I_{\varepsilon})$, and $\delta(E_+ \setminus \tau(I_{\varepsilon}))$ are pairwise disjoint, and since the maps $\delta_{(E_+ \setminus \tau(I_{\varepsilon}))}$, $\delta_{I_{\varepsilon}}$, and $\delta_{(E_+ \setminus \tau(I_{\varepsilon}))}$ are injective, it follows that $\delta_{G_1}$ is a measurable injective map. Thus $G_1$ has the desired properties.

To construct $G_2$, we observe first that the collection $\{2^{-n}E_- + 2\pi\}_{n \in \mathbb{N}} \cup \{2^{-n}E_+ - 2\pi\}_{n \in \mathbb{N}}$ is an interval partition of the set $E \setminus \{-2\pi\}$. Moreover $\tau(I_{\varepsilon})$ is
either an interval of length $2\varepsilon$ or the union of two intervals of combined lengths $2\varepsilon$. Since $\varepsilon < \pi/4$, there exists an $n_0 \in \mathbb{N}$ such that $\tau(2^{-n_0}E) \cap \tau(I_\varepsilon) = \emptyset$. In other words, $\tau(2^{-n_0}E) \subset E\setminus \tau(I_\varepsilon)$. We define $G_2 = I_\varepsilon \cup 2^{-n_0}(E\setminus \delta(I_\varepsilon))$. Using arguments similar to those above, one shows that $\delta_{|G_2} : G_2 \to E$ is a measurable bijective map, and using the fact that

\[
\tau(2^{-n_0}(E\setminus \delta(I_\varepsilon))) = (2^{-n_0}(E_+\setminus \delta(I_\varepsilon)) + 2\pi) \\
\cup (2^{-n_0}(E_+\setminus \delta(I_\varepsilon)) - 2\pi) \subset \tau(2^{-n_0}E) \subset E\setminus \tau(I_\varepsilon),
\]

we obtain that $\tau_{|G_2} : G_2 \to E$ is a measurable injective map. Thus $G_2$ has the desired properties, and the proof is complete.

A regularized wavelet set $W$ is called a regularized MRA-wavelet set [2] if the family $\{\tilde{W} + 2k\pi\}_{k \in \mathbb{Z}}$ is a partition of $\mathbb{R}\setminus \{2k\pi : k \in \mathbb{Z}\}$, where $\tilde{W} = \bigcup_{n \in \mathbb{N}} 2^{-n}(W)$. A set is called an MRA-subwavelet set if it is a subset of a regularized MRA-wavelet set.

Question 4. Is there a characterization of MRA-subwavelet sets similar to that given in Theorem 2 for subwavelet sets?

References


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