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REAL FORMS OF A RIEMANN SURFACE OF EVEN GENUS

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Abstract. Natanzon proved that a Riemann surface X of genus $g \geq 2$ has at most $2(\sqrt{g}+1)$ conjugacy classes of symmetries, and this bound is attained for infinitely many genera g. The aim of this note is to prove that a Riemann surface of even genus g has at most four conjugacy classes of symmetries and this bound is attained for an arbitrary even g as well. An equivalent formulation in terms of algebraic curves is that a complex curve of an even genus g has at most four real forms which are not birationally equivalent.

1. Introduction

Natanzon [4] (see also [3]) proved that a Riemann surface X of genus $g \geq 2$ has at most $2(\sqrt{g}+1)$ conjugacy classes of symmetries, and this bound is attained for infinitely many odd genera g. Singerman [5] showed that if X is hyperelliptic, then the number of non-conjugate pairs of symmetries does not exceed 3 if g is even and 4 if g is odd. The aim of this note is to prove that a Riemann surface of even genus g has at most 4 conjugacy classes of symmetries and this bound is attained for an arbitrary even g as well. An equivalent formulation is that a Riemann surface of even genus g is the complex double of at most 4 bordered Klein surfaces, or in terms of algebraic curves, that a complex curve of an even genus g has at most four real forms which are not birationally equivalent (see [1]).

2. Preliminaries

Let X be a Riemann surface of genus $g \geq 2$. By a symmetry of X we mean an anticonformal involution σ of X with fixed points and a surface admitting a symmetry is said to be *symmetric*. A symmetric Riemann surface X corresponds to a real algebraic curve. In the group $\mathrm{Aut}^{\pm}(X)$, of all conformal and anticonformal automorphisms of X, non-conjugate symmetries correspond to different real models of the curve.

Arbitrary compact Riemann surfaces of genus $g \geq 2$ can be represented as the orbit space $\mathcal{H}/\Gamma$ of the hyperbolic plane $\mathcal{H}$ with respect to the action of a Fuchsian surface group Γ, a discrete subgroup of $\mathrm{Aut}^{+}(\mathcal{H}) = \mathrm{PSL}(2, \mathbf{R})$ without elliptic elements. A discrete subgroup Λ of $\mathrm{Aut}^{\pm}(\mathcal{H})$ with compact orbit space is called an

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NEC (*non-euclidean crystallographic*) *group.* The algebraic structure of an NEC group Λ is determined by the signature:

$$s(\Lambda) = (h; \pm; [m_1, ..., m_r]; \{(n_{11}, ..., n_{1s_1}), ..., (n_{k1}, ..., n_{ks_k})\}). \tag{1}$$

The orbit space $\mathcal{H}/\Lambda$ is an orbifold with underlying surface of genus h, having r cone points and k boundary components, each with $s_j \geq 0$ corner points. The signs "+" and "−" correspond to orientable and non-orientable orbifolds respectively. The integers m_i are called the proper periods of Λ and they are the orders of the cone points of $\mathcal{H}/\Lambda$. The brackets $(n_{i1}, ..., n_{is_i})$ are the period cycles of Λ and the integers n_{ij} are the link periods of Λ; they are the orders of the corner points of $\mathcal{H}/\Lambda$. Λ is called the *group* (or *fundamental group*) *of the orbifold* $\mathcal{H}/\Lambda$.

A group Λ with signature (1) has the presentation with generators: $x_1, ..., x_r$, $e_1, ..., e_k$, $c_{ij}, 1 \leq i \leq k, 0 \leq j \leq s_i$ and $a_1, b_1, ..., a_h, b_h$ if $\mathcal{H}/\Lambda$ is orientable or $d_1, ..., d_h$ otherwise, and relators: $x_i^{m_i}, i = 1, ..., r$, $c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}$, $c_{i0}e_i^{-1}c_{is_i}e_i$, $i = 1, ..., k, j = 0, ..., s_i$ and $x_1...x_r e_1...e_k a_1 b_1 a_1^{-1} b_1^{-1} ... a_h b_h a_h^{-1} b_h^{-1}$ or $x_1...x_r e_1...e_k d_1^2...d_h^2$, according to whether $\mathcal{H}/\Lambda$ is orientable or not.

The hyperbolic area of the orbifold $\mathcal{H}/\Lambda$ coincides with the hyperbolic area of an arbitrary fundamental region of Λ and equals:

$$\mu(\Lambda) = 2\pi(\varepsilon h - 2 + k + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}(1 - \frac{1}{n_{ij}})), \tag{2}$$

where $\varepsilon = 2$ if there is a "+" sign and $\varepsilon = 1$ otherwise. If Λ' is a subgroup of Λ of finite index, then it is an NEC group itself and the following Riemann-Hurwitz formula holds:

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda). \tag{3}$$

Given an NEC group Λ the subgroup Λ^+ of Λ consisting of the orientation-preserving elements is called the *cannonical Fuchsian subgroup of* Λ.

An NEC group Γ without elliptic elements is called a *surface group* and it has signature $(h; \pm; [-], \{(-), \overset{k}{\ldots}, (-)\})$. In such a case $\mathcal{H}/\Gamma$ is a *Klein surface*, i.e., a surface with a dianalytic structure of topological genus h, orientable or not according to whether the sign is "+" or "−" and having k boundary components. Conversely, a Klein surface whose complex double has genus greater than one can be expressed as $\mathcal{H}/\Gamma$ for some NEC surface group Γ. Furthermore, given a Riemann (resp. Klein) surface represented as the orbit space $X = \mathcal{H}/\Gamma$, with Γ a surface group, a finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ.

3. 2-groups of automorphisms of surfaces of even genus

We start this section with the following lemma which allows us to restrict ourselves to finite 2-groups when we work with conjugacy classes of symmetries of a Riemann surface.

Lemma 3.1. *Let g be a positive integer greater than or equal to 2. There exists a Riemann surface X of genus g having k non-conjugate symmetries which generate a 2-subgroup of* $\mathrm{Aut}^{\pm}(X)$ *such that any Riemann surface X' of genus g has no more than k conjugacy classes of symmetries.*

Proof. Let X be a Riemann surface of genus g with maximal number k of non-conjugate symmetries $\tau_1, \ldots, \tau_k$ and let G_2 be a 2-Sylow subgroup of $G = \mathrm{Aut}^{\pm}(X)$. Then by the Sylow theorem $\tau_1^{\alpha_1}, \ldots, \tau_m^{\alpha_m} \in G_2$, for some $\alpha_1, \ldots, \alpha_m \in G$, and hence the result.

Lemma 3.2. *Let X be a Riemann surface of even genus g and let G be a 2-group of automorphisms of X. Then G is an extension of a cyclic or dihedral group by Z_2. In particular, if G is generated by symmetries, then either G is a cyclic group, a dihedral group, or a semidirect product of a cyclic or dihedral group by Z_2.*

Proof. Let $X = \mathcal{H}/\Gamma$ for some surface NEC group Γ and let $G = \Lambda/\Gamma$. Assume that Λ has a signature of a general form (1). By Theorems 2.2.4 and 2.3.3 of [2] , G is cyclic or dihedral if Λ has a proper period is equal to $|G|$ or a link period equal to $|G|/2$ respectively. So assume that neither a proper period of Λ is equal to $|G|$ nor a link period is equal to $|G|/2$. Let $M = \{i \mid 2m_i = |G|\}$ and $N = \{(i,j) \mid 4n_{ij} = |G|\}$, and let N' and M' be the complementary sets. Then

$$\frac{|G|}{2}\left(\alpha h - 2 + k + \sum_{i\in M'}\left(1-\frac{1}{m_i}\right) + \sum_{(i,j)\in N'}\frac{1}{2}\left(1-\frac{1}{n_{ij}}\right)\right)$$

is even and, since g is even, we obtain from the Riemann-Hurwitz formula (3) that

$$\frac{|G|}{2}\left(\sum_{i\in M}\left(1-\frac{1}{m_i}\right) + \sum_{(i,j)\in N}\frac{1}{2}\left(1-\frac{1}{n_{ij}}\right)\right)$$

is odd. But the last is equal to $|G||M|/2 + |G||N|/4 - |M| - |N|$ and therefore $|M| + |N|$ is odd. So in particular $|M| \neq 0$ or $|N| \neq 0$. In the first case G contains $H = Z_{|G|/2}$ while in the second one $H = D_{|G|/4}$ as a subgroup of index 2, hence the first part of the theorem. Now, if in addition G is generated by elements of order 2, then there exists an element $g \in G \setminus H$ of order 2 and so $G = H \ltimes Z_2$. This completes the proof.

Theorem 3.3 (Main result)**.** *A Riemann surface of even genus g has at most 4 conjugacy classes of symmetries. Furthermore this bound is attained for every even genus g.*

Proof. The dihedral 2-group has three conjugacy classes of elements of order 2 and it is easy to check that all semidirect products $G = Z_n \ltimes Z_2$ have at most three such classes. So it remains to count the number of conjugacy classes of symmetries of Riemann surfaces whose groups of automorphisms are semidirect products $G = D_n \ltimes Z_2$, where n is a power of 2. These groups have the presentations:

$$G_{\alpha,\beta} = \langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^n = 1, zxz = (xy)^\alpha x, zyz = (xy)^\beta x\rangle,$$

where $\alpha - \beta \equiv 1 \mod 2$, $\alpha(\alpha - \beta + 1) \equiv 0 \mod n$, $\beta(\alpha-\beta) + \alpha + 1 \equiv 0 \mod n$.

From the proof of the previous lemma it follows that we can assume that x, y and z are symmetries of X. So neither $(xy)^{n/2}$ nor an element of the form $z(xy)^\delta x$ can be a symmetry since they preserve the orientation of X as compositions of an even number of symmetries.

Finally an element of the form $z(xy)^\gamma$ has order two if and only if

$$\gamma(\alpha - \beta + 1) \equiv 0 \mod n.$$

We have

$$(xy)^m z(xy)^{-m} = z(xy)^{m(\alpha-\beta-1)} \text{ and } x(xy)^m z(xy)^{-m}x = z(xy)^{m(\beta+1-\alpha)+\alpha}$$

and so $z(xy)^{2s\alpha}$ and $z(xy)^{(2s+1)\alpha}$ are conjugate to z for all s. In particular for α odd there is only one conjugacy class of elements of order 2 of the form $z(xy)^{\gamma}$.

Now let α be even and let $\alpha - \beta + 1 = 2^s t$, where t is odd.

If $s = 0$, then $\alpha - \beta + 1 = 0$ and so arbitrary $z(xy)^{\gamma}$ has order 2. Furthermore, $(xy)^{-m}z(xy)^m = z(xy)^{2m}$, $(xy)^{-m}z(xy)(xy)^m = z(xy)^{2m+1}$ and so there are at most two conjugacy classes of elements of order 2 of this form with representatives z and zxy. But actually these elements are non-conjugate in G since also $x(xy)^m z(x(xy)^m)^{-1} = z(xy)^{2m+\alpha}$, where $2m+\alpha$ is even. If $s = 1$, then $\alpha = n/2$ or $\alpha = 0$ and $z, z(xy)^{n/2}$ are the only elements of order 2 of this form and furthermore they are non-conjugate just for $\alpha = 0$ and $\beta = n-1$, i.e., for $G = D_n \times Z_2$. Indeed for $\alpha \neq 0$, $xzx = z(xy)^{\alpha}$ and, for $\alpha = 0, \beta \neq n-1$, $yzy = z(xy)^{\beta+1}$. Finally, if $s \geq 2$, then every element of order 2 of this form is equal to $z(xy)^{2v}$ for some v. On the other hand $\alpha - \beta - 1 = 2u$, for some odd u, and thus every element of the form $z(xy)^{2v}$ is conjugate to z. This completes the proof of the first part.

To prove the second part of the theorem, observe that by [6] the signature of the canonical Fuchsian subgroup of an NEC-group with signature $(0;+;[-];\{(2,\overset{g+3}{\ldots},2)\})$ is maximal. So by Remark 5.1.1(1) and Theorem 5.1.2 of [2] there exists a maximal NEC-group Λ with this signature. Let $\theta : \Lambda \to G = Z_2 \times Z_2 \times Z_2 = \langle x, y, z\rangle$ defined by $\theta(c_{2j}) = x$, $\theta(c_{2j+1}) = y$ for $0 \leq j \leq (g-2)/2$, $\theta(c_g) = x$, $\theta(c_{g+1}) = z$ and finally $\theta(c_{g+2}) = zxy$. Then by [2] , $\Gamma = \ker\theta$ is a surface Fuchsian group, and by the Riemann-Hurwitz formula $X = \mathcal{H}/\Gamma$ is a surface of genus g. Finally its symmetries x, y, z and zxy are non-conjugate since $G = \mathrm{Aut}^{\pm}(X)$. □

Notice that the groups $G_{0,n-1} = D_n \times Z_2$ and $G_{\alpha,\alpha+1}$, where α is even, are isomorphic. In fact, the application

$$\varphi(x) = x, \quad \varphi(y) = y, \quad \varphi(z) = z(xy)^{\alpha/2}x$$

induces an isomorphism $\varphi : G_{\alpha,\alpha+1} \to D_n \times Z_2$.

In this way, apart from the quantitative result concerning the number of non-conjugate symmetries obtained above, we have obtained also the following qualitative result.

Corollary 3.4. *Let X be a Riemann surface of even genus and let G be a subgroup of $\mathrm{Aut}^{\pm}(X)$ generated by non-conjugate symmetries $\sigma_1, \sigma_2, \sigma_3$ and σ_4. Then $G = D_n \times Z_2$.*

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