SEMIGROUP PROPERTIES OF FACTORS IN THE POLAR DECOMPOSITION OR THE OPERATOR DE-MOIVRE FORMULA

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(Communicated by Palle E. T. Jorgensen)

Abstract. We give necessary and sufficient conditions under which the factors in the polar decomposition of a semigroup homomorphism are themselves semigroup homomorphisms.

1. Introduction

For a complex number \(z\) its polar form is \(z = |z|e^{i\theta}\) with a real number \(\theta\). The polar decomposition of a bounded Hilbert space operator \(T = |T|V\), where \(|T| = (T^*T)^{1/2}\) and \(V\) is an appropriate partial isometry, is the non-commutative generalization of the polar form of a complex number. \(\mathbb{N}\) is the additive semigroup of natural numbers.

De-Moivre’s formula says that \(z^n = |z|^ne^{in\theta}\), \(n \in \mathbb{N}\).

It is therefore a natural question to ask:

When does \(T^n = V^n|T|^n\), \(n \in \mathbb{N}\), for an operator \(T\) with the polar decomposition \(T = |T|V\)?

A sufficient condition is, clearly, that \(T\) is quasinormal (see definition below). From general results of this paper it will follow that quasinormality is also necessary.

The above question is equivalent to:

When does \(|T^n| = |T|^n\) and the partial isometric factor of \(T^n\) equal \(V^n\)?

Upon a closer examination, this question can be formulated using the semigroup language as follows: The mapping \(n \to T^n\), \(n \in \mathbb{N}\), is, obviously, a semigroup homomorphism. Hence our question is equivalent to the question: When are the mappings \(n \to |T^n|\) and \(n \to \) (the partial isometric factor of \(T^n\)) semigroup homomorphisms?

The only answer in this context has been given by B. Morrel and P. Muhly in [4], Theorem I, who proved that if the operator \(T\) is centered (see definition below), then the partial isometric factor of \(T^n\) is \(V^n\). Their proof uses the principle of mathematical induction and thus cannot be generalized to any other semigroup.

Before we state the general problem let us introduce some terminology.

\(B(H)\) is the algebra of all linear, bounded operators in a Hilbert space \(H\). \(I\) denotes the identity operator on \(H\). An operator \(V \in B(H)\) is called a partial isometry if \(V^*V\) is an orthogonal projection, or, equivalently, if \(VV^*V = V\). By
Theorem 1. (1) \( (\star) \) cf. \([4]\).

The polar decomposition of \( T \in B(H) \) we understand the decomposition \( T = V \) with \( |T| = (T^* T)^{1/2} \) and \( V: \mathbb{R}(|T|) \to \mathbb{R}(T) \), \( V |T| x = Tx, \ x \in H, \ ker T = ker V \). \( T \in B(H) \) is quasinormal if \( T^* T \) commutes with \( T \) or, equivalently, if the factors in the polar decomposition of \( T \) commute. \( T \in B(H) \) is called centered if the set \( \{ T^n (T^*)^m, (T^*)^m T^m, n, m \in \mathbb{N}, n, m \neq 0 \} \) is commutative. \( T \in B(H) \) is hyponormal if \( T^* T - TT^* \geq 0 \).

Let \( S \) be a commutative semigroup with 1.

A mapping \( \pi: S \to B(H) \) is called a semigroup homomorphism if \( \pi(s + t) = \pi(s) \pi(t), s, t \in S \), and \( \pi(1) = I \), a normal homomorphism if the set \( \{ \pi(s), \pi(t)^* \}, s, t \in S \} \) is commutative, a quasinormal homomorphism if the set \( \{ \pi(s)^* \pi(s), \pi(t), s, t \in S \} \) is commutative, a subnormal homomorphism if there is a Hilbert space \( K \) containing \( H \) and a normal homomorphism \( \tau: S \to B(H) \) such that \( H \) is invariant for each \( \pi(s) \) and \( \pi(s)_H = \pi(s), s \in S \), and a centered homomorphism if the set \( \{ \pi(s)^* \pi(s), \pi(t) \pi(t)^*, s, t \in S \} \) is commutative. All these special kinds of homomorphisms are, clearly, assumed to be semigroup homomorphisms. Clearly, each quasinormal homomorphism is centered.

Let \( \pi: S \to B(H) \) be a semigroup homomorphism. Let \( \pi(s) = \mu(s) \theta(s) \) be the polar decomposition of the operator \( \pi(s), s \in S \).

In this paper we will answer the question under which conditions \( \mu \) and \( \theta \) are semigroup homomorphisms. This result applied to \( S = \mathbb{N} \) will answer our first question, and Theorem 1 applied to the semigroup \([0, \infty) \) will generalize a result of M. Ennery in [2], Theorem 6, which she proved under the additional assumption of strong continuity.

2. The results

Let \( S \) be a commutative semigroup with 1.

Let \( \pi: S \to B(H) \) be a semigroup homomorphism, and let \( \pi(s) = \theta(s) \mu(s) \) be the polar decomposition of \( \pi(s), s \in S \). \( E(t) = \theta(t)^* \theta(t) \) is the initial projection for \( \theta(t) \) and \( F(t) = \theta(t) \theta(t)^* \) is the final projection for \( \theta(t) \).

It is known in general that

\[
\theta(t)^* \theta(t) = \theta(t),
\]
\( \theta(t)^* \theta(t) \) is the projection onto \((ker \pi(t))^\perp = (ker \mu(t))^\perp = ran \mu(t) \), and

\[
(1) \quad \theta(t)^* \theta(t) \mu(t) = \mu(t).
\]

Theorem 1. (a) If \( \pi \) is centered, then \( \theta \) is a semigroup homomorphism.
(b) Assume additionally that for each \( s, t \in S \) there exists \( r \in S \) such that \( s = t + r \) or \( t = s + r \). If \( \theta \) is a semigroup homomorphism, then \( \pi \) is centered.

Proof. (a) As in the case of one operator, \( \mu(s), E(t), F(r) \) commute for all \( s, t, r \in S \) (cf. [4]).

\[
(2) \quad \mu(t) \text{ and } \theta(r) \mu(r) \theta(r)^* \text{ commute,}
\]

because \( \mu(t)^2 = \pi(t)^* \pi(t) \) and \( \theta(r) \mu(r)^2 \theta(r)^* = \pi(r) \pi(r)^* \) commute.

Notice that

\[
\theta(r) \mu(r) \theta(r)^* = (\theta(r) \mu(r)^2 \theta(r)^*)^{1/2}.
\]

Moreover,

\[
(3) \quad \theta(t + r) \mu(t + r) = \pi(t + r) = \pi(t) \pi(r) = \theta(t) \mu(t) \theta(r) \mu(r).
\]
Since $E(t+r)$ commutes with $\mu(t+r)$ and $E(t+r) \leq E(r)$ because $\text{ran} \mu(t+r) \subseteq \text{ran} \mu(r)$, it follows that 

$$
\mu(t+r) = \mu(t+r)E(t+r) = \mu(t+r)E(t+r)E(r) = \mu(t+r)E(r),
$$

and by using (3) and (2) we get that 

$$
\theta(t+r)\mu(t+r) = \theta(t+r)\mu(t+r)E(r) = \theta(t+r)\mu(t+r)\theta(r)\theta(r) = \theta(t)\mu(t)\theta(r)\theta(r)\mu(t)\theta(r).
$$

Thus,

$$
\theta(t+r)\mu(t+r) = \theta(t)\theta(r)[\mu(r)\theta(r)\mu(t)\theta(r)].
$$

We claim that the right hand side in this equality is the polar decomposition of $\pi(t+r)$.

First let us prove that $\mu(r)$ and $\theta(r)\theta(r)\mu(t)\theta(r)$ commute. Using (1) and (2) we get 

$$
\mu(r)\theta(r)\mu(t)\theta(r) = \theta(r)\theta(r)\mu(t)\theta(r)\mu(t)\theta(r)
$$

Consequently, 

$$
\mu(r)\theta(r)\mu(t)\theta(r) \geq 0.
$$

Moreover, 

$$
(\mu(r)\theta(r)\mu(t)\theta(r))^2 = \pi(t+r)^*\pi(t+r),
$$

because 

$$
\mu(r)\theta(r)\mu(t)\theta(r)\mu(r)\theta(r)\mu(t)\theta(r) = \theta(r)\theta(r)\mu(t)\theta(r)\mu(t)\theta(r)\mu(t)
$$

Hence,

$$
\mu(r)\theta(r)\mu(t)\theta(r) = |\pi(t+r)| = \mu(t+r).
$$

It remains to be proved that $\theta(t)\theta(r) = \theta(t+r)$. Since $E(t), F(r)$ commute, $\theta(t)\theta(r)$ is a partial isometry. Once it is proved that $\ker \pi(t+r) = \ker \theta(t)\theta(r)$, the uniqueness of the polar decomposition will finish the proof. Since $\ker \pi(t+r) = \ker \mu(t+r)$ and $\mu(t+r) = \mu(r)\theta(r)\mu(t)\theta(r)$ as shown above, taking into account that $\ker \mu(s) = \ker \theta(s), s \in S$, we get ($x \in H$) 

$$
\mu(r)\theta(r)\mu(t)\theta(r)x = 0 \iff \theta(r)\theta(r)\mu(t)\theta(r)x = 0
$$

(b) Let $r, t \in S$. Then 

$$
\theta(t+r)\mu(t+r) = \pi(t+r) = \pi(t)\pi(r) = \theta(t)\mu(t)\theta(r)\mu(r).
$$

Multiplying on the left by $\theta(t)^*$ and using (1), we get 

$$
\theta(t)^*\theta(t+r)\mu(t+r) = \mu(t)\theta(r)\mu(r).
$$

On the other hand, $\theta$ is assumed to be a homomorphism. Therefore, 

$$
\theta(t)^*\theta(t+r) = \theta(t)^*\theta(t)\theta(r) = E(t)\theta(r)\theta(r)^*\theta(r) = E(t)F(r)\theta(r) = F(r)E(t)\theta(r)
$$

Therefore, 

$$
\theta(t)^*\theta(t+r)\theta(t)\theta(r)\theta(t)\theta(r) = \theta(r)\theta(r)^*\theta(t)\theta(r)\theta(t)\theta(r).
$$
If we substitute this into (4) and use (1) for \( t + r \), we get

\[ \theta(r)\mu(t + r) = \mu(t)\theta(r)\mu(r). \]  \hspace{1cm} (5)

Now, multiplying on the right by \( \theta(r)^* \) we obtain

\[ \theta(r)\mu(t + r)\theta(r)^* = \mu(t)[\theta(r)\mu(r)\theta(r)^*]. \]

Since the operator on the left hand side, \( \mu(t) \), and \( \theta(r)\mu(r)\theta(r)^* \) are all self-adjoint, we conclude that \( \mu(t) \) commutes with \( \theta(r)\mu(r)\theta(r)^* \).

Therefore if we multiply the equality (5) on the left by \( \theta(r)^* \) and use the inclusion \( \text{ran} \mu(t + r) \subset \text{ran} \mu(r) \), we get

\[ \mu(t + r) = \theta(r)^*\theta(r)\mu(t + r) = \theta(r)^*\mu(t)\theta(r)\mu(r). \]

Therefore, since \( \mu(t + r) \) is self-adjoint, \( \mu(r) \) commutes with \( \theta(r)^*\mu(t)\theta(r) \).

Finally,

\[ \mu(t + r)\mu(r) = \theta(r)^*\mu(t)\theta(r)\mu(r)^2 = \mu(r)\theta(r)^*\mu(t)\theta(r)\mu(r) = \mu(r)\mu(t + r). \]

Hence, by our additional assumption in (b),

\[ \mu(t)\mu(s) = \mu(s)\mu(t), \quad s, t \in S. \]

Now we are ready to prove the claim, i.e., three commutativity relations. The proofs follow.

First:

\[ \pi(s)^*\pi(s)\pi(t)^*\pi(t) = \mu(s)^2\mu(t)^2 = \pi(t)^*\pi(t)\pi(s)^*\pi(s). \]

Second:

\[ \pi(t)^*\pi(t)\pi(r)^* = \mu(t)^2\theta(r)\mu(r)^2\theta(r)^* = \mu(t)[\mu(t)\theta(r)\mu(r)]\mu(r)\theta(r)^* \]

\[ = \mu(t)\theta(r)\mu(t + r)\mu(r)\theta(r)^* = \theta(r)^*\mu(t + r)\theta(r)\] commutes with \( \pi(r)\pi(t)^* \).

Third:

\[ \pi(t + r)\pi(t + r)^*\pi(t)\pi(t)^* \]

\[ = \theta(t + r)\mu(t + r)^2\theta(t + r)^*\theta(t)\mu(t)^2\theta(t) \]

\[ = \theta(t + r)\mu(t + r)^2\theta(r)^*\theta(t)^*\theta(t)\mu(t)^2\theta(t) \]

\[ = \theta(t)\theta(r)\mu(t + r)^2\theta(r)^*\mu(t)^2\theta(t) \]

\[ = \theta(t)\mu(t)\theta(r)\mu(t + r)\theta(r)^*\mu(t)^2\theta(t) \]

\[ = \theta(t)\mu(t)\theta(r)\mu(t + r)\theta(r)^*\mu(t)^2\theta(t)^* \]

\[ = \theta(t)\mu(t)\mu(t)\theta(r)\mu(r)\theta(r)^*\mu(t)^2\theta(t)^* \]

\[ = \theta(t)\mu(t)^2\theta(r)\mu(r)^2\theta(r)^*\mu(t)^2\theta(t)^*. \]

Since the last operator is self-adjoint, \( \pi(t + r)\pi(t + r)^* \) commutes with \( \pi(t)\pi(t)^* \).

Using the additional assumption in (b) we get that \( \pi(s)\pi(s)^* \) commutes with \( \pi(t)\pi(t)^* \). \hfill \Box
Our proof of part (b) follows the idea of the proof of Theorem 6 of M. Embry in [2].

In part (b) we made an additional assumption about the semigroup $S$. Unfortunately, we were not able to prove part (b) without that assumption, which not all semigroups satisfy, e.g., $\mathbb{N} \times \mathbb{N}$ does not. This assumption has a certain order-type flavor, which is, perhaps, essential.

**Theorem 2.** Let $S$ be a commutative semigroup with unit and let $\pi: S \rightarrow B(H)$ be a semigroup homomorphism. $\pi$ is a quasinormal homomorphism if and only if $\mu$ is a semigroup homomorphism.

**Proof.** Suppose $\pi$ is a quasinormal homomorphism. Notice that $\theta(t), \mu(t), \theta(s)$, and $\mu(s)$ commute, $s, t \in S$.

Thus $\pi(s + t) = \pi(s)\pi(t) = \theta(s)\mu(s)\theta(t)\mu(t) = \theta(s)\theta(t)\mu(s)\mu(t)$. Now, since $\pi$ is quasinormal, it is centered and therefore $\theta$ is also a homomorphism by Theorem 1. Hence, $\pi(t + s) = \theta(s + t)\mu(s)\mu(t)$ and by the uniqueness of the polar decomposition we get that $\mu(s + t) = \mu(s)\mu(t)$. This part of the proof can essentially be found in A. Lubin [3], proof of Theorem 4.5.

Suppose now that $\mu$ is a semigroup homomorphism. To prove that $\pi$ is quasinormal we use one of the equivalent conditions for quasinormality from Theorem 3.10 in W. Szymański [5], namely, $\pi$ is a quasinormal homomorphism if and only if each operator $\pi(s)$, $s \in S$, is quasinormal and $\pi$ is a subnormal homomorphism.

(i) To prove that $\pi$ is subnormal we use the following criterion to characterize subnormal homomorphisms (W. Szymański [5]):

(EL) $\pi$ is subnormal if and only if

$$\Sigma(\pi(s + t)f(t), \pi(s + t)f(s)) \geq 0$$

for all functions $f: S \rightarrow H$ with finite support.

Let $f$ be such a function. Then

$$\Sigma(\pi(s + t)f(t), \pi(s + t)f(s)) = \Sigma(\theta(s + t)\mu(s + t)f(t), \theta(s + t)\mu(s + t)f(s))$$

$$= \Sigma(\theta(s + t)\theta(s + t)\mu(s + t)f(t), \mu(s + t)f(s))$$

$$= \Sigma(\mu(s + t)f(t), \mu(s + t)f(s)) = \Sigma(\mu(s + t)f(t), f(s))$$

$$= \Sigma(\mu(s + s + t)f(t), f(s)) = \Sigma(\mu(s + s)f(t), f(s)) = \Sigma(\mu(s + s)f(s), f(s)) = \|\Sigma(\mu(s + s)f(s))\|^2 \geq 0.$$

Thus $\pi$ is subnormal.

(ii) Now we only need to prove that each $\pi(s)$, $s \in S$, is quasinormal. Each $\pi(s)$, $s \in S$, is subnormal, because $\pi$ is a subnormal homomorphism, as shown in (i) above. Hence each $\pi(s)$ is hyponormal, $s \in S$. A simple observation made by M. Embry in [1], page 63, that $T \in B(H)$ is quasinormal if and only if $T$ is hyponormal and $|T|^2 = |T|^2$, completes the proof. Indeed, let $T = \pi(s)$, $s \in S$. Since $\mu$ is a semigroup homomorphism, it follows that $|T|^2 = |\pi(2s)| = \mu(2s) = \mu(s)^2 = |T|^2$.

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