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CYCLE RANK OF LYAPUNOV GRAPHS AND THE GENERA OF MANIFOLDS

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Abstract. We show that the cycle-rank $r(L)$ of a Lyapunov graph L on a manifold M satisfies: $r(L) \leq g(M)$, where $g(M)$ is the genus of M. This generalizes a theorem of Franks. We also show that given any integer r with $0 \leq r \leq g(M)$, $r = r(L)$ for some Lyapunov graph L on $M, \dim M > 2$.

1. Introduction

Let M be a smooth, compact, connected n-manifold with boundary. The genus of M, $g(M)$, [1] is the maximal number of mutually disjoint, smooth, compact, connected, two-sided codimension one submanifolds that do not disconnect M. This definition coincides with the classical definition of genus of a compact orientable 2-manifold.

Let $f : M \to \mathbb{R}$ be a Lyapunov function associated to a flow and define the following equivalence relation on M: $x \sim_f y$ if and only if x and y belong to the same connected component of a level set of f. We call $M/\sim_f$ a *Lyapunov graph*. The *cycle rank* of a graph is the maximum number of edges that can be removed without disconnecting the graph.

We will generalize the following theorem of Franks [4].

Theorem 1.1. *Let φ_t be a smooth flow on a closed, connected n-manifold M with Lyapunov function f. Let L be the Lyapunov graph of f. Assume that L is finite. If M is orientable and $\beta_1(M) = 0$ (the first Betti number), then the cycle-rank of L, $r(L)$, is 0. Since L is connected, this implies that L is a tree.*

We will show that $r(L) \leq g(M)$ if L is a Lyapunov graph of a Lyapunov function f associated to a smooth flow φ_t on M. As a consequence of Corollary 2.4, which asserts that $\beta_1(M) = 0$ if and only if $g(M) = 0$, we prove Theorem 1.1 without the assumption: M orientable. We will also prove that given an integer $r, 0 \leq r \leq g(M)$, there exists a Lyapunov graph L associated to a Morse function f with gradient flow φ_t on M such that $r(L) = r$.

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2. The genus of a manifold

The genus of a group $\pi, g(\pi)$, is the maximal rank r of a free group F_r (r is the number of generators) such that there exists an epimorphism $\pi \to F_r$. If π is finitely presented, $g(\pi) < \infty$. In [1] Cornea proves the following theorem whose proof we sketch in Appendix A:

Theorem 2.1. *Let M be a smooth compact connected n-manifold with boundary. Then*

1. $g(M) \leq g(\pi_1 M)$,
2. $g(M) = g(\pi_1 M)$ *if* $\partial M = \emptyset$.

By using the fact that a subgroup of a free group is free, Cornea [1] also proves the following theorem:

Theorem 2.2. *Let π_1, π_2 be two finitely presented groups. Then $g(\pi_1 * \pi_2) = g(\pi_1) + g(\pi_2)$.*

In [1], Cornea has the following result which we will prove here using $\mathbb{Z}$ coefficients.

Proposition 2.3. *Let M be a smooth, connected closed n-manifold. Then*

1. $g(M) \leq \beta_1(M)$ *(the first Betti number),*
2. $\beta_1(M) > 0 \Rightarrow g(M) > 0$.

Proof. There is an epimorphism $h : \pi_1 M \to F_g, F_g$ the free group on $g = g(M) = g(\pi_1 M)$ generators. We have the following commutative diagram:

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{\quad h \quad} & F_g \\ \downarrow & & \downarrow \\ H_1(M;\mathbb{Z}) & \xrightarrow[\quad h_{ab} \quad]{} & \mathbb{Z}^g \end{array}$$

where the vertical maps are abelianization homomorphisms and h_{ab} is h abelianized. The homomorphism h_{ab} is surjective because the other homomorphisms above are. Thus, result 1. follows.

If $\beta_1(M) > 0$, then $H_1(M;\mathbb{Z})$ contains a direct summand isomorphic to $\mathbb{Z}$. In particular, there is an epimorphism $H_1(M;\mathbb{Z}) \to \mathbb{Z}$. Composing this epimorphism with the abelianization $\pi_1(M) \to H_1(M;\mathbb{Z})$ gives an epimorphism $\pi_1(M) \to \mathbb{Z}$. Thus, $g(M) \geq 1$. □

Corollary 2.4. *Let M be a smooth, connected closed n-manifold. Then, $g(M) = 0$ if and only if $\beta_1(M) = 0$.*

Proof. Straightforward. □

Example 2.5. Let π be an abelian group. Given an epimorphism $\pi \to F_r$, we conclude $r = 0$ or 1. Therefore, $g(\pi) = 0$ or 1. By the Structure Theorem for finitely generated abelian groups it follows that $g(\pi) = 0$ if and only if π is finite. Note that $g(\mathbb{Z}^n) = 1$. Thus, we have the following examples:

1. $T^n = S^1 \times \ldots \times S^1$ (n times) $g(T^n) = 1$.
2. $g((S^1 \times S^1 \times S^2)\#(S^1 \times S^3)\#T^4) = 3$.
3. $g((T^3 \times S^2 \times S^3)\#T^8\#(T^6 \times S^2)\#(S^2 \times S^6)) = 3$.

3. Dynamics and the genus

Proposition 3.1. *Let M be a connected, closed smooth n-manifold. Let φ_t be a smooth flow on M with associated Lyapunov function f. Let L be a Lyapunov graph associated to f. Assume that L is finite. Then*

$$r(L) \leq g(M).$$

Proof. Set $r = r(L)$. Let $T \subset L$ be a maximal tree and $q : M \to L$ the quotient map. $L - T$ is the disjoint union of r edges $e_1, \ldots, e_r$. By Sard's theorem it is possible to take points $x_1, \ldots, x_r, x_1 \in e_1, \ldots, x_r \in e_r$ with the following properties: $q^{-1}(x_1), \ldots, q^{-1}(x_r)$ are submanifolds. These submanifolds are mutually disjoint, smooth, closed, connected, two-sided, codimension one submanifolds that do not disconnect M. The result follows. □

Proposition 3.1 together with Corollary 2.4 provide a proof of Theorem 1.1 without the orientability assumption.

Example 3.2. Let n be any *large* integer > 1, $M = T^n$ and L be the Lyapunov graph of a Lyapunov function f associated to a smooth flow φ_t on T^n. Proposition 3.1 implies that $r(L) \leq 1$. Thus, L can have at most one cycle! However, the first Betti number of T^n increases with n, therefore, genus, and not the first Betti number, is the right tool to study the cycle-rank of Lyapunov graphs.

Theorem 3.3. *Let M be a smooth, connected, closed n-manifold, $n > 2$. Let r be an integer with $0 \leq r \leq g(M)$. Then, there is a gradient flow φ_t on M with associated Morse function f such that $r(L) = r, L$ the Lyapunov graph of f. In particular, $g(M)$ is the largest cycle-rank for Lyapunov graphs on M.*

The fact that this theorem does not hold for $n = 2$ is shown in [3].

Before we proceed with the proof, we will define a Lyapunov graph associated to a smooth flow on a smooth compact n-manifold with boundary W. Let φ_t be a smooth flow on W with associated Lyapunov function $\tilde{f}$. Let $\partial_+ W$ be the component of ∂W for which $\dfrac{\partial}{\partial t}\varphi_t$ points inward. Similarly, $\partial_- W$ is the component of ∂W for which $\dfrac{\partial}{\partial t}\varphi_t$ points outward. The Lyapunov graph $\tilde{L}$ of $\tilde{f}$ is defined exactly as in [4]. However, since $\partial W \neq \emptyset$, the vertices correspond to components of the chain recurrent of φ_t together with the boundary components of W. Vertices that correspond to boundary components will be referred to as boundary vertices. In what follows we assume $\tilde{L}$ is finite. We will need a lemma.

Lemma 3.4. *Let W be as above with $n > 2$. Assume that $\tilde{f}$ is the Morse function associated to an ordered handle decomposition [5] $\tilde{\mathcal{H}}$ of W. In addition, assume $\partial_+ W = \emptyset$ and $\tilde{\mathcal{H}}$ contains a single n-handle (equivalently we could assume $\partial_- W = \emptyset$ and $\mathcal{H}$ contains a single 0-handle). Then, $\tilde{L}$ is a tree.*

Proof (of Theorem 3.3). Let $N_1, \ldots, N_r$, $r \geq 1$, be mutually disjoint, smooth, closed, connected, two-sided codimension one submanifolds that do not disconnect M. Let $T_1, \ldots, T_r$ be mutually disjoint closed tubular neighborhoods of $N_1, \ldots, N_r$ respectively. Set $W = M - \bigcup_{i=1}^{r} \operatorname{int} T_i$. See Figure 1. As W is connected, we can choose an ordered handle decomposition $\widetilde{\mathcal{H}}$ for $(W; \partial W, \emptyset)$ containing a single n-handle [5]. Choose handle decompositions $\mathcal{H}_i$ for $(T_i; \emptyset, \partial T_i)$ which are ordered and

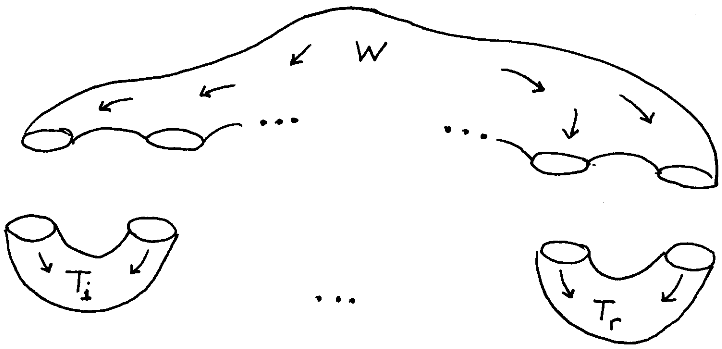

FIGURE 1. Decomposition of M.

contains just one 0-handle, $i = 1, \ldots, r$. Let $f_i, i = 1, \ldots, r$, be Morse functions associated to these handle decompositions $\mathcal{H}_i$.

Define a handle decomposition $\mathcal{H}$ for M as follows: $\mathcal{H} = (\bigcup_{i=1}^{r} \mathcal{H}_i) \cup \widetilde{\mathcal{H}}$. The order of attachment is: attach the handles of $\mathcal{H}_1$ respecting the order of an attachment of $\mathcal{H}_1$. Do the same for $\mathcal{H}_2, \ldots, \mathcal{H}_r, \widetilde{\mathcal{H}}$. Let f be the Morse function that corresponds to $\mathcal{H}$, such that $f\Big|_W = \widetilde{f}$ and $f\Big|_{T_i} = f_i, i = 1, \ldots, r$. Let L be a Lyapunov graph of f and $\widetilde{L}, L_i$ be Lyapunov graphs of $\widetilde{f}, f_i, i = 1, \ldots, r$. L is obtained by "glueing" $\widetilde{L}$ and $L_i, i = 1, \ldots, r$, along boundary vertices that correspond to the $2r$ components of ∂W and to the two components of $\partial T_i, i = 1, \ldots, r$. Since ∂W has $2r$ components and $\widetilde{\mathcal{H}}$ is an ordered handle decomposition with one n-handle, then by Lemma 3.4 the corresponding graph $\tilde{L}$ is a tree as shown in Figure 2. Similarly, ∂T_i has 2 components and the handle decomposition $\mathcal{H}_i$ is ordered and has one 0-handle. Hence, by Lemma 3.4 the corresponding graph L_i is a tree as shown below in Figure 2. The "glueing" of the graphs will be performed so that the edge labellings incident to a pair of boundary vertices that will be identified match. Furthermore, a pair of identified boundary vertices becomes an edge point (more precisely, if e_-, e_+ are edges incident to a boundary vertex $v, e = e_- \cup \{v\} \cup e_+$ is the edge in L for which v is an edge point). Hence, $r(L) = r$. We now prove the case $r = 0$. In view of a result of Smale's [6] we can take an ordered Morse function f on M with one index 0 singularity and one index n singularity. By results in [2] as summarized in Table 1, all level sets are connected and the graph L associated to the flow is linear. Hence $r(L) = 0$. □

Proof (of Lemma 3.4). Assume $\partial_+ W = \emptyset$. The case $\partial_- W = \emptyset$ is analogous. L is oriented by the direction of flow lines. Let v be a vertex of L. Denote by $e^-(v), e^+(v)$ the number of outgoing, ingoing edges of v, respectively. If v corresponds to a component of $\partial_- W$, then $e^-(v) = 0, e^+(v) = 1$. If v corresponds to an index i singularity of φ_t, we set ind $v = i$ (the index of v). By Corollary 3.1 in [2] we have:

L is path-connected because W is. Let v_n be the unique vertex with ind $v_n = n$. Since we are assuming an ordered handle decomposition, this implies that $e^+(v) = 1$ if v is a vertex with ind $v = n - 1$.

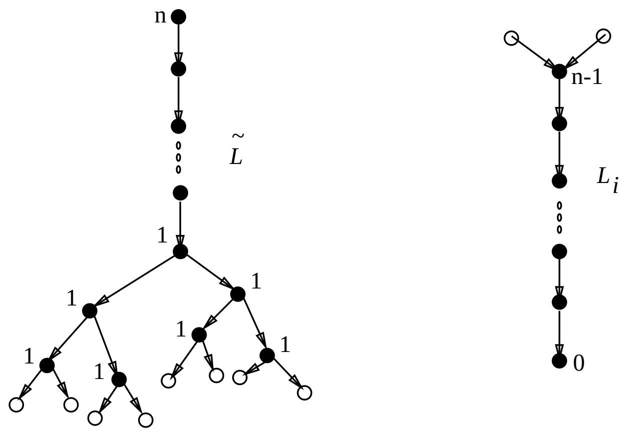

FIGURE 2. Lyapunov graphs $\tilde{L}$ and L_i. Here, "●" indicates a vertex labelled with the index of a singularity of φ_t and "○" indicates a boundary vertex.

TABLE 1

ind v	0	1	$1 < i < n-1$	$n-1$	n
$e^-(v)$	0	1 or 2	1	1	1
$e^+(v)$	1	1	1	1 or 2	0

Otherwise, if $e^+(v) = 2$ we would necessarily have to attach an index $n-1$ handle before attaching an index 1 handle. The orientation of L is equivalent to a partial order relation for its vertices such that $v \leq u$ if $\partial e = \{v, u\}$, e an edge which is ingoing for v (equivalently e is outgoing for u). We define $\partial^+ e = v$ and $\partial^- e = u$.

Let $\mathcal{V}$, $\mathcal{E}$ be the vertices and edges of L respectively. By the previous table and paragraph, $e^+(v) = 1$ for all $v \in \mathcal{V} - \{v_n\}$. This fact allows us to define a bijection $\mathcal{F}\colon \mathcal{V} - \{v_n\} \to \mathcal{E}$ as follows: for $v \in \mathcal{V} - \{v_n\}$, $\mathcal{F}(v) = e$ where $\partial^+ e = v$, i.e., e is the edge ingoing for v. Conclusion: the Euler characteristic of L is one. Thus, L is a tree. □

APPENDIX A

We sketch here a proof of the following theorem of Cornea's [1]:

Theorem 3.5. *Let M be a smooth compact connected n-manifold with boundary. Then*

1. $g(M) \leq g(\pi_1 M)$,
2. $g(M) = g(\pi_1 M)$ *if* $\partial M = \emptyset$.

Proof. (Sketch) Part 1. is proved as follows. Let $N_1, \ldots, N_r, r \leq g(M)$ (at this point we do not know that $g(M) < \infty$), be mutually disjoint, smooth, compact, connected, two-sided codimension one submanifolds that do not disconnect M. Let $T_1, \ldots, T_r$ be mutually disjoint closed tubular neighborhoods of $N_1, \ldots, N_r$ respectively. There are diffeomorphisms of pairs $\varphi_i : (T_i, N_i) \to (N_i \times D^1, N_i \times$

$\{0\}), i = 1, \ldots, r$. Collapsing $W = M - \bigcup_{i=1}^{r} T_i$ to a point and T_i to $\{x_i\} \times D^1, x_i \in N_i, i = 1, \ldots, r$, defines an equivalence relation $\sim$ on M. Hence $M/\sim = B_r$, a bouquet of r circles. Let $c : M \to B_r$ be the quotient map. The circles of B_r are $c(\varphi_i^{-1}(\{x_i\} \times D^1)), i = 1, \ldots, r$. As W is path-connected, it follows that $c_* : \pi_1(M) \to \pi_1(B_r) \cong F_r$ is an epimorphism. Conclusion: $r \leq g(\pi_1 M)$. As $g(M)$ is the maximal number of mutually disjoint, smooth, compact, connected, two-sided codimension one submanifolds that do not disconnect M, it follows that $g(M) \leq g(\pi_1 M)$ $(< \infty)$.

Part 2. is proved as follows. Let $g = g(\pi_1 M)$. Start with an epimorphism $\pi_1 M \to \pi_1(B_g) \cong F_g$. As B_g is an Eilenberg-Mac Lane space (see [7], p. 225), the above epimorphism is induced by a continuous map $h : M \to B_g$. The basepoint of B_g, b, is the intersection of all circles of B_g. Let x_0 be any point of M. We may assume that the pre-image of b is x_0. Thus, h is the one point union of g maps from M to circles containing b. Homotopying rel x_0 if necessary, we may assume that these maps are smooth. Now, by Sard's theorem there are points $b_1, \ldots, b_g \in B_g - \{b\}$ lying in disjoint circles which are regular values for the functions that make up h. Conclusion: $h^{-1}(b_1), \ldots, h^{-1}(b_g)$ are mutually disjoint, smooth, closed, connected codimension one submanifolds that do not disconnect M. Thus, $g(\pi_1 M) \leq g(M)$. Part 2. follows. □

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