CYCLE RANK OF LYAPUNOV GRAPHS AND THE GENERA OF MANIFOLDS

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Abstract. We show that the cycle-rank \( r(L) \) of a Lyapunov graph \( L \) on a manifold \( M \) satisfies: \( r(L) \leq g(M) \), where \( g(M) \) is the genus of \( M \). This generalizes a theorem of Franks. We also show that given any integer \( r \) with \( 0 \leq r \leq g(M) \), \( r = r(L) \) for some Lyapunov graph \( L \) on \( M \), \( \dim M > 2 \).

1. Introduction

Let \( M \) be a smooth, compact, connected \( n \)-manifold with boundary. The genus of \( M \), \( g(M) \), [1] is the maximal number of mutually disjoint, smooth, compact, connected, two-sided codimension one submanifolds that do not disconnect \( M \). This definition coincides with the classical definition of genus of a compact orientable 2-manifold.

Let \( f : M \to \mathbb{R} \) be a Lyapunov function associated to a flow and define the following equivalence relation on \( M \): \( x \sim_f y \) if and only if \( x \) and \( y \) belong to the same connected component of a level set of \( f \). We call \( M/\sim_f \) a Lyapunov graph. The cycle rank of a graph is the maximum number of edges that can be removed without disconnecting the graph.

We will generalize the following theorem of Franks [4].

Theorem 1.1. Let \( \varphi_t \) be a smooth flow on a closed, connected \( n \)-manifold \( M \) with Lyapunov function \( f \). Let \( L \) be the Lyapunov graph of \( f \). Assume that \( L \) is finite. If \( M \) is orientable and \( \beta_1(M) = 0 \) (the first Betti number), then the cycle-rank of \( L \), \( r(L) \), is 0. Since \( L \) is connected, this implies that \( L \) is a tree.

We will show that \( r(L) \leq g(M) \) if \( L \) is a Lyapunov graph of a Lyapunov function \( f \) associated to a smooth flow \( \varphi_t \) on \( M \). As a consequence of Corollary 2.4, which asserts that \( \beta_1(M) = 0 \) if and only if \( g(M) = 0 \), we prove Theorem 1.1 without the assumption: \( M \) orientable. We will also prove that given an integer \( r, 0 \leq r \leq g(M) \), there exists a Lyapunov graph \( L \) associated to a Morse function \( f \) with gradient flow \( \varphi_t \) on \( M \) such that \( r(L) = r \).

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2. The genus of a manifold

The genus of a group \( \pi, g(\pi) \), is the maximal rank \( r \) of a free group \( F_r \) (\( r \) is the number of generators) such that there exists an epimorphism \( \pi \to F_r \). If \( \pi \) is finitely presented, \( g(\pi) < \infty \). In [1] Cornea proves the following theorem whose proof we sketch in Appendix A:

Theorem 2.1. Let \( M \) be a smooth compact connected \( n \)-manifold with boundary. Then

1. \( g(M) \leq g(\pi_1 M) \),
2. \( g(M) = g(\pi_1 M) \) if \( \partial M = \emptyset \).

By using the fact that a subgroup of a free group is free, Cornea [1] also proves the following theorem:

Theorem 2.2. Let \( \pi_1, \pi_2 \) be two finitely presented groups. Then

\[ g(\pi_1 * \pi_2) = g(\pi_1) + g(\pi_2) \]

In [1], Cornea has the following result which we will prove here using \( \mathbb{Z} \) coefficients.

Proposition 2.3. Let \( M \) be a smooth, connected closed \( n \)-manifold. Then

1. \( g(M) \leq \beta_1(M) \) (the first Betti number),
2. \( \beta_1(M) > 0 \Rightarrow g(M) > 0 \).

Proof. There is an epimorphism \( h : \pi_1 M \to F_g \), the free group on \( g = g(M) = g(\pi_1 M) \) generators. We have the following commutative diagram:

\[
\begin{array}{ccc}
\pi_1(M) & \xrightarrow{h} & F_g \\
\downarrow & & \downarrow \\
H_1(M; \mathbb{Z}) & \xrightarrow{h_{ab}} & \mathbb{Z}^g 
\end{array}
\]

where the vertical maps are abelianization homomorphisms and \( h_{ab} \) is \( h \) abelianized. The homomorphism \( h_{ab} \) is surjective because the other homomorphisms above are. Thus, result 1. follows.

If \( \beta_1(M) > 0 \), then \( H_1(M; \mathbb{Z}) \) contains a direct summand isomorphic to \( \mathbb{Z} \). In particular, there is an epimorphism \( H_1(M; \mathbb{Z}) \to \mathbb{Z} \). Composing this epimorphism with the abelianization \( \pi_1(M) \to H_1(M; \mathbb{Z}) \) gives an epimorphism \( \pi_1(M) \to \mathbb{Z} \). Thus, \( g(M) \geq 1 \).

Corollary 2.4. Let \( M \) be a smooth, connected closed \( n \)-manifold. Then, \( g(M) = 0 \) if and only if \( \beta_1(M) = 0 \).

Proof. Straightforward.

Example 2.5. Let \( \pi \) be an abelian group. Given an epimorphism \( \pi \to F_r \), we conclude \( r = 0 \) or 1. Therefore, \( g(\pi) = 0 \) or 1. By the Structure Theorem for finitely generated abelian groups it follows that \( g(\pi) = 0 \) if and only if \( \pi \) is finite. Note that \( g(\mathbb{Z}^n) = 1 \). Thus, we have the following examples:

1. \( T^n = S^1 \times \ldots \times S^1 \) (\( n \) times) \( g(T^n) = 1 \).
2. \( g((S^1 \times S^1 \times S^2) \# (S^1 \times S^3) \# T^4) = 3 \).
3. \( g((T^3 \times S^2 \times S^3) \# T^8 \# (T^6 \times S^2) \# (S^2 \times S^6)) = 3 \).
3. Dynamics and the genus

**Proposition 3.1.** Let $M$ be a connected, closed smooth $n$-manifold. Let $\varphi_t$ be a smooth flow on $M$ with associated Lyapunov function $f$. Let $L$ be a Lyapunov graph associated to $f$. Assume that $L$ is finite. Then

$$r(L) \leq g(M).$$

**Proof.** Set $r = r(L)$. Let $T \subset L$ be a maximal tree and $q : M \to L$ the quotient map. $L - T$ is the disjoint union of $r$ edges $e_1, \ldots, e_r$. By Sard’s theorem it is possible to take points $x_1, \ldots, x_r, x_1 \in e_1, \ldots, x_r \in e_r$ with the following properties: $q^{-1}(x_1), \ldots, q^{-1}(x_r)$ are submanifolds. These submanifolds are mutually disjoint, smooth, closed, connected, two-sided, codimension one submanifolds that do not disconnect $M$. The result follows. \hfill $\square$

Proposition 3.1 together with Corollary 2.4 provide a proof of Theorem 1.1 without the orientability assumption.

**Example 3.2.** Let $n$ be any large integer $> 1$, $M = T^n$ and $L$ be the Lyapunov graph of a Lyapunov function $f$ associated to a smooth flow $\varphi_t$ on $T^n$. Proposition 3.1 implies that $r(L) \leq 1$. Thus, $L$ can have at most one cycle! However, the first Betti number of $T^n$ increases with $n$, therefore, genus, and not the first Betti number, is the right tool to study the cycle-rank of Lyapunov graphs.

**Theorem 3.3.** Let $M$ be a smooth, connected, closed $n$-manifold, $n > 2$. Let $r$ be an integer with $0 \leq r \leq g(M)$. Then, there is a gradient flow $\varphi_t$ on $M$ with associated Morse function $f$ such that $r(L) = r, L$ the Lyapunov graph of $f$. In particular, $g(M)$ is the largest cycle-rank for Lyapunov graphs on $M$.

The fact that this theorem does not hold for $n = 2$ is shown in [3].

Before we proceed with the proof, we will define a Lyapunov graph associated to a smooth flow on a smooth compact $n$-manifold with boundary $W$. Let $\varphi_t$ be a smooth flow on $W$ with associated Lyapunov function $\hat{f}$. Let $\partial_t W$ be the component of $\partial W$ for which $\frac{\partial}{\partial t} \varphi_t$ points inward. Similarly, $\partial - W$ is the component of $\partial W$ for which $\frac{\partial}{\partial t} \varphi_t$ points outward. The Lyapunov graph $\hat{L}$ of $\hat{f}$ is defined exactly as in [4]. However, since $\partial W \neq \emptyset$, the vertices correspond to components of the chain recurrent of $\varphi_t$ together with the boundary components of $W$. Vertices that correspond to boundary components will be referred to as boundary vertices. In what follows we assume $\hat{L}$ is finite. We will need a lemma.

**Lemma 3.4.** Let $W$ be as above with $n > 2$. Assume that $\hat{f}$ is the Morse function associated to an ordered handle decomposition [5] $\mathcal{H}$ of $W$. In addition, assume $\partial_+ W = \emptyset$ and $\mathcal{H}$ contains a single $n$-handle (equivalently we could assume $\partial_- W = \emptyset$ and $\mathcal{H}$ contains a single 0-handle). Then, $\hat{L}$ is a tree.

**Proof (of Theorem 3.3).** Let $N_1, \ldots, N_r, r \geq 1$, be mutually disjoint, smooth, closed, connected, two-sided codimension one submanifolds that do not disconnect $M$. Let $T_1, \ldots, T_r$ be mutually disjoint closed tubular neighborhoods of $N_1, \ldots, N_r$ respectively. Set $W = M - \bigcup_{i=1}^r \operatorname{int} T_i$. See Figure 1. As $W$ is connected, we can choose an ordered handle decomposition $\tilde{\mathcal{H}}$ for $(W; \partial W, \emptyset)$ containing a single $n$-handle [5]. Choose handle decompositions $\mathcal{H}_i$ for $(T_i; \emptyset, \partial T_i)$ which are ordered and
contains just one 0-handle, $i = 1, \ldots, r$. Let $f_i, i = 1, \ldots, r$, be Morse functions associated to these handle decompositions $H_i$.

Define a handle decomposition $H$ for $M$ as follows: $H = \bigcup_{i=1}^r H_i \cup \tilde{H}$. The order of attachment is: attach the handles of $H_1$ respecting the order of an attachment of $H_1$. Do the same for $H_2, \ldots, H_r, \tilde{H}$. Let $f$ be the Morse function that corresponds to $H$, such that $f \bigg|_W = \tilde{f}$ and $f \bigg|_{T_i} = f_i, i = 1, \ldots, r$. Let $L$ be a Lyapunov graph of $f$ and $\tilde{L}, L_i$ be Lyapunov graphs of $\tilde{f}, f_i, i = 1, \ldots, r$. $L$ is obtained by “glueing” $\tilde{L}$ and $L_i, i = 1, \ldots, r$, along boundary vertices that correspond to the $2r$ components of $\partial W$ and to the two components of $\partial T_i, i = 1, \ldots, r$. Since $\partial W$ has $2r$ components and $\tilde{H}$ is an ordered handle decomposition with one $n$-handle, then by Lemma 3.4 the corresponding graph $\tilde{L}$ is a tree as shown in Figure 2. Similarly, $\partial T_i$ has 2 components and the handle decomposition $H_i$ is ordered and has one 0-handle. Hence, by Lemma 3.4 the corresponding graph $L_i$ is a tree as shown below in Figure 2. The “glueing” of the graphs will be performed so that the edge labellings incident to a pair of boundary vertices that will be identified match. Furthermore, a pair of identified boundary vertices becomes an edge point (more precisely, if $e_-, e_+$ are edges incident to a boundary vertex $v, e = e_- \cup \{v\} \cup e_+$ is the edge in $L$ for which $v$ is an edge point). Hence, $r(L) = r$. We now prove the case $r = 0$. In view of a result of Smale’s [6] we can take an ordered Morse function $f$ on $M$ with one index 0 singularity and one index $n$ singularity. By results in [2] as summarized in Table 1, all level sets are connected and the graph $L$ associated to the flow is linear. Hence $r(L) = 0$.

**Proof (of Lemma 3.4).** Assume $\partial_+ W = \emptyset$. The case $\partial_- W = \emptyset$ is analogous. $L$ is oriented by the direction of flow lines. Let $v$ be a vertex of $L$. Denote by $e^-(v), e^+(v)$ the number of outgoing, ingoing edges of $v$, respectively. If $v$ corresponds to a component of $\partial_+ W$, then $e^-(v) = 0, e^+(v) = 1$. If $v$ corresponds to an index $i$ singularity of $\varphi_t$, we set $\text{ind } v = i$ (the index of $v$). By Corollary 3.1 in [2] we have: $L$ is path-connected because $W$ is. Let $v_n$ be the unique vertex with $\text{ind } v_n = n$. Since we are assuming an ordered handle decomposition, this implies that $e^+(v) = 1$ if $v$ is a vertex with $\text{ind } v = n - 1$. 

![Figure 1. Decomposition of $M$.](image)
Figure 2. Lyapunov graphs $\tilde{L}$ and $L_i$. Here, "•" indicates a vertex labelled with the index of a singularity of $\varphi_t$ and "O" indicates a boundary vertex.

Table 1

<table>
<thead>
<tr>
<th>ind $v$</th>
<th>0</th>
<th>1</th>
<th>$1 &lt; i &lt; n - 1$</th>
<th>$n - 1$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^-(v)$</td>
<td>0</td>
<td>1 or 2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$e^+(v)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1 or 2</td>
<td>0</td>
</tr>
</tbody>
</table>

Otherwise, if $e^+(v) = 2$ we would necessarily have to attach an index $n - 1$ handle before attaching an index 1 handle. The orientation of $L$ is equivalent to a partial order relation for its vertices such that $v \leq u$ if $\partial e = \{v, u\}$, $e$ an edge which is ingoing for $v$ (equivalently $e$ is outgoing for $u$). We define $\partial^+ e = v$ and $\partial^- e = u$.

Let $V, E$ be the vertices and edges of $L$ respectively. By the previous table and paragraph, $e^+(v) = 1$ for all $v \in V - \{v_n\}$. This fact allows us to define a bijection $F: V - \{v_n\} \rightarrow E$ as follows: for $v \in V - \{v_n\}$, $F(v) = e$ where $\partial^+ e = v$, i.e., $e$ is the edge ingoing for $v$. Conclusion: the Euler characteristic of $L$ is one. Thus, $L$ is a tree.

Appendix A

We sketch here a proof of the following theorem of Cornea’s [1]:

**Theorem 3.5.** Let $M$ be a smooth compact connected $n$-manifold with boundary. Then

1. $g(M) \leq g(\pi_1 M)$,
2. $g(M) = g(\pi_1 M)$ if $\partial M = \emptyset$.

**Proof.** (Sketch) Part 1. is proved as follows. Let $N_1, \ldots, N_r, r \leq g(M)$ (at this point we do not know that $g(M) < \infty$), be mutually disjoint, smooth, compact, connected, two-sided codimension one submanifolds that do not disconnect $M$. Let $T_1, \ldots, T_r$ be mutually disjoint closed tubular neighborhoods of $N_1, \ldots, N_r$ respectively. There are diffeomorphisms of pairs $\varphi_i : (T_i, N_i) \rightarrow (N_i \times D^1, N_i \times D^1)$.
\{0\}, i = 1, \ldots, r. Collapsing \( W = M - \bigcup_{i=1}^{r} T_i \) to a point and \( T_i \) to \( \{x_i\} \times D^1, x_i \in N_i, i = 1, \ldots, r \), defines an equivalence relation \( \sim \) on \( M \). Hence \( M/\sim = B_r \), a bouquet of \( r \) circles. Let \( c: M \to B_r \) be the quotient map. The circles of \( B_r \) are \( c(\varphi_i^{-1}(\{x_i\} \times D^1)), i = 1, \ldots, r \). As \( W \) is path-connected, it follows that \( c_*: \pi_1(M) \to \pi_1(B_r) \cong F_r \) is an epimorphism. Conclusion: \( r \leq g(\pi_1 M) \). As \( g(M) \) is the maximal number of mutually disjoint, smooth, compact, connected, two-sided codimension one submanifolds that do not disconnect \( M \), it follows that \( g(M) \leq g(\pi_1 M) (\leq \infty) \).

Part 2. is proved as follows. Let \( g = g(\pi_1 M) \). Start with an epimorphism \( \pi_1 M \to \pi_1(B_g) \cong F_g \). As \( B_g \) is an Eilenberg-Mac Lane space (see [7], p. 225), the above epimorphism is induced by a continuous map \( h: M \to B_g \). The basepoint of \( B_g \), \( b \), is the intersection of all circles of \( B_g \). Let \( x_0 \) be any point of \( M \). We may assume that the pre-image of \( b \) is \( x_0 \). Thus, \( h \) is the one point union of \( g \) maps from \( M \) to circles containing \( b \). Homotopying rel \( x_0 \) if necessary, we may assume that these maps are smooth. Now, by Sard’s theorem there are points \( b_1, \ldots, b_g \in B_g - \{b\} \) lying in disjoint circles which are regular values for the functions that make up \( h \). Conclusion: \( h^{-1}(b_1), \ldots, h^{-1}(b_g) \) are mutually disjoint, smooth, closed, connected codimension one submanifolds that do not disconnect \( M \). Thus, \( g(\pi_1 M) \leq g(M) \). Part 2. follows.

References


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