

POSITIVE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS IN THE EUCLIDEAN PLANE

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ABSTRACT. In the present paper, we study the existence of solutions to the problem

$$\begin{cases} \Delta u + f(x, u) = 0 & \text{in } D \\ u > 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

where D is an unbounded domain in \mathbb{R}^2 with a compact nonempty boundary ∂D consisting of finitely many Jordan curves. The goal is to prove an existence theorem for the above problem in a general setting by using Brownian path integration and potential theory.

1. INTRODUCTION

In the present paper, we study the existence of solutions to the problem

$$(1.1) \quad \begin{cases} \Delta u + f(x, u) = 0 & \text{in } D \\ u > 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

where D is an unbounded domain in \mathbb{R}^2 with a compact nonempty boundary ∂D consisting of finitely many Jordan curves. The goal is to prove an existence theorem for problem (1.1) in a general setting by using Brownian path integration and potential theory.

For $D \subset \mathbb{R}^n$, $n \geq 3$, D unbounded and with a compact Lipschitz boundary, Z. Zhao [13] proved the following problem:

$$(1.2) \quad \begin{cases} \Delta u + K(x)f(u) = 0 & \text{in } D \\ u > 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

if K is a Borel measurable function in D satisfying that the family $\left\{ \frac{K(\cdot)}{|\cdot-x|^{d-2}} \right\}$ is uniformly integrable over D with a parameter $x \in D$, and f is a continuous function in $(0, b)$ for some $0 < b \leq \infty$ satisfying that:

$$\lim_{w \rightarrow 0^+} \frac{f(w)}{w} = 0.$$

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Then the problem (1.2) has infinitely many bounded solutions. More precisely, there exists $b_0 \in (0, b]$ such that for each $c \in (0, b_0]$ there exists a solution u of (1.2) satisfying

$$\lim_{|x| \rightarrow \infty} u(x) = c.$$

A similar nonlinear problem for the second order ordinary differential equations is studied in [14]. We state our main result as

Theorem 1.1 (The Main Theorem). *Let D be an unbounded domain in \mathbb{R}^2 with a compact nonempty boundary ∂D consisting of finitely many Jordan curves. Suppose that $f(x, s)$ is a Borel measurable function in $\mathbb{R}^2 \times \mathbb{R}^+$ and $f(x, \cdot)$ is a continuous function in \mathbb{R}^+ for each fixed $x \in \mathbb{R}^2$. If there exists a positive, convex and continuously differentiable function $F(x, s)$ in $\mathbb{R}^2 \times \mathbb{R}^+$ satisfying the conditions*

$$(1.3) \quad |f(x, s)| \leq F(x, s), \quad (x, s) \in \mathbb{R}^2 \times \mathbb{R}^+,$$

$$(1.4) \quad F(x, 0) = F_s(x, 0) = 0,$$

and

$$(1.5) \quad F_s(x, \ln(|x| + 1)) \in K_2^\infty,$$

then the problem

$$(1.6) \quad \begin{cases} \Delta u + f(x, u) = 0 & \text{in } D \\ u > 0 & \text{in } D \\ u = 0 & \text{on } \partial D \end{cases}$$

has infinitely many solutions. More precisely there exists a number $b > 0$ such that for each $c \in (0, b]$, (1.6) has a solution u satisfying

$$(1.7) \quad \lim_{|x| \rightarrow \infty} \frac{u(x)}{\ln |x|} = c.$$

Remark. A special case of the above general setting is the semilinear problem for any V in K_2 , and $p > 1$: If we take $f(x, s) = V(x)s^p$ and $F(x, s) = |V(x)|s^p$, then problem (1.1) becomes the corresponding problem for the semilinear elliptic equation

$$(1.8) \quad \begin{cases} \Delta u + V(x)u^p = 0 & \text{in } D \\ u > 0 & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases}$$

Then f and F satisfy (1.3) and (1.4) obviously. Condition (1.5) on F is equivalent to

$$\int_{|x| \geq 1} |V(x)|[\ln(1 + |x|)]^p dx < \infty.$$

2. GREEN FUNCTIONS AND h -CONDITIONAL BROWNIAN MOTION

In this section we collect the results on Green functions in \mathbb{R}^2 and h -conditional Brownian motion.

The Kato class K_2 is defined to be the set of all Borel measurable functions on \mathbb{R}^2 satisfying

$$(2.1) \quad \lim_{\alpha \downarrow 0} \left[\sup_{x \in \mathbb{R}^2} \int_{|y-x| < \alpha} \ln \frac{1}{|y-x|} |q(y)| dy \right] = 0$$

where $q : \mathbb{R}^2 \rightarrow \mathbb{R}$ is measurable. K_2 is an important class of the potential functions for the Schrödinger operator. Another potential class K_2^∞ which was introduced in [5] is as follows:

$$(2.2) \quad K_2^\infty = \left\{ q \in K_2 : \int_{|y| > 1} \ln |y| |q(y)| dy < \infty \right\}.$$

Let $\{X_t\}$ be 2-dimensional Brownian motion (see [2]). For any open or closed set A in \mathbb{R}^2 , let

$$T_A(\omega) = \inf\{t > 0 : X_t(\omega) \in A\}.$$

This is called the hitting time of A . The hitting time of A^c is called the exit time from A and denoted by τ_A . We shall use P^x to denote the probability measure on the Brownian continuous paths starting at x . E^x is the expectation on P^x . Let $G_D(x, y)$ be the Green function for D . We list some known results.

For any Borel function φ in D the Green operator is defined as

$$\begin{aligned} G_D \varphi(x) &= E^x \left[\int_0^{\tau_D} \varphi(X_t) dt \right] \\ &= \int_D G_D(x, y) \varphi(y) dy, \end{aligned}$$

and

$$(2.3) \quad \Delta(G_D \varphi) = -2\varphi.$$

Proposition 2.1. *Let D be an unbounded domain in \mathbb{R}^2 with a compact nonempty boundary ∂D consisting of finitely many Jordan curves. Then there exists a harmonic function $h > 0$ in D such that*

$$(2.4) \quad \lim_{D \ni x \rightarrow z} h(x) = 0 \quad \forall z \in \partial D,$$

and

$$(2.5) \quad \lim_{|x| \rightarrow \infty} \frac{h(x)}{\ln |x|} = 1.$$

Proof. Pick a point $a \in \mathbb{R}^2 \setminus \bar{D}$ and $r > 0$ (small enough) such that $\bar{B}(a, r) \subset \mathbb{R}^2 \setminus \bar{D}$. Let $x^* = a + r^2 \frac{(x-a)}{|x-a|^2}$ be the Kelvin inversion from $D \cup \{\infty\}$ onto D^* , where $D^* = \{x^* \in B(a, r) : x \in D \cup \{\infty\}\}$. Since the Kelvin inversion preserves harmonic functions and D^* is bounded Jordan domain, we have

$$G_D(x, y) = G_{D^*}(x^*, y^*), \quad \forall x, y \text{ in } D.$$

Let

$$h(x) = \pi G_{D^*}(x^*, a).$$

Thus, we have

$$(2.6) \quad \lim_{y \rightarrow \infty} \pi G_D(x, y) = \lim_{y^* \rightarrow a} \pi G_{D^*}(x^*, y^*) = \pi G_{D^*}(x^*, a).$$

The rest of the proof is just the properties of the Green function in the bounded domain. \square

Now we can develop the notion of conditional Brownian motion introduced by Doob for a general theory. Let $h > 0$ be a harmonic function in D . We define

$$p_h^D(t; x, y) = h(x)^{-1} p^D(t; x, y) h(y), \quad t > 0, \quad x, y \in D,$$

where p^D is the transition density of the killed Brownian motion. $p_h^D(t; x, y)$ satisfies the conditions for a transition density. Let P_h^x and E_h^x denote respectively the probability and expectation determined by the h -conditional Brownian motion starting at x . Then we have for all $f \in \mathcal{B}^+(D)$:

$$E_h^x[t < \tau_D; f(X_t)] = h(x)^{-1} E^x[t < \tau_D; f(X_t)h(X_t)], \quad x \in D.$$

Theorem 2.2 (3-G). *Let $D \subset \mathbb{R}^2$ be as in Proposition 2.1. Then there exists a constant $\mathbf{C} > 0$ depending on D only such that $\forall x, y$, and $z \in D$:*

$$(2.7) \quad \frac{G_D(x, y)G_D(y, z)}{G_D(x, z)} \leq \mathbf{C}[G_D(x, y) + G_D(y, z) + 1].$$

Proof. We used Kelvin transformation to reduce to the bounded Jordan domain case (see [2]). For $n \geq 3$, the 3-G theorem for the unbounded Lipschitz domain was proved by Herbst and Zhao [6] based on [4]. \square

Proposition 2.3. *If $q \in K_2^\infty$, then $q \in L^1(\mathbb{R}^2)$.*

Proof. Let $q \in K_2^\infty$. Then by definition of K_2^∞ , $q \in K_2$. So there exists $\alpha_1 > 0$ such that

$$(2.8) \quad \sup_{x \in \mathbb{R}^2} \int_{|y-x| \leq \alpha_1} \ln \frac{1}{|x-y|} |q(y)| dy < 1$$

and

$$(2.9) \quad \int_{|y| \geq e} \ln |y| |q(y)| dy < \infty.$$

Let $\alpha = \min(\alpha_1, e^{-1})$. For the compact set $\bar{B} = \bar{B}(0, e)$, there exists finite points $x_i, i = 1, \dots, L$, such that $\bar{B} \subseteq \bigcup_{i=1}^L B(x_i, \alpha)$ and $\forall i = 1, \dots, L, y \in B(x_i, \alpha)$ we have $\ln \frac{1}{|x_i - y|} \geq \ln \frac{1}{e^{-1}} = 1$. Then by (2.8)

$$(2.10) \quad \begin{aligned} \int_{|y| \leq e} |q(y)| dy &\leq \sum_{i=1}^L \int_{B(x_i, \alpha)} |q(y)| dy \\ &\leq \sum_{i=1}^L \int_{B(x_i, \alpha)} \ln \frac{1}{|x_i - y|} |q(y)| dy \\ &\leq L. \end{aligned}$$

Since $q \in K_2^\infty$, we have

$$(2.11) \quad \int_{|y|>e} |q(y)|dy \leq \int_{|y|>e} \ln |y||q(y)|dy < \infty.$$

Thus, by (2.10) and (2.11)

$$\int_{\mathbb{R}^2} |q(y)|dy \leq \int_{|y|>e} |q(y)|dy + L < \infty.$$

□

3. UNIFORMLY INTEGRABLE FUNCTIONS

This is the main technical section in which we investigate uniform integrability of the family of the functions $\{G_D(x, \cdot)|q(\cdot)|\}$ and $\{\frac{1}{h(x)}G_D(x, \cdot)h(\cdot)|q(\cdot)|\}$.

Since $D \supset B_r^* = \mathbb{R}^2 \setminus \overline{B(a, r)}$, using the explicit formula of $G_{B_r^*}(\cdot, \cdot)$ and the definition of K_2^∞ , we can prove:

Proposition 3.1. *For $q \in K_2^\infty$, the family of functions $\{G_D(x, \cdot)|q(\cdot)|\}$ with parameter $x \in D$ is uniformly integrable over D .*

We define

$$(3.1) \quad \|q\|_D = \sup_{x \in D} \int_D G(x, y)|q(y)|dy,$$

and

$$(3.2) \quad \|q\|_1 = \int_D |q(y)|dy.$$

Proposition 3.2. *Let D and h be as in Proposition 2.1. Then for any $q \in K_2^\infty$, we have*

- (a) $\{\frac{1}{h(x)}G_D(x, \cdot)h(\cdot)|q(\cdot)| : x \in D\}$ is uniformly integrable.
- (b) There exists a constant c_D such that $\forall x \in D$:

$$(3.3) \quad \frac{1}{h(x)} \int_D G_D(x, y)h(y)|q(y)|dy \leq c_D(\|q\|_D + \|q\|_1).$$

Proof. By Theorem (3-G), there exists $\mathbf{C} > 0$ such that

$$(3.4) \quad \frac{G_D(x, y)G_D(y, z)|q(y)|}{G_D(x, z)} \leq \mathbf{C}(G_D(x, y) + G_D(y, z) + 1)|q(y)|.$$

Recalling $h(x) = \lim_{z \rightarrow \infty} \pi G_D(x, z)$, and using Proposition 3.1 and the fact $q \in L^1(D)$, we can obtain that $\{\frac{1}{h(x)}G_D(x, \cdot)h(\cdot)|q(\cdot)| : x \in D\}$ is uniformly integrable. Thus, (a) follows.

By taking the integral of (3.4) with respect to y , we have

$$\begin{aligned}
 & \frac{1}{G_D(x, z)} \int_D G_D(x, y)G_D(y, z)|q(y)|dy \\
 (3.5) \quad & \leq \mathbf{C} \left(\int_D G_D(x, y)|q(y)|dy + \int_D G_D(y, z)|q(y)|dy + \int_D |q(y)|dy \right) \\
 & \leq c_D(\|q\|_D + \|q\|_1),
 \end{aligned}$$

where $c_D = 2\mathbf{C}$. By taking the limit of (3.5) as $z \rightarrow \infty$, we have by Fatou’s lemma:

$$\frac{1}{h(x)} \int_D G_D(x, y)h(y)|q(y)|dy \leq c_D(\|q\|_D + \|q\|_1).$$

□

Proposition 3.3. *Let D and h be as in Proposition 2.1. Let $q \in K_2^\infty$. If*

$$(3.6) \quad \sup_{x \in D} E_h^x \left[\int_0^{\tau_D} |q(X_t)|dt \right] \leq \frac{1}{2},$$

then the function

$$(3.7) \quad w(x) = h(x)E_h^x \left[e^{\int_0^{\tau_D} q(X_s)ds} \right], \quad \forall x \in D,$$

is well defined in D and satisfies that $\forall x \in D$

$$(3.8) \quad e^{-\frac{1}{2}}h(x) \leq w(x) \leq 2h(x),$$

and

$$(3.9) \quad w(x) = h(x) + \int_D G_D(x, y)q(y)w(y)dy.$$

Proof. By Khas’minskii’s lemma, we have by (3.6),

$$(3.10) \quad E_h^x \left[e^{\int_0^{\tau_D} q(X_s)ds} \right] \leq 2.$$

By Jensen’s inequality and (3.6), we have

$$(3.11) \quad e^{-\frac{1}{2}} \leq E_h^x \left[e^{\int_0^{\tau_D} q(X_s)ds} \right].$$

Thus, it follows by (3.7), (3.10), and (3.11) that

$$h(x)e^{-\frac{1}{2}} \leq w(x) \leq 2h(x).$$

The equality (3.9) will be obtained by calculation on the conditional Brownian paths:

$$\begin{aligned} w(x) - h(x) &= h(x)E_h^x \left[e^{\int_0^{\tau_D} q(X_s)ds} - 1 \right] \\ &= h(x)E_h^x \left[\int_0^{\tau_D} q(X_t)e^{\int_t^{\tau_D} q(X_s)ds} dt \right] \\ &= h(x) \int_0^\infty E_h^x \left[t < \tau_D : q(X_t)e^{\int_t^{\tau_D} q(X_s)ds} \right] dt \\ &= h(x) \int_0^\infty E_h^x \left[t < \tau_D : q(X_t)E_h^{X_t} \left[e^{\int_0^{\tau_D} q(X_s)ds} \right] \right] dt \end{aligned}$$

(by the Markov property for the conditional Brownian motion)

$$\begin{aligned} &= \int_0^\infty E^x \left[t < \tau_D; q(X_t)h(X_t)E_h^{X_t} \left[e^{\int_0^{\tau_D} q(X_s)ds} \right] \right] dt \\ &= \int_0^\infty E^x [t < \tau_D; q(X_t)w(X_t)] dt \\ &= E^x \left[\int_0^{\tau_D} q(X_t)w(X_t) dt \right] \\ &= \int_D G_D(x, y)q(y)w(y)dy. \end{aligned}$$

□

4. PROOF OF THE MAIN THEOREM

Let $D^\infty = \bar{D} \cup \{\infty\}$ be the compactification of D . Let

$$(4.1) \quad \mathcal{P}(D) = \{ \psi \in C(D) : \lim_{x \rightarrow z \in \partial D} \psi(x) \text{ and } \lim_{|x| \rightarrow \infty} \psi(x) \text{ exist and are finite} \}.$$

Obviously $\mathcal{P}(D)$ is isometric to $C(D^\infty)$. $\mathcal{P}(D)$ is a Banach space, with the norm

$$\| \psi \| = \sup_{x \in D} | \psi(x) |.$$

Let q_0 be a positive function belonging to K_2^∞ , and let

$$(4.2) \quad Q_0 = \{ q \in K_2^\infty : |q(x)| \leq q_0(x), \forall x \in D \}.$$

Proposition 4.1. *For a fixed $q_0 > 0$ and $q_0 \in K_2^\infty$, and the harmonic function h as defined in Proposition 2.1, the family of the functions*

$$G_D^h[Q_0] = \left\{ \frac{1}{h(\cdot)} \int_D G_D(\cdot, y)h(y)q(y)dy : q \in Q_0 \right\}$$

is uniformly bounded and equicontinuous in D , and consequently it is relatively compact in $\mathcal{P}(D)$.

Remark. In order to imply the relative compactness of the family in the supremum norm, the equicontinuity of the family in $\mathcal{P}(D)$ should include that the two limits in (4.1) converge uniformly for all functions in the family. This is a modification of the classical Ascoli-Arzelà theorem.

Proof. The uniform integrability of $\left\{ \frac{1}{h(x)}G(x, \cdot)h(\cdot)|q(\cdot)| : x \in D \right\}$, by Proposition 3.2, justifies the interchange of the limit and integration, hence we obtain that the function

$$L_q(x) \equiv \frac{1}{h(x)} \int_D G_D(x, y)h(y)q(y)dy$$

is continuous. $\lim_{|x| \rightarrow \infty} L_q(x)$ and $\lim_{x \rightarrow z \in \partial D} L_q(x)$ exist and are finite. Thus, L_q belongs to $\mathcal{P}(D)$. For any $q \in Q_0$, by (4.2) and 3.2 we have

$$\begin{aligned} \sup_{x \in \bar{D}} \int_D \frac{G_D(x, y)}{h(x)}h(y)|q(y)|dy &\leq \sup_{x \in D} \int_D \frac{G_D(x, y)}{h(x)}h(y)q_0(y)dy \\ &\leq c_D(\|q_0\|_D + \|q_0\|_1). \end{aligned}$$

Thus, $G_D^h[Q_0]$ is uniformly bounded. By using the uniform integrability of $\left\{ \frac{1}{h(x)}G_D(x, \cdot)h(\cdot)q_0(\cdot) \right\}$, we have

i) $\forall q \in Q_0, \forall z \in \bar{D}$, as $x, x' \rightarrow z$:

$$\begin{aligned} &\left| \int_D \frac{G_D(x, y)}{h(x)}h(y)q(y)dy - \int_D \frac{G_D(x', y)}{h(x')}h(y)q(y)dy \right| \\ (4.3) \quad &\leq \int_D \left| \frac{G_D(x, y)}{h(x)} - \frac{G_D(x', y)}{h(x')} \right| h(y)|q(y)|dy \\ &\leq \int_D \left| \frac{G_D(x, y)}{h(x)} - \frac{G_D(x', y)}{h(x')} \right| h(y)q_0(y)dy \rightarrow 0, \end{aligned}$$

since $x - x' \rightarrow 0$.

ii) $\forall q \in Q_0$,

$$\int_D \frac{G_D(x, y)}{h(x)}h(y)|q(y)|dy \leq \int_D \frac{G_D(x, y)}{h(x)}h(y)q_0(y)dy \rightarrow 0$$

as $|x| \rightarrow \infty$.

Thus, $G_D^h[Q_0]$ is equicontinuous in $\mathcal{P}(D)$, and the limits in (4.1) converge uniformly for all functions in $G_D^h[Q_0]$. Then by the Ascoli-Arzelà theorem $G_D^h[Q_0]$ is relatively compact in $\mathcal{P}(D)$. \square

Proof of the Main Theorem. Let $b > 0$ be the real number determined later and, for $c \in (0, b]$, let

$$(4.4) \quad \Lambda = \left\{ v \in \mathcal{P}(D) : ce^{-\frac{1}{2}} \leq v(x) \leq 2c, \quad \forall x \in D \right\}.$$

For each $v \in \Lambda$ define

$$(4.5) \quad Tv(x) = cE_h^x \left[e^{\int_0^{\tau_D} q_v(X_t)dt} \right], \quad \forall x \in D,$$

where

$$(4.6) \quad q_v(x) = \frac{f(x, v(x)h(x))}{2v(x)h(x)}.$$

For each $x \in D$, by using condition (1.3), (1.4) and the mean value theorem we showed that

$$(4.7) \quad \left| \frac{f(x, v(x)h(x))}{2v(x)h(x)} \right| \leq F_s(x, 2c\eta \ln(|x| + 1)).$$

We now have

$$\begin{aligned} E_h^x \left[\int_0^{\tau_D} |q_v(X_t)| dt \right] &= \frac{1}{h(x)} E^x \left[\int_0^{\tau_D} \frac{h(X_t) |f(X_t, v(X_t)h(X_t))|}{2v(X_t)h(X_t)} dt \right] \\ &= \frac{1}{h(x)} \int_D \frac{G_D(x, y) h(y) |f(y, v(y)h(y))|}{2v(y)h(y)} dy \end{aligned}$$

(by (4.7) and the 3-G Theorem)

$$(4.8) \quad \leq \mathbf{C} \int_D (G_D(x, y) + h(y) + 1) F_s(y, 2c\eta \ln(|y| + 1)) dy.$$

Since $F_s(y, \ln(|y| + 1)) \in K_2^\infty \subseteq K_2$, there exists $\alpha \in (0, 1)$ such that

$$(4.9) \quad \sup_{x \in D} \frac{1}{\pi} \int_{|y-x| < \alpha} \ln \frac{1}{|y-x|} F_s(y, \ln(|y| + 1)) dy < \frac{1}{4\mathbf{C}}.$$

By using $F_s(y, \ln(|y| + 1)) \in K_2^\infty$ again, we have by Proposition 2.3

$$\left\{ h(y) + 1 + \frac{1}{\pi} \left[C_r + \ln \frac{1}{\alpha} + 2 \ln |y - a| \right] \right\} F_s(y, \ln(|y| + 1)) \in L^1(D).$$

Thus, using the monotone convergence theorem and noting that $F_s(y, 0) = 0$ and that $F_s(y, \cdot)$ is a nondecreasing function, there exists $\gamma \in (0, 1)$ such that

$$(4.10) \quad \int_D \left[h(y) + 1 + \frac{1}{\pi} \left(C_r + \ln \frac{1}{\alpha} + 2 \ln |y - a| \right) \right] F_s(y, \gamma \ln(|y| + 1)) dy < \frac{1}{4\mathbf{C}}.$$

Now, we determine b by letting

$$(4.11) \quad b = \frac{\gamma}{2\eta}.$$

Thus, $\forall x \in D$, for $c \in (0, b]$, $2\eta c \leq \gamma < 1$, we have

$$(4.12) \quad \begin{aligned} \int_D G_D(x, y) F_s(y, 2\eta c \ln(|y| + 1)) dy &\leq \int_D G_{B_r^+(a)}(x, y) F_s(y, \gamma \ln(|y| + 1)) dy \\ &\leq \frac{1}{\pi} \int_D (C_r + 2 \ln |y - a| + \ln |x - y|^{-1}) F_s(y, \gamma \ln(|y| + 1)) dy \end{aligned}$$

by (4.9)

$$\leq \frac{1}{4\mathbf{C}} + \frac{1}{\pi} \int_D \left(C_r + 2 \ln |y - a| + \ln \frac{1}{\alpha} \right) F_s(y, \gamma \ln(|y| + 1)) dy.$$

It follows from (4.8), (4.10), and (4.12) that

$$(4.13) \quad \sup_{x \in D} E_h^x \left[\int_0^{\tau_D} |q_v(X_t)| dt \right] \leq \frac{1}{2}.$$

Similarly,

$$(4.14) \quad \sup_{x \in D} E_h^x \left[\int_0^{\tau_D} F_s(X_t, 2c\eta \ln(|x_t| + 1)) \right] \leq \frac{1}{2}.$$

Thus by (4.5), (4.13) and Proposition 3.3 we have

$$(4.15) \quad ce^{-\frac{1}{2}} \leq Tv(x) \leq 2c,$$

and

$$(4.16) \quad Tv(x) = c + \frac{1}{h(x)} \int_D G_D(x, y) h(y) q_v(y) Tv(y) dy.$$

By (4.15) and the uniform integrability of $\left\{ \frac{1}{h(x)} G_D(x, \cdot) h(\cdot) q(\cdot) : x \in D \right\}$, we see that $\lim_{x \rightarrow z \in \partial D} Tv(x)$ and $\lim_{|x| \rightarrow \infty} Tv(x)$ exist and are finite. Tv is continuous in D . Thus, $Tv \in \mathcal{P}(D)$ and $Tv \in \Lambda$, so

$$(4.17) \quad T\Lambda \subseteq \Lambda.$$

For any $v \in \Lambda$, by (4.7):

$$|q_v(y)Tv(y)| \leq 2cF_s(y, \ln(|y| + 1)).$$

Let $q_0(y) = 2cF_s(y, \ln(|y| + 1))$ and let Q_0 be given as in (4.1). It follows that

$$(4.18) \quad q_v(\cdot)Tv(\cdot) \in Q_0,$$

and

$$\left\{ \frac{1}{h(\cdot)} \int_D G_D(\cdot, y) h(y) q_v(y) Tv(y) dy : v \in \Lambda \right\} \subseteq G_D^h[Q_0].$$

We thus have by Proposition 4.1 that $T\Lambda$ is a relatively compact set in $\mathcal{P}(D)$. We shall prove the continuity of T in Λ in the supremum norm. Let $v_n \rightarrow v$ in Λ as $n \rightarrow \infty$. Since $\tau < \infty$ for P^x -almost Brownian paths, we have by the continuity of function f and the bounded convergence theorem that for almost every Brownian path:

$$(4.19) \quad \int_0^{\tau_D} \frac{f(X_t, v_n(X_t)h(X_t))}{2v_n(X_t)h(X_t)} dt \rightarrow \int_0^{\tau_D} \frac{f(X_t, v(X_t)h(X_t))}{2v(X_t)h(X_t)} dt.$$

Since $\{v_n\} \subset \Lambda$, each term of the sequence in (4.19) is bounded by

$$\int_0^{\tau_D} F_s(X_t, 2\eta c \ln(|X_t| + 1)) dt.$$

By (4.14) and Khas'minskii's lemma

$$(4.20) \quad E_h^x \left[e^{\int_0^{\tau_D} F_s(X_t, 2\eta c \ln(|X_t| + 1)) dt} \right] \leq 2 < \infty,$$

and hence it follows from (4.19), (4.20), and the dominated convergence theorem that $\forall x \in D$

$$E_h^x \left[e^{\int_0^{\tau_D} q_{v_n}(X_t) dt} \right] \rightarrow E_h^x \left[e^{\int_0^{\tau_D} q_v(X_t) dt} \right].$$

Thus by (4.5)

$$(4.21) \quad Tv_n(x) \rightarrow Tv(x)$$

as $n \rightarrow \infty$. We have proved that $T\Lambda$ is a relatively compact family, therefore the pointwise convergence in (4.21) implies the uniform convergence, namely, $\|Tv_n - Tv\| \rightarrow 0$ as $n \rightarrow \infty$. Thus, we proved that T is a compact and continuous mapping from Λ to itself. By the definition of Λ , Λ is obviously a nonempty, closed, bounded, and convex set in $\mathcal{P}(D)$. Hence by the Schauder fixed-point theorem, there exists a function $v \in \Lambda$ such that

$$(4.22) \quad Tv = v.$$

It follows from (4.16) and (4.22) that

$$(4.23) \quad v(x) = c + \frac{1}{h(x)} \int_D G_D(x, y)h(y)q_v(y)v(y)dy.$$

Now let

$$(4.24) \quad u(x) = h(x)v(x).$$

Since $v \geq ce^{-\frac{1}{2}}$ and $h > 0$ in D , $u > 0$ in D . By (4.6), (4.23), and (4.24), we have

$$(4.25) \quad u(x) = ch(x) + \frac{1}{2} \int_D G_D(x, y)f(y, u(y))dy.$$

Since h is harmonic in D , applying Δ to both sides of (4.25) and using $\Delta(G_D f) = -2f$, we have

$$(4.26) \quad \begin{aligned} \Delta u(x) &= \frac{1}{2} \Delta \left[\int_D G_D(x, y)f(y, u(y))dy \right] \\ &= -f(x, u(x)). \end{aligned}$$

By the property of h and the boundedness of v , we have

$$(4.27) \quad \lim_{x \rightarrow \partial D} u(x) = 0$$

and

$$(4.28) \quad \lim_{|x| \rightarrow \infty} \frac{u(x)}{\ln|x|} = \lim_{|x| \rightarrow \infty} \frac{v(x)h(x)}{\ln|x|} = c.$$

We thus complete the proof. □

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