

ALGEBRAS OF INVARIANT FUNCTIONS ON THE SHILOV BOUNDARIES OF SIEGEL DOMAINS

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ABSTRACT. Let $D = G/K$ be a bounded symmetric domain and K/L the Shilov boundary of D . Let \mathcal{N} be the Shilov boundary of the Siegel domain realization of G/K . We consider the case when D is the exceptional non-tube type domain of the type $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \times \mathfrak{so}(2))$. We prove that $(\mathcal{N} \rtimes L, L)$ is not a Gelfand pair and thus resolve an open question of G. Carcano.

§0. INTRODUCTION

Let $D = G/K$ be a Hermitian symmetric space in a complex space V in its standard Harish-Chandra realization, and let K/L be the Shilov boundary of D . Then G/K can also be realized as a Siegel domain. Let \mathcal{N} be the Shilov boundary of the Siegel domain. Then L is acting on \mathcal{N} . Then L acts on V and leaves the Shilov boundary \mathcal{N} invariant. We can thus form the semidirect product $\mathcal{N} \rtimes L$. It is of considerable interest to study the question of determining whether $(\mathcal{N} \rtimes L, L)$ is a Gelfand pair; see [C2] and [HR]. When G/K is a tube domain, then N is commutative and the question is trivial. There are three types of non-tube domains, two of them are classical domains and one is exceptional. The corresponding symmetric pairs (G, K) are $(SU(p, p+q), S(U(p) \times U(p+q)))$, $(SO^*(4n+2), U(2n+1))$ and $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \times \mathfrak{so}(2))$. In [C2] Carcano considered the classical non-tube type domains $(SU(p, p+q), S(U(p) \times U(p+q)))$, $(SO^*(4n+2), U(2n+1))$ and proved that the answer is yes except for $(SU(p, p+q), S(U(p) \times U(p+q)))$ in the case $p > 2$ and $q \geq 2$. The case for the exceptional case is left open. In the present paper we will study the remaining case of the exceptional domain and thus resolve the open question.

Our main theorem is the following

Theorem A. *Let D be the bounded symmetric domain of type $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \times \mathfrak{so}(2))$ and K/L be its Shilov boundary of D . Let \mathcal{N} be the Shilov boundary of the Siegel domain realization of G/K . Then $(\mathcal{N} \rtimes L, L)$ is not a Gelfand pair.*

We remark that there are several open questions [BJLR] about Gelfand pairs related to the $\mathfrak{so}(10) \times \mathfrak{so}(2)$ actions on the tangent space of the symmetric domain $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \times \mathfrak{so}(2))$. In a forthcoming paper we will examine some of those.

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§1. BOUNDED SYMMETRIC DOMAINS
AND THEIR SIEGEL DOMAIN REALIZATIONS

In this section we recall the Jordan algebraic characterization of bounded symmetric domain, which is very convenient for our purpose.

Let D be an irreducible bounded symmetric domain in a complex space V . Let $Aut(D)$ be the group of all biholomorphic automorphisms of D , let $G = Aut(D)_0$ be the connected component of the identity in $Aut(D)$, and let K be the isotropy subgroup of G at the point 0. Then, as a Hermitian symmetric space, $D = G/K$. The Lie algebra \mathfrak{g} of G is identified with the Lie algebra $aut(D)$ of all completely integrable holomorphic vector fields on D , equipped with the Lie product $[X, Y](z) := X'(z)Y(z) - Y'(z)X(z)$, $X, Y \in aut(D)$, $z \in D$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the *Cartan decomposition* of \mathfrak{g} with respect to the involution $\theta(X)(z) := -X(-z)$. There exists a quadratic form $Q : V \rightarrow End(\bar{V}, V)$ (where \bar{V} is the complex conjugate of V), such that $\mathfrak{p} = \{\xi_v; v \in V\}$, where $\xi_v(z) := v - Q(z)v$.

Let $\{z, v, w\}$ be the polarization of the $Q(z)v$, i.e.,

$$\{z, v, w\} = (Q(z+w)v - Q(z)v - Q(w)v).$$

This defines a *triple product* $V \times \bar{V} \times V \rightarrow V$, with respect to which V is a *JB*-triple*; see [Up].

We define $D(z, v) \in End(V, V)$ by $D(z, v)w = \{z, v, w\}$. Then we have $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. Explicitly, $[X, \xi_z] = \xi_{Xz}$, for all $X \in \mathfrak{k}$, and $z \in V$, and $[\xi_z, \xi_v] = D(z, v) - D(v, z)$, for all $z, v \in V$.

We define

$$\langle z, w \rangle = \text{Tr}D(z, w).$$

Then $\langle \cdot, \cdot \rangle$ is a Hermitian product on V and it is K -invariant, where “Tr” is the trace functional on $End(V)$.

Thus, K acts on V by unitary transformations. The domain D is realized as the open unit ball of V with respect to the spectral norm, i.e.

$$D = \{z \in V : \|D(z, z)\|^2 < 1\}$$

where $\|D(z, z)\|$ is the operator norm of $D(z, z)$ on the Hilbert space $(V, \langle \cdot, \cdot \rangle)$.

An element $v \in V$ is a *tripotent* if $\{v, v, v\} = v$. Two tripotents v and u are *orthogonal* if $D(v, u) = 0$. Orthogonality is a symmetric relation. A tripotent v is *minimal* if it cannot be written as the sum of two non-zero orthogonal tripotents. A *frame* is a maximal family of pairwise orthogonal, minimal tripotents. It is known that the group K acts transitively on frames. In particular, the cardinality of all frames is the same, and is equal to the rank r of D . Every $z \in V$ admits a *spectral decomposition* $z = \sum_{j=1}^r s_j v_j$, where $\{v_j\}_{j=1}^r$ is a frame and $s_1 \geq s_2 \geq \dots \geq s_r \geq 0$ are the *singular numbers* of z . The spectral norm of z is equal to the largest singular value s_1 .

Let us choose and fix a frame $\{e_j\}_{j=1}^r$ in V . Then, by the transitivity of K on the frames, each element $z \in V$ admits a *polar decomposition* $z = k \sum_{j=1}^r s_j e_j$, where $k \in K$ and $s_j = s_j(z)$ are the singular numbers of z .

Let $e := e_1 + \dots + e_r$. Then e is a *maximal tripotent*, i.e. the only tripotent which is orthogonal to e is 0. Let

$$V = \sum_{0 \leq j \leq k \leq r} \oplus V_{j,k}$$

be the *joint Peirce decomposition* of V associated with $\{e_j\}_{j=1}^r$, where

$$V_{j,k} = \{v \in V; D(e_l, e_l)v = \delta_{l,j} + \delta_{l,k}v, 1 \leq l \leq r\},$$

for $(j, k) \neq (0, 0)$ and $V_{0,0} = \{0\}$. By the minimality of the $\{e_j\}_{j=1}^r$, $V_{j,j} = \mathbb{C}e_j$, $1 \leq j \leq r$. The triple of integers (r, a, b) with r the rank, and

$$a := \dim V_{j,k} (1 \leq j < k \leq r), b := \dim V_{0,j} (1 \leq j \leq r)$$

is independent of the choice of the frame and uniquely determines the Jordan triple.

The Shilov boundary of D is $S = K/L$ and $L = \{k \in K; k \cdot e = e\}$.

Let

$$V_1 = \sum_{1 \leq j < k \leq r} V_{j,k}, V_{1/2} = \sum_{j=1}^r V_{0,j}.$$

Then $V = V_1 \oplus V_{1/2}$ is the Peirce decomposition of V , namely,

$$V_j = \{v \in V; D(e, e)v = 2jv\}, j = 1, 1/2.$$

Moreover the map $z \mapsto Q(e)\bar{v}$ is an involution on V_1 . Let A be the fixed point set. Then $V_1 = A + iA$. Furthermore A is a real Jordan algebra. Let Ω be the open cone of positive elements in A . Then Ω is a convex symmetric cone. See [L1].

There exists a biholomorphic mapping (called the Cayley transform) from D onto an unbounded domain $D(\Omega, F)$ in V ,

$$D(\Omega, F) = \{(Z, W) \in V_1 \times V_{1/2} : \Re(Z) - F(W, W) \in \Omega\}$$

where $\Re(Z)$ is taken in V_1 with respect to a real form of A of V_1 and $F : V_{1/2} \times V_{1/2} \rightarrow V_1$ is defined by

$$F(w, w) = \{w\bar{w}e\}.$$

The Shilov boundary of the domain is the $\mathcal{N} = \{(Z, W) \in V_1 \times V_{1/2} : \Re(Z) = F(W, W) \in \Omega\}$ with can be further identified with $A \times V_{1/2}$. The group L is acting on N and thus we can form the semidirect product $\mathcal{N} \rtimes L$.

There are six types of bounded symmetric domains or Siegel domains according to the types of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. See [H]. Among them there are three types of domains that are of non-tube type. The corresponding symmetric pairs (G, K) or $(\mathfrak{g}, \mathfrak{k})$ are $(SU(p, q), S(U(p) \times U(q)))$, $(O^*(4n + 2), U(2n + 1))$ and $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \times \mathfrak{so}(2))$, where $\mathfrak{e}_{6(-14)}$ is the exceptional Lie algebra. Let N be the Shilov boundary realized as a homogeneous space K/L . In [C2] Carcano raised the question of determining whether $(\mathcal{N} \rtimes L, L)$ is a Gelfand pair. When D is of tube type this is a trivial question. Carcano considered the classical non-tube domains $(SU(p, p + q), S(U(p) \times U(p + q)))$, $(SO^*(4n + 2), U(2n + 1))$ and proved that this is so except for $(SU(p, p + q), S(U(p) \times U(p + q)))$ in the case $p > 2$ and $q \geq 2$. We prove in the next two sections that the answer is NO for the exceptional case.

§2. CLASS-TWO REPRESENTATIONS OF \mathcal{N}

In this section we characterize all the class-two representations of the nilpotent group \mathcal{N} .

We identify A^* with A by the scalar product $\langle \cdot, \cdot \rangle$.

Lemma. *Let $\lambda \in A^* = A$. The quadratic form*

$$F_\lambda(w, w) = \langle F(w, w), \lambda \rangle$$

is non-degenerate if and only if λ is invertible in the Jordan algebra A .

Proof. It is clear ([L1]) that λ is invertible in A if and only if the eigenvalues of λ are all non-zero. Let $\lambda = s_1 E_1 + \cdots + s_j E_j$ be the spectral decomposition of λ for some real numbers $s_k \neq 0$. By adding more zero terms we may write $\lambda = s_1 E_1 + \cdots + s_r E_r$ where E_1, \dots, E_r form a frame of A . Let W_{jk} be the Peirce decomposition of E_1, \dots, E_r . Then we have

$$W_{1/2} = \sum_{j=1}^r W_{j0} = V_{1/2}$$

and

$$W_1 = \sum_{j \geq k \neq 0} W_{jk} = V_1$$

where $(V_{1/2}, V_1)$ is as in §1 with respect to a fixed frame e_1, \dots, e_r , since both are the Peirce decomposition of e , and it is unique. Now for any $w \in W_{1/2}$ we write $w = \sum_{j=1}^r w_j$ with $w_j \in W_{j0}$. We have

$$\begin{aligned} F_\lambda(w, w) &= \langle \{w\bar{w}e\}, \lambda \rangle \\ &= \sum_{j=1}^r \lambda_j \langle \{w_j \bar{w}_j e\}, \lambda \rangle \\ &= \sum_{j=1}^r \lambda_j \langle w_j, w_j \rangle \end{aligned}$$

where the second equality is obtained by the Peirce rule [L1, Theorem 3.14] and the third equality is obtained by using

$$\langle \{x\bar{y}z\}, w \rangle = \langle x, \{y\bar{z}w\} \rangle.$$

See [L1, §3.5]. Thus the form F_λ is non-degenerate if and only if all s_j are non-zero, i.e., if and only if λ is invertible. \square

Following [C2] we denote by Λ the set of all $\lambda \in A^*$ so that F_λ is non-degenerate. For each $\lambda \in \Lambda$ let $\lambda = \lambda_1 e_1 + \cdots + \lambda_r e_r$ be the spectral decomposition of λ . Associated to λ is a complex structure J_λ on $V_{1/2}$. See [C2]. We can give a simple form of J_λ in terms of the Jordan triple structure as follows. We define $\text{sgn } \lambda = \text{sgn } \lambda_1 e_1 + \cdots + \text{sgn } \lambda_r e_r$, and define J_λ on $V_{1/2}$ by

$$J_\lambda = iD(\text{sgn}(\lambda), e).$$

Clearly $J_\lambda^2 = -1$ on $V_{1/2}$. Let V_{ij} be the spaces in the Peirce decomposition of e_1, \dots, e_r ; we thus have $V_{1/2} = \sum_{j=1}^r V_{j0}$. Now the complex structure J_λ on $V_{1/2}$ is the given one on V_{j0} if $\lambda_j > 0$ and is the opposite one if $\lambda_j < 0$.

Let $\mathcal{F}_\lambda(V_{1/2})$ be the Fock space of J_λ -holomorphic functions f on $V_{1/2}$ such that

$$\|f\|_\lambda = \int_{V_{1/2}} |f(w)|^2 \exp(-|w|_\lambda^2) dw$$

where $|w|_\lambda^2$ is the J_λ -Hermitian form on $V_{1/2}$,

$$|w|_\lambda^2 = \Im F_\lambda(J_\lambda w, w),$$

where \Im denote the imaginary part. The group \mathcal{N} is acting on $\mathcal{F}_\lambda(V_{1/2})$ by the standard formula; see [C2, Theorem 1]. Let L^λ be the isotropy group of λ in L , that is,

$$L^\lambda = \{k \in L; k\lambda = \lambda\}.$$

The group L^λ is acting on $\mathcal{F}_\lambda(V_{1/2})$ by a change of variables, since L^λ fixes the $V_{1/2}$ and the complex structure.

Then Theorem 1 in [C2] implies that the reduced dual of \mathcal{N} is Λ , namely the Plancherel measure of $\hat{\mathcal{N}}$, the dual of \mathcal{N} , is concentrated on Λ . It follows further from our Lemma above that Λ is actually the set of invertible elements in the Jordan algebra A .

We recall further the following fact proved in [C1].

Theorem. *($\mathcal{N} \rtimes L, L$) is a Gelfand pair if and only if for each $\lambda \in \Lambda$ the L^λ action on $\mathcal{F}_\lambda(V_{1/2})$ is of multiplicity one.*

§3. THE DECOMPOSITION OF THE POLYNOMIAL SPACES

In this section we prove our main theorem stated in the introduction.

Retaining the notation in §1 we let $(\mathfrak{g}, \mathfrak{k})$ be the symmetric pair $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \times \mathfrak{so}(2))$ and $(\mathfrak{k}, \mathfrak{l})$ be the corresponding pair of Lie algebras of K and L where K/L is the Shilov boundary of the domain $(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \times \mathfrak{so}(2))$. The triple (r, a, b) of the Jordan pair is $(2, 6, 4)$ [L2]. The underlying space V is now the half-spin representation of $\mathfrak{so}(10)$.

We fix a frame e_1, e_2 . Let $V = V_1 \oplus V_{1/2}$ be the Peirce decomposition as in §1. We put $\mathfrak{p}_1 = \{\xi_v, v \in v_2\}$ and let \mathfrak{k} be the span of $\{[\xi_{v_1}, \xi_{v_2}], v_1, v_2 \in V_2\}$ and $\mathfrak{g}_1 = \mathfrak{p}_1 + \mathfrak{k}_1$. Thus \mathfrak{g}_1 is a semisimple Lie algebra and $(\mathfrak{g}_1, \mathfrak{k}_1)$ is a Hermitian symmetric pair of tube type. Its Shilov boundary is a symmetric space and $(\mathfrak{k}_1, \mathfrak{l}_1)$ is the corresponding symmetric pair. Then clearly $\mathfrak{l}_1 \subset \mathfrak{l}$ where \mathfrak{l} is the Lie algebra of L . Denote by L_1 the Lie subgroup of L with Lie algebra \mathfrak{l}_1 . Then $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$ where \mathfrak{l}_2 is a Lie algebra of \mathfrak{l} that centralizes \mathfrak{g}_1 . Let L_2 be the Lie subgroup of L with Lie algebra \mathfrak{l}_2 . Then $L = L_1 L_2$; see [C2].

The triple (r, a, b) for the symmetric space $(\mathfrak{g}_1, \mathfrak{k}_1)$ is now $(2, 6, 0)$. Thus $(\mathfrak{g}_1, \mathfrak{k}_1)$ is $(\mathfrak{so}(8, 2), \mathfrak{so}(8) \times \mathfrak{so}(2))$, since all rank 2 tube domains are of the type $(\mathfrak{so}(n, 2), \mathfrak{so}(n) \times \mathfrak{so}(2))$. The corresponding Hermitian symmetric domain $(\mathfrak{g}_1, \mathfrak{k}_1)$ is the Lie ball $(\mathfrak{so}(8, 2), \mathfrak{so}(8) \times \mathfrak{so}(2))$ in $V_1 = \mathbb{C}^8$. Its Shilov boundary is $(\mathfrak{k}_1, \mathfrak{l}_1)$, which is isomorphic to $(\mathfrak{so}(8) \times \mathfrak{so}(2), \mathfrak{so}(7))$.

Following [Up] we let

$$\mathfrak{h}_{-1} = \mathbb{C}D(e_1, e_1) + \mathbb{C}D(e_2, e_2).$$

Then \mathfrak{h}_1 is an abelian subspace of $\mathfrak{k}^{\mathbb{C}}$. We extend it to a Cartan subalgebra \mathfrak{h} of $\mathfrak{k}^{\mathbb{C}}$ and let $\mathfrak{h} = \mathfrak{h}_{-1} + \mathfrak{h}_1$. Define γ_1 and γ_2 on \mathfrak{h} by letting $\gamma_k = 0$ on \mathfrak{h}_1 and $\gamma_k(D(e_j, e_j)) = 2\delta_{jk}$, $j, k = 1, 2$. We let $\gamma_1 > \gamma_2$. Thus V , the underlying space, is a module of \mathfrak{k} with highest weight γ_1 . Furthermore it follows from the Peirce decomposition of e_1, e_2 that

$$(3.1) \quad V_1 = \mathbb{C}e_1 + \mathbb{C}e_2 + V_{1,2}$$

and

$$(3.2) \quad V_{1/2} = V_{10} + V_{20}.$$

Now V_1 and $V_{1/2}$ are both invariant subspaces of $\mathfrak{k}_1 = \mathfrak{so}(8) \times \mathfrak{so}(2)$. In fact \mathfrak{k}_1 is generated by $D(u, \bar{v}) - D(v, \bar{u})$ with $u, v \in V_1$. By the Peirce rule we know that V_1 and $V_{1/2}$ are invariant subspaces of $D(u, \bar{v})$ and $D(v, \bar{u})$. The decomposition $V = V_1 \oplus V_{1/2}$ is just the decomposition of the half-spin representation of $\mathfrak{k} = \mathfrak{so}(10) \times \mathfrak{so}(2)$ restricted to $\mathfrak{k}_1 = \mathfrak{so}(8) \times \mathfrak{so}(2)$. V_1 is the defining representation of $\mathfrak{so}(8) \times \mathfrak{so}(2)$ with highest weight γ_1 . (Note that γ_1 is non-zero on the center $\mathfrak{so}(2)$ of \mathfrak{k}_1 , and $D(e_1, e_1)$ is not in the $\mathfrak{so}(8)$.) However $D(e_1, e_1) - D(e_2, e_2)$ is in the semisimple part $\mathfrak{so}(8)$ of \mathfrak{k}_1 and it is dual to the highest weight of V_1 of $\mathfrak{so}(8)$. The above decompositions (3.1) and (3.2) imply that $D(e_1, e_1) - D(e_2, e_2)$ acts on $V_{1/2}$ with half the weight of that of V_1 . Thus $V_{1/2}$ is the half-spin representation of \mathfrak{k}_1 . Namely the restriction to $\mathfrak{so}(8) \times \mathfrak{so}(2)$ of the spin representation of $\mathfrak{k} = \mathfrak{so}(10) \times \mathfrak{so}(2)$ splits into the defining representation and the spin representation. (See also [HU, §11.11] where the restriction to $\mathfrak{so}(7)$ is further studied.)

Now, from $S = K/L$ we have $14 = \dim S = \dim \mathfrak{k} - \dim \mathfrak{l} = \dim \mathfrak{k} - \dim \mathfrak{l}_1 - \dim \mathfrak{l}_2$. However $\dim \mathfrak{k} = \dim \mathfrak{so}(10) + 1 = 46$ and $\dim \mathfrak{l}_1 = \dim \mathfrak{so}(7) = 21$. Thus we have $\dim \mathfrak{l}_2 = 1$. We write $\mathfrak{l}_2 = \mathbb{R}H$. (In fact we have $H = \pm(\frac{i}{2}D(e, e) - i)$, here i is the element in the center of \mathfrak{k} which defines the complex structure on \mathfrak{p}^+ ; see also [FK, p.71], we can take H to be w' in their notation.)

The space $V_{1/2}$ is the spin representation (or Clifford module) of $\mathfrak{so}(8)$. Of course H is acting on $V_{1/2}$ by a constant multiple. Moreover this constant is not zero; otherwise H is zero on $V = V_1 + V_{1/2}$ being zero on V_1 , which implies that $H = 0$, a contradiction.

We now prove

Proposition. *Let $\lambda \in A^*$ be such that F_λ is non-degenerate and suppose that in the spectral decomposition of λ , $\lambda = s_1e_1 + s_2e_2$, s_1 and s_2 have the same sign and are different. Then the Fock space $\mathcal{F}_\lambda(V_{1/2})$ is not of multiplicity one under L^λ .*

Proof. Suppose that λ is invertible. Without loss of generality we may assume that $s_1 = 1$. Thus $\lambda = e_1 + se_2$ with $s \neq 1$ and $s > 0$. Then $\mathfrak{l}^\lambda = \mathfrak{l}_1^\lambda + \mathfrak{l}_2$ with \mathfrak{l}_1^λ isomorphic to $\mathfrak{so}(6)$. Now L^λ fixes $e = e_1 + e_2$ and $\lambda = e_1 + se_2$, thus it fixes e_1 and e_2 . The Peirce decomposition of $V_{1/2}$ with respect to e_1 and e_2 is (3.2). Thus both V_{10} and V_{20} are the invariant subspaces of \mathfrak{l}^λ . However it follows from [BtD, p.290] that it is just the decomposition of the spin representations of $\mathfrak{so}(8)$ under $\mathfrak{so}(6)$; V_{10} and V_{20} are the two non-isomorphic half-spin representations of $\mathfrak{so}(6)$. Now under $\mathfrak{l}^\lambda = \mathfrak{l}_1^\lambda + \mathfrak{l}_2 = \mathfrak{so}(6) + \mathbb{R}H$ we have the space of holomorphic polynomials

$$\mathcal{P}(V_{1/2}) = \mathcal{P}(V_{10}) \otimes \mathcal{P}(V_{20}).$$

We identify V_{10} and V_{20} with \mathbb{C}^4 and identify the representation on $V_{1/2}$ of $\mathfrak{l}^\lambda = \mathfrak{so}(6) \times \mathbb{R}H = \mathfrak{su}(4) \times \mathbb{R}H$ with the following explicit action: for any $u \in \mathfrak{su}(4)$,

$$(u, H) : V_{10} \oplus V_{20} \rightarrow V_{10} \oplus V_{20}, v_1 \oplus v_2 \mapsto (iUv_1) \oplus (-i\bar{U}v_2).$$

We consider the space $P_1(V_{10}) \otimes P_1(V_{20})$ where $P_1(V_{10})$ is the space of polynomials of degree one on V_{10} ; similarly for $P_1(V_{20})$. Now we take the polynomial

$$f : (v_1, v_2) \mapsto v_2^t v_1.$$

It is clear that f is in $P_1(V_{10}) \otimes P_1(V_{20})$ and is \mathfrak{l}^λ invariant. Thus $P_1(V_{10}) \otimes P_1(V_{20})$ is not an irreducible module of \mathfrak{l}^λ . Therefore, by [BJR, §2, Remark 1] we know that $\mathcal{P}(V_{1/2})$ is not of multiplicity one under \mathfrak{l}^λ and thus L^λ . □

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