ALGEBRAS OF INVARIANT FUNCTIONS
ON THE SHILOV BOUNDARIES OF SIEGEL DOMAINS

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Abstract. Let $D = G/K$ be a bounded symmetric domain and $K/L$ the Shilov boundary of $D$. Let $\mathcal{N}$ be the Shilov boundary of the Siegel domain realization of $G/K$. We consider the case when $D$ is the exceptional non-tube type domain of the type $(\mathfrak{e}_6(-14), \mathfrak{so}(10) \times \mathfrak{so}(2))$. We prove that $(\mathcal{N} \rtimes L, L)$ is not a Gelfand pair and thus resolve an open question of G. Carcano.

§0. Introduction

Let $D = G/K$ be a Hermitian symmetric space in a complex space $V$ in its standard Harish-Chandra realization, and let $K/L$ be the Shilov boundary of $D$. Then $G/K$ can also be realized as a Siegel domain. Let $\mathcal{N}$ be the Shilov boundary of the Siegel domain. Then $L$ is acting on $\mathcal{N}$. Then $L$ acts on $V$ and leaves the Shilov boundary $\mathcal{N}$ invariant. We can thus form the semidirect product $\mathcal{N} \rtimes L$. It is of considerable interest to study the question of determining whether $(\mathcal{N} \rtimes L, L)$ is a Gelfand pair; see [C2] and [HR]. When $G/K$ is a tube domain, then $N$ is commutative and the question is trivial. There are three types of non-tube domains, two of them are classical domains and one is exceptional. The corresponding symmetric pairs $(G, K)$ are $(SU(p, p + q), SU(p) \times SU(p + q))$, $(SO^*(4n + 2), U(2n + 1))$ and $(\mathfrak{e}_6(-14), \mathfrak{so}(10) \times \mathfrak{so}(2))$. In [C2] Carcano considered the classical non-tube type domains $(SU(p, p + q), SU(p) \times SU(p + q))$, $(SO^*(4n + 2), U(2n + 1))$ and proved that the answer is yes except for $(SU(p, p + q), SU(p) \times SU(p + q))$ in the case $p > 2$ and $q \geq 2$. The case for the exceptional case is left open. In the present paper we will study the remaining case of the exceptional domain and thus resolve the open question.

Our main theorem is the following

Theorem A. Let $D$ be the bounded symmetric domain of type $(\mathfrak{e}_6(-14), \mathfrak{so}(10) \times \mathfrak{so}(2))$ and $K/L$ be its Shilov boundary of $D$. Let $\mathcal{N}$ be the Shilov boundary of the Siegel domain realization of $G/K$. Then $(\mathcal{N} \rtimes L, L)$ is not a Gelfand pair.

We remark that there are several open questions [BJLR] about Gelfand pairs related to the $\mathfrak{so}(10) \times \mathfrak{so}(2)$ actions on the tangent space of the symmetric domain $(\mathfrak{e}_6(-14), \mathfrak{so}(10) \times \mathfrak{so}(2))$. In a forthcoming paper we will examine some of those.
§1. BOUNDED SYMMETRIC DOMAINS
AND THEIR SIEGEL DOMAIN REALIZATIONS

In this section we recall the Jordan algebraic characterization of bounded symmetric domain, which is very convenient for our purpose.

Let $D$ be an irreducible bounded symmetric domain in a complex space $V$. Let $Aut(D)$ be the group of all biholomorphic automorphisms of $D$, let $G = Aut(D)_0$ be the connected component of the identity in $Aut(D)$, and let $K$ be the isotropy subgroup of $G$ at the point 0. Then, as a Hermitian symmetric space, $D = G/K$.

The Lie algebra $g$ of $G$ is identified with the Lie algebra $aut(D)$ of all completely integrable holomorphic vector fields on $D$, equipped with the Lie product $[X,Y] := X'(z)Y(z) - Y'(z)X(z)$, $X,Y ∈ aut(D)$, $z ∈ D$.

Let $g = t + p$ be the Cartan decomposition of $g$ with respect to the involution $θ(X)(z) := −X(−z)$. There exists a quadratic form $Q : V → End(V,V)$ (where $V$ is the complex conjugate of $V$), such that $p = \{ξ_v ; v ∈ V\}$, where $ξ_v(z) := v − Q(z)v$.

Let $\{z,v,w\}$ be the polarization of the $Q(z)v$, i.e.,

$$\{z,v,w\} = (Q(z+w)v − Q(z)v − Q(w)v).$$

This defines a triple product $V × V × V → V$, with respect to which $V$ is a $JB^*$-triple; see [Up].

We define $D(z,v) ∈ End(V,V)$ by $D(z,v)w = \{z,v,w\}$. Then we have $[t,t] ⊆ t$, $[t,p] ⊆ p$ and $[p,p] ⊂ t$. Explicitly, $[X,ξ_z] = ξ_{Xz}$, for all $X ∈ t$, and $z ∈ V$, and $[ξ_z,ξ_v] = D(z,v) − D(v,z)$, for all $z,v ∈ V$.

We define

$$\langle z,w \rangle = TrD(z,w).$$

Then $\langle · , · \rangle$ is a Hermitian product on $V$ and it is $K$-invariant, where “Tr” is the trace functional on $End(V)$.

Thus, $K$ acts on $V$ by unitary transformations. The domain $D$ is realized as the open unit ball of $V$ with respect to the spectral norm, i.e.

$$D = \{z ∈ V : \|D(z,z)\|^2 < 1\}$$

where $\|D(z,z)\|$ is the operator norm of $D(z,z)$ on the Hilbert space $(V, ⟨·,·⟩)$.

An element $v ∈ V$ is a tripotent if $\{v,v,v\} = v$. Two tripotents $v$ and $u$ are orthogonal if $D(v,u) = 0$. Orthogonality is a symmetric relation. A tripotent $v$ is minimal if it cannot be written as the sum of two non-zero orthogonal tripotents. A frame is a maximal family of pairwise orthogonal, minimal tripotents. It is known that the group $K$ acts transitively on frames. In particular, the cardinality of all frames is the same, and is equal to the rank $r$ of $D$. Every $z ∈ V$ admits a spectral decomposition $z = \sum_{j=1}^r s_jv_j$, where $\{v_j\}_{j=1}^r$ is a frame and $s_1 ≥ s_2 ≥ \cdots ≥ s_r ≥ 0$ are the singular numbers of $z$. The spectral norm of $z$ is equal to the largest singular value $s_1$.

Let us choose and fix a frame $\{e_j\}_{j=1}^r$ in $V$. Then, by the transitivity of $K$ on the frames, each element $z ∈ V$ admits a polar decomposition $z = k \sum_{j=1}^r s_je_j$, where $k ∈ K$ and $s_j = s_j(z)$ are the singular numbers of $z$.

Let $e := e_1 + \cdots + e_r$. Then $e$ is a maximal tripotent, i.e. the only tripotent which is orthogonal to $e$ is 0. Let

$$V = \bigoplus_{0 ≤ j ≤ k ≤ r} V_{j,k}$$
be the joint Peirce decomposition of $V$ associated with $\{e_j\}_{j=1}^r$, where

$$V_{j,k} = \{ v \in V; D(e_i, e_j)v = \delta_{i,j} + \delta_{i,k}v, \ 1 \leq l \leq r \},$$

for $(j, k) \neq (0, 0)$ and $V_{0,0} = \{0\}$. By the minimality of the $\{e_j\}_{j=1}^r$, $V_{j,j} = \mathbb{C}e_j$, $1 \leq j \leq r$. The triple of integers $(r, a, b)$ with $r$ the rank, and

$$a := \dim V_{j,k} \ (1 \leq j < k \leq r), \ b := \dim V_{0,j} \ (1 \leq j \leq r)$$

is independent of the choice of the frame and uniquely determines the Jordan triple.

The Shilov boundary of $D$ is $S = K/L$ and $L = \{k \in K; k \cdot e = e\}$.

Let

$$V_1 = \bigoplus_{1 \leq j < k \leq r} V_{j,k}, \ V_{1/2} = \bigoplus_{j=1}^r V_{0,j}.$$ 

Then $V = V_1 \oplus V_{1/2}$ is the Peirce decomposition of $V$, namely,

$$V_j = \{ v \in V; D(e_i, e_j)v = 2jv \}, \ j = 1, 1/2.$$ 

Moreover the map $z \mapsto Q(e)\overline{z}$ is an involution on $V_1$. Let $A$ be the fixed point set. Then $V_1 = A + iA$. Furthermore $A$ is a real Jordan algebra. Let $\Omega$ be the open cone of positive elements in $A$. Then $\Omega$ is a convex symmetric cone. See [L1].

There exists a biholomorphic mapping (called the Cayley transform) from $D$ onto an unbounded domain $D(\Omega, F)$ in $V$,

$$D(\Omega, F) = \{(Z, W) \in V_1 \times V_{1/2} : \Re(Z) = F(W, W) \in \Omega\}.$$ 

where $\Re(Z)$ is taken in $V_1$ with respect to a real form of $A$ of $V_1$ and $F : V_{1/2} \times V_{1/2} \to V_1$ is defined by

$$F(w, w) = \{w\overline{w}e\}.$$ 

The Shilov boundary of the domain is the $\mathcal{N} = \{(Z, W) \in V_1 \times V_{1/2} : \Re(Z) = F(W, W) \in \Omega\}$ with can be further identified with $A \times V_{1/2}$. The group $L$ is acting on $\mathcal{N}$ and thus we can form the semidirect product $\mathcal{N} \rtimes L$.

There are six types of bounded symmetric domains or Siegel domains according to the types of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. See [H]. Among them there are three types of domains that are of non-tube type. The corresponding symmetric pairs $(G, K)$ or $(\mathfrak{g}, \mathfrak{k})$ are $\{(SU(p, q), S(U(p) \times U(q))), (O^+(4n + 2), U(2n + 1)) \text{ and } (\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) \times \mathfrak{so}(2))\}$, where $\mathfrak{e}_{6(-14)}$ is the exceptional space $K/L$. In [C2] Carcano raised the question of determining whether $(\mathcal{N} \rtimes L, L)$ is a Gelfand pair. When $D$ is of tube type this is a trivial question. Carcano considered the classical non-tube domains $(SU(p, p + q), S(U(p) \times U(p + q)))$, $(SO^+(4n + 2), U(2n + 1))$ and proved that this is so except for $(SU(p, p + q), S(U(p) \times U(p + q)))$ in the case $p > 2$ and $q \geq 2$.

We prove in the next two sections that the answer is NO for the exceptional case.

§2. Class-two representations of $\mathcal{N}$

In this section we characterize all the class-two representations of the nilpotent group $\mathcal{N}$.

We identify $A^*$ with $A$ by the scalar product $(\cdot, \cdot)$.

Lemma. Let $\lambda \in A^* = A$. The quadratic form

$$F_\lambda(w, w) = \langle F(w, w), \lambda \rangle$$

is non-degenerate if and only if $\lambda$ is invertible in the Jordan algebra $A$. 

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Proof. It is clear ([L1]) that \( \lambda \) is invertible in \( A \) if and only if the eigenvalues of \( \lambda \) are all non-zero. Let \( \lambda = s_1E_1 + \cdots + s_mE_m \) be the spectral decomposition of \( \lambda \) for some real numbers \( s_k \neq 0 \). By adding more zero terms we may write \( \lambda = s_1E_1 + \cdots + s_mE_m \) where \( E_1, \ldots, E_m \) form a frame of \( A \). Let \( W_{jk} \) be the Peirce decomposition of \( E_1, \ldots, E_m \). Then we have

\[
W_{1/2} = \sum_{j=1}^r W_{j0} = V_{1/2}
\]

and

\[
W_1 = \sum_{j,k \neq 0} W_{jk} = V_1
\]

where \( (V_{1/2}, V_1) \) is as in §1 with respect to a fixed frame \( e_1, \ldots, e_r \), since both are the Peirce decomposition of \( e \), and it is unique. Now for any \( w \in W_{1/2} \) we write \( w = \sum_{j=1}^r w_j \) with \( w_j \in W_{j0} \). We have

\[
F_\lambda(w, w) = \langle \{w\bar{w}e\}, \lambda \rangle = \sum_{j=1}^r \lambda_j \langle \{w_j\bar{w}j\}, \lambda \rangle = \sum_{j=1}^r \lambda_j \langle w_j, w \rangle
\]

where the second equality is obtained by the Peirce rule [L1, Theorem 3.14] and the third equality is obtained by using

\[
\langle \{x\bar{y}z\}, w \rangle = \langle x, \{y\bar{z}w\} \rangle.
\]

See [L1, §3.5]. Thus the form \( F_\lambda \) is non-degenerate if and only if all \( s_j \) are non-zero, i.e., if and only if \( \lambda \) is invertible. \( \Box \)

Following [C2] we denote by \( \Lambda \) the set of all \( \lambda \in A^* \) so that \( F_\lambda \) is non-degenerate. For each \( \lambda \in \Lambda \) let \( \lambda = \lambda_1e_1 + \cdots + \lambda_re_r \) be the spectral decomposition of \( \lambda \). Associated to \( \lambda \) is a complex structure \( J_\lambda \) on \( V_{1/2} \). See [C2]. We can give a simple form of \( J_\lambda \) in terms of the Jordan triple structure as follows. We define \( \text{sgn} \lambda = \text{sgn} \lambda_1e_1 + \cdots + \text{sgn} \lambda_re_r \), and define \( J_\lambda \) on \( V_{1/2} \) by

\[
J_\lambda = iD(\text{sgn}(\lambda), e).
\]

Clearly \( J_\lambda^2 = -1 \) on \( V_{1/2} \). Let \( V_{ij} \) be the spaces in the Peirce decomposition of \( e_1, \ldots, e_r \); we thus have \( V_{1/2} = \bigoplus_{j=1}^r V_{j0} \). Now the complex structure \( J_\lambda \) on \( V_{1/2} \) is the given one on \( V_{j0} \) if \( \lambda_j > 0 \) and is the opposite one if \( \lambda_j < 0 \).

Let \( \mathcal{F}_\lambda(V_{1/2}) \) be the Fock space of \( J_\lambda \)-holomorphic functions \( f \) on \( V_{1/2} \) such that

\[
\|f\|_\lambda = \int_{V_{1/2}} |f(w)|^2 \exp(-|w|^2_\lambda)dw
\]

where \( |w|^2_\lambda \) is the \( J_\lambda \)-Hermitian form on \( V_{1/2} \);

\[
|w|^2_\lambda = \Im F_\lambda(J_\lambda w, w),
\]
where $\Im$ denote the imaginary part. The group $\mathcal{N}$ is acting on $\mathcal{F}_\lambda(V_{1/2})$ by the standard formula; see [C2, Theorem 1]. Let $L^\lambda$ be the isotropy group of $\lambda$ in $L$, that is,

$$L^\lambda = \{k \in L; k\lambda = \lambda\}.$$ 

The group $L^\lambda$ is acting on $\mathcal{F}_\lambda(V_{1/2})$ by a change of variables, since $L^\lambda$ fixes the $V_{1/2}$ and the complex structure.

Then Theorem 1 in [C2] implies that the reduced dual of $\mathcal{N}$ is $\Lambda$, namely the Plancherel measure of $\mathcal{N}$, the dual of $\mathcal{N}$, is concentrated on $\Lambda$. It follows further from our Lemma above that $\Lambda$ is actually the set of invertible elements in the Jordan algebra $A$.

We recall further the following fact proved in [C1].

**Theorem.** $(\mathcal{N} \ltimes L, L)$ is a Gelfand pair if and only if for each $\lambda \in \Lambda$ the $L^\lambda$ action on $\mathcal{F}_\lambda(V_{1/2})$ is of multiplicity one.

### §3. The decomposition of the polynomial spaces

In this section we prove our main theorem stated in the introduction.

Retaining the notation in §1 we let $(g, \mathfrak{t})$ be the symmetric pair $(\mathfrak{so}(10) \times \mathfrak{so}(10), \mathfrak{so}(2))$ and $(\mathfrak{t}, \mathfrak{l})$ be the corresponding pair of Lie algebras of $K$ and $L$ where $K/L$ is the Shilov boundary of the domain $(\mathfrak{so}(10) \times \mathfrak{so}(10), \mathfrak{so}(2))$. The triple $(r, a, b)$ of the Jordan pair is $(2, 6, 4)$ [L2]. The underlying space $V$ is now the half-spin representation of $\mathfrak{so}(10)$.

We fix a frame $e_1, e_2$. Let $V = V_1 \oplus V_{1/2}$ be the Peirce decomposition as in §1. We put $p_1 = \{\xi, v \in V_2\}$ and let $\mathfrak{k}$ be the span of $\{[\xi, \xi], v_1, v_2 \in V_2\}$ and $g_1 = p_1 + \mathfrak{k}_1$. Thus $g_1$ is a semisimple Lie algebra and $(g_1, \mathfrak{k}_1)$ is a Hermitian symmetric pair of tube type. Its Shilov boundary is a symmetric space and $(\mathfrak{t}_1, \mathfrak{l}_1)$ is the corresponding symmetric pair. Then clearly $\mathfrak{l}_1 \subset \mathfrak{l}$ is the Lie algebra of $L$. Denote by $L_1$ the Lie subgroup of $L$ with Lie algebra $\mathfrak{l}_1$. Then $\mathfrak{l} = \mathfrak{l}_1 + \mathfrak{l}_2$ where $\mathfrak{l}_2$ is a Lie algebra of $\mathfrak{l}$ that centralizes $g_1$. Let $L_2$ be the Lie subgroup of $L$ with Lie algebra $\mathfrak{l}_2$. Then $L = L_1 L_2$; see [C2].

The triple $(r, a, b)$ for the symmetric space $(g_1, \mathfrak{k}_1)$ is now $(2, 6, 0)$. Thus $(g_1, \mathfrak{k}_1)$ is $(\mathfrak{so}(8, 2), \mathfrak{so}(8) \times \mathfrak{so}(2))$, since all rank 2 tube domains are of the type $(\mathfrak{so}(n, 2), \mathfrak{so}(n) \times \mathfrak{so}(2))$. The corresponding Hermitian symmetric domain $(g_1, \mathfrak{k}_1)$ is the Lie ball $(\mathfrak{so}(8, 2), \mathfrak{so}(8) \times \mathfrak{so}(2))$ in $V_1 = \mathbb{C}^8$. Its Shilov boundary is $\mathfrak{t}_1$, which is isomorphic to $(\mathfrak{so}(8) \times \mathfrak{so}(2), \mathfrak{so}(7))$.

Following [Up] we let

$$\mathfrak{h}_{-1} = \mathbb{C}D(e_1, e_1) + \mathbb{C}D(e_2, e_2).$$

Then $\mathfrak{h}_1$ is an abelian subspace of $\mathfrak{f}^\mathbb{C}$. We extend it to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{f}^\mathbb{C}$ and let $\mathfrak{h} = \mathfrak{h}_{-1} + \mathfrak{h}_1$. Define $\gamma_1$ and $\gamma_2$ on $\mathfrak{h}$ by letting $\gamma_k = 0$ on $\mathfrak{h}_1$ and $\gamma_k(D(e_j, e_j)) = 2\delta_{jk}$, $j, k = 1, 2$. We let $\gamma_1 > \gamma_2$. Thus $V$, the underlying space, is a module of $\mathfrak{f}$ with highest weight $\gamma_1$. Furthermore it follows from the Peirce decomposition of $e_1, e_2$ that

$$V_1 = \mathbb{C}e_1 + \mathbb{C}e_2 + V_{1,2}$$

and

$$V_{1/2} = V_{10} + V_{20}.$$
Now $V_1$ and $V_{1/2}$ are both invariant subspaces of $\mathfrak{t}_1 = \mathfrak{so}(8) \times \mathfrak{so}(2)$. In fact $\mathfrak{t}_1$ is generated by $D(u, \bar{v}) - D(v, \bar{u})$ with $u, v \in V_1$. By the Peirce rule we know that $V_1$ and $V_{1/2}$ are invariant subspaces of $D(u, \bar{v})$ and $D(v, \bar{u})$. The decomposition $V = V_1 \oplus V_{1/2}$ is just the decomposition of the half-spin representation of $\mathfrak{t} = \mathfrak{so}(10) \times \mathfrak{so}(2)$ restricted to $\mathfrak{t}_1 = \mathfrak{so}(8) \times \mathfrak{so}(2)$. $V_1$ is the defining representation of $\mathfrak{so}(8) \times \mathfrak{so}(2)$ with highest weight $\gamma_1$. (Note that $\gamma_1$ is non-zero on the center $\mathfrak{so}(2)$ of $\mathfrak{t}_1$, and $D(e_1, e_1)$ is not in the $\mathfrak{so}(8)$.) However $D(e_1, e_1) - D(e_2, e_2)$ is in the semisimple part $\mathfrak{so}(8)$ of $\mathfrak{t}_1$ and it is dual to the highest weight of $V_1$ of $\mathfrak{so}(8)$. The above decompositions (3.1) and (3.2) imply that $D(e_1, e_1) - D(e_2, e_2)$ acts on $V_{1/2}$ with half the weight of that of $V_1$. Thus $V_{1/2}$ is the half-spin representation of $\mathfrak{t}_1$. Namely the restriction to $\mathfrak{so}(8) \times \mathfrak{so}(2)$ of the spin representation of $\mathfrak{t} = \mathfrak{so}(10) \times \mathfrak{so}(2)$ splits into the defining representation and the spin representation. (See also [HU, §11.11] where the restriction to $\mathfrak{so}(7)$ is further studied.)

Now, from $S = K/L$ we have $14 = \dim S = \dim \mathfrak{t} = \dim \mathfrak{t}_1 = \dim \mathfrak{t}_2$. However $\dim \mathfrak{t} = \dim \mathfrak{so}(10) + 1 = 46$ and $\dim \mathfrak{t}_1 = \dim \mathfrak{so}(7) = 21$. Thus we have $\dim \mathfrak{t}_2 = 1$. We write $\mathfrak{t}_2 = \mathbb{R}H$. (In fact we have $H = \pm (\frac{1}{2} D(e, e) - i)$, here $i$ is the element in the center of $\mathfrak{t}$ which defines the complex structure on $\mathfrak{p}^*$; see also [FK, p.71], we can take $H$ to be $u'$ in their notation.)

The space $V_{1/2}$ is the spin representation (or Clifford module) of $\mathfrak{so}(8)$. Of course $H$ is acting on $V_{1/2}$ by a constant multiple. Moreover this constant is not zero; otherwise $H$ is zero on $V = V_1 + V_{1/2}$ being zero on $V_1$, which implies that $H = 0$, a contradiction.

We now prove

**Proposition.** Let $\lambda \in A^*$ be such that $F_\lambda$ is non-degenerate and suppose that in the spectral decomposition of $\lambda$, $\lambda = s_1e_1 + s_2e_2$, $s_1$ and $s_2$ have the same sign and are different. Then the Fock space $F_\lambda(V_{1/2})$ is not of multiplicity one under $L^\lambda$.

**Proof.** Suppose that $\lambda$ is invertible. Without loss of generality we may assume that $s_1 = 1$. Thus $\lambda = e_1 + s_2e_2$ with $s \neq 1$ and $s > 0$. Then $\mathfrak{t}^\lambda = \mathfrak{t}_1^0 + \mathfrak{t}_2$ with $\mathfrak{t}_1^0$ isomorphic to $\mathfrak{so}(6)$. Now $L^\lambda$ fixes $e = e_1 + e_2$ and $\lambda = e_1 + s_2e_2$, thus it fixes $e_1$ and $e_2$. The Peirce decomposition of $V_{1/2}$ with respect to $e_1$ and $e_2$ is (3.2). Thus both $V_{10}$ and $V_{20}$ are the invariant subspaces of $\mathfrak{t}^\lambda$. However it follows from [BtD, p.290] that it is just the decomposition of the spin representations of $\mathfrak{so}(8)$ under $\mathfrak{so}(6)$: $V_{10}$ and $V_{20}$ are the two non-isomorphic half-spin representations of $\mathfrak{so}(6)$. Now under $\mathfrak{t}^\lambda = \mathfrak{t}_1^0 + \mathfrak{t}_2 = \mathfrak{so}(6) + \mathbb{R}H$ we have the space of holomorphic polynomials

$$\mathcal{P}(V_{1/2}) = \mathcal{P}(V_{10}) \otimes \mathcal{P}(V_{20}).$$

We identify $V_{10}$ and $V_{20}$ with $\mathbb{C}^4$ and identify the representation on $V_{1/2}$ of $\mathfrak{t}^\lambda = \mathfrak{so}(6) \times \mathbb{R}H = \mathfrak{su}(4) \times \mathbb{R}H$ with the following explicit action: for any $u \in \mathfrak{su}(4)$,

$$(u, H) : V_{10} \oplus V_{20} \rightarrow V_{10} \oplus V_{20}, v_1 \oplus v_2 \mapsto (iUv_1) \oplus (-i\bar{U}v_2).$$

We consider the space $P_1(V_{10}) \otimes P_1(V_{20})$ where $P_1(V_{10})$ is the space of polynomials of degree one on $V_{10}$; similarly for $P_1(V_{20})$. Now we take the polynomial

$$f : (v_1, v_2) \mapsto v_2^2 v_1.$$

It is clear that $f$ is in $P_1(V_{10}) \otimes P_1(V_{20})$ and is $\mathfrak{t}^\lambda$ invariant. Thus $P_1(V_{10}) \otimes P_1(V_{20})$ is not an irreducible module of $\mathfrak{t}^\lambda$. Therefore, by [BJR, §2, Remark 1] we know that $\mathcal{P}(V_{1/2})$ is not of multiplicity one under $\mathfrak{t}^\lambda$ and thus $L^\lambda$. \qed
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