A CLASS OF M-DILATION SCALING FUNCTIONS WITH REGULARITY GROWING PROPORTIONALLY TO FILTER SUPPORT WIDTH

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Abstract. In this paper, a class of M-dilation scaling functions with regularity growing proportionally to filter support width is constructed. This answers a question proposed by Daubechies on p.338 of her book Ten Lectures on Wavelets (1992).

1. Introduction

Let $M \geq 2$ be a fixed integer. A multiresolution analysis for dilation $M$ consists of a sequence of closed subspaces $V_j$ of $L^2(\mathbb{R})$ that satisfy the following conditions (see [C], [D], [M]):

i) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$;

ii) $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R})$;

iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;

iv) $f \in V_j \iff f(2^{-j}) \in V_0$;

v) there exists a function $\phi$ in $V_0$ such that \{\phi(\cdot - n); n \in \mathbb{Z}\} is an orthonormal basis of $V_0$.

The function $\phi$ is called an $M$-dilation scaling function. It is easy to see that $\phi$ satisfies the refinement equation

$$\phi(x) = \sum_{n \in \mathbb{Z}} c_n \phi(Mx - n),$$

where the sequence \{c_n\} satisfies

$$\sum_{n \in \mathbb{Z}} c_n = M.$$
In this paper we shall only deal with compactly supported $M$-dilation scaling functions. In this case the sequence $\{c_n\}$ must have finite length. The function
\begin{equation}
H(\xi) = \frac{1}{M} \sum_{n \in \mathbb{Z}} c_n e^{in\xi}
\end{equation}
is called a symbol corresponding to the refinement equation (1).

The filter support width $W(\phi)$ of a $M$-dilation scaling function $\phi$ is defined as the difference of the largest and the smallest indices of the nonzero $c_n$. The regularity $R(\phi)$ of $\phi$ is defined as the supremum of $\alpha$ such that $\phi \in C^\alpha$, where $C^\alpha$ denotes the Hölder class of index $\alpha$.

In her book [D, p.338], Daubechies remarks that:

At present, I know of no explicit scheme that provides an infinite family of $H$ (i.e., symbol $H$), for dilation 3 (i.e., $M = 3$), with regularity growing proportionally to the filter support width.

To our knowledge, this question is still open. The purpose of this paper is to construct a class of $M$-dilation scaling functions $\phi_N$ for which there exists a constant $\lambda_M$ independent of $N$ such that
\begin{equation}
R(\phi_N) \geq \lambda_M W(\phi_N),
\end{equation}
where $M \geq 3$. On the other hand it is already known that (see [DL])
\begin{equation}
W(\phi_N) \geq R(\phi_N).
\end{equation}

These facts give an affirmative answer to Daubechies’ question.

The regularity of $\phi$ has been studied widely; see for instance [CL], [BDS], [D], [HW1], [HW2], [So], [S] and [WL]. In general, to study the regularity of $\phi$ we need to consider the symbol (2) first. By the Fourier transform, we see that all the symbols $H$ satisfy
\begin{equation}
\sum_{l=0}^{M-1} |H(\xi + \frac{2l\pi}{M})|^2 = 1.
\end{equation}
The solutions $H$ of the equation (4) are determined by (see [BDS], [H], [HSZ])
\begin{equation}
|H(\xi)|^2 = \left(\frac{\sin^2(M\xi/2)}{M^2 \sin^2 \xi/2}\right)^N \sum_{s=0}^{N-1} M(a(s) \sin^{2s} \frac{\xi}{2} + (\sin \frac{M\xi}{2})^{2s} R(\xi),
\end{equation}
where
\[ M(a(s) = \sum_{s_1 + \cdots + s_{M-1} = s} \prod_{j=1}^{M-1} \left( \frac{N - 1 + s_j}{s_j} \right) \frac{1}{\sin^{2s_j} \frac{j\pi}{M}} \]
and $R$ is a real-valued trigonometric polynomial such that $\sum_{l=0}^{M-1} R(\xi + 2l\pi/M) = 0$ and the right hand side of (5) is nonnegative.

By the Riesz Lemma (see [D, p.172]), such symbol $H$ exists. Let $\phi_N$ be a solution of (5) with $R = 0$, and let $\phi_N$ be the solution of (1) corresponding to the symbol $\phi_N$. In [BDS], Bi, Dai and Sun prove the following estimates on the regularity of $\phi_N$:
\[ |R(\phi_N) - \frac{\ln N}{4\ln M} | \leq C \]
when $M$ is odd, and
\[ |R(N\phi) - 4N \ln \left( \frac{\ln N}{4\ln M} \right) | \leq C \]
when $M$ is even. A more precise estimate of $R(N\phi)$ can be found in [S]. For the special cases $M = 3, 4, 5$, similar results are obtained by Soardi ([So]) and Heller and Wells ([HW2]) independently. This result shows that for these special $N\phi$ the regularity does not grow proportionally to the filter support width when $M$ is odd.

To construct $M$-dilation scaling functions with regularity growing proportionally to the filter support width we use the symbol $H_N$ determined by
\[ |H_N(\xi)|^2 = \sum_{k_0 + \cdots + k_{M-1} = MN-M+1} \alpha_N(k_0, \cdots, k_{M-1}) \frac{(MN-M+1)!}{k_0! \cdots k_{M-1}!} \]
\[ \times \prod_{l=0}^{M-1} \left( \frac{\sin M\xi/2}{M\sin(\xi/2+\ln M)} \right)^{2k_l}, \]
where $N \geq 1$, and $\alpha_N(k_0, \cdots, k_{M-1})$ is defined by
\[ \alpha_N(k_0, \cdots, k_{M-1}) = \begin{cases} 0, & \text{if } k_0 \leq N-1, \\
\frac{1}{\pi(k_0)}, & \text{if } k_0 \geq N, \end{cases} \]
where $E = \{ j : k_j \geq N \}$ and $\#(E)$ is the cardinality of $E$. Let $\phi_N$ be the solution of (1) corresponding to a symbol $H_N$. Then we have the following

**Theorem.** Let $M \geq 3$ and $N \geq 2$ be any natural numbers. Then $\phi_N$ is an $M$-dilation scaling function and there exists a constant $C$ independent of $N$ such that
\[ \left( \frac{1}{2} - \frac{(M-1)\ln(1 + \frac{1}{M-1})}{2\ln M} \right)N - \frac{\ln N}{4\ln M} - C \]
\[ \leq R(\phi_N) \leq \left( \frac{1}{2} - \frac{(M-1)\ln(1 + \frac{1}{M})}{2\ln M} \right)N - \frac{\ln N}{4\ln M} + C. \]

**Remark 1.** Observe that $W(\phi_N) \leq 2(M-1)MN$. Therefore the regularity $R(\phi_N)$ of $\phi_N$ grows proportionally to the filter support width $W(\phi_N)$, i.e., (3) holds.

**Remark 2.** Let $D(\phi) = R(\phi)/W(\phi)$ be the rate of regularity and filter support width of a scaling function $\phi$. Then
\[ D(\phi_N) \geq \frac{1}{4M(M-1)} \left( 1 - \frac{(M-1)\ln(1 + \frac{1}{M-1})}{\ln M} \right) - C \frac{\ln N}{N}, \]
and
\[ D(N\phi) \leq \frac{\ln N}{4NM\ln M} + \frac{C}{N} \]
when $M$ is odd, and
\[ D(N\phi) \leq \frac{\ln \left( \sin M\pi/(2M+2) \right)^{-1}}{M\ln M} + C \frac{\ln N}{N} \]
when $M$ is even. Therefore we get
\[ D(\phi_N)/D(N\phi) \geq \frac{N}{\ln N} \left( \frac{\ln M}{M-1} - \ln(1 + \frac{1}{M-1}) \right) - C \]
when $M$ is odd, and
\[ D(\phi_N)/D(N\phi) \geq \frac{\ln M - (M-1)\ln(1 + \frac{1}{M-1})}{4(M-1)\ln(\sin M\pi/(2M+2))^{-1}} - C \frac{\ln N}{N} \]
when \( M \) is even. This shows that \( D(\phi_N) \) of the \( M \)-dilation scaling function \( \phi_N \) is larger than the one of \( N \phi \) even when \( M \) is an even integer larger than 4.

2. Proof of the Theorem

To prove the Theorem, we estimate \( H_N(\xi) \) first. Let
\[
h(\xi) = \frac{\sin^2 M\xi/2}{M^2 \sin^2 \xi/2}
\]
and
\[
B_N(\xi) = (h(\xi))^{-N} |H_N(\xi)|^2.
\]
Then for all real valued \( \xi \) we have
\[
B_N(-\xi) = \sum_{k_0 + \cdots + k_{M-1} = MN-M+1} \alpha_N(k_0, k_{M-1}, \cdots, k_1) \frac{(MN-M+1)!}{k_0! k_1! \cdots k_{M-1}!}
\times (h(\xi))^{k_0-N} \prod_{l=1}^{M-1} (h(\xi + \frac{2l\pi}{M}))^{k_l}
= B_N(\xi)
\]
and
\[
B_N(\xi) \geq 0.
\]
Therefore by the Riesz Lemma ([D, p.172]) we obtain the existence of \( H_N(\xi) \) with
\[
H_N(\xi) = \left( \frac{1 - e^{iM\xi}}{M(1 - e^{i\xi})} \right)^N \tilde{H}_N(\xi)
\]
and
\[
|\tilde{H}_N(\xi)|^2 = B_N(\xi).
\]
From the definition of \( \alpha_N(k_0, \cdots, k_{M-1}) \) and from
\[
\sum_{l=0}^{M-1} h(\xi + \frac{2l\pi}{M}) = 1,
\]
we get
\[
\alpha_N(k_0, k_1, \cdots, k_{M-1}) + \alpha_N(k_1, k_2, \cdots, k_{M-1}, k_0)
+ \cdots + \alpha_N(k_{M-1}, k_0, \cdots, k_{M-2}) = 1
\]
and
\[
\sum_{l=0}^{M-1} |H_N(\xi + \frac{2l\pi}{M})|^2
= \sum_{k_0 + \cdots + k_{M-1} = MN-M+1} \alpha_N(k_0, k_1, \cdots, k_{M-1}) + \alpha_N(k_1, k_2, \cdots, k_{M-1}, k_0)
+ \cdots + \alpha_N(k_{M-1}, k_0, \cdots, k_{M-2}) \times \frac{(MN-M+1)!}{k_0! k_1! \cdots k_{M-1}!} \prod_{l=0}^{M-1} (h(\xi + \frac{2l\pi}{M}))^{k_l}
= (\sum_{l=0}^{M-1} h(\xi + \frac{2l\pi}{M}))^{MN-M+1}
= 1.
\]
Therefore (4) holds for $H_N(\xi)$. Recall that $H_N(\xi) \neq 0$ when $|\xi| \leq \pi/M$. Hence the solution $\phi_N$ of (1) corresponding to the symbol $H_N(\xi)$ is an $M$-dilation scaling function by an elementary argument ([D, p.182, Theorem 6.3.1] with $K = [-\pi, \pi]$).

To estimate the regularity of $\phi_N$, we need some estimates on $B_N(\xi)$. From the Stirling formula, which says that $n!$ is equivalent to $n^n e^{-n} \sqrt{n}$, from $1/(M-1) \leq \alpha_N(k_0, \cdots, k_{M-1}) \leq 1$ when $k_0 \geq N$ and from $h(2\pi/(M-1)) = 1/M^2$, we get

$$B_N(\xi) \leq \sum_{k_0+\cdots+k_{M-1}=MN-M+1, k_0 \geq N} \frac{(MN-M+1)!}{k_0! \cdots k_{M-1}!} \times (h(\xi))^{k_0-N} \prod_{l=1}^{M-1} (h(\xi + 2l\pi/M)^{k_l} \leq (MN-M+1)! \sum_{0 \leq k_0 \leq (N-1)(M-1)} \frac{(k_0 + N)!((N-1)(M-1) - k_0)!}{(N-1)(M-1)!) \times (h(\xi))^{k_0(1 - h(\xi))^{(N-1)(M-1) - k_0} \leq \frac{(MN-M+1)!}{N!(N-1)(M-1)!} \leq CM^N(1 + \frac{1}{M-1})^{(M-1)N} N^{-1/2}$$

and

$$B_N(2\pi/(M-1)) \geq \frac{1}{M-1} \sum_{k_1+\cdots+k_{M-1}=(M-1)(N-1)} \frac{(MN-M+1)!}{N!k_1! \cdots k_{M-1}!} \times \prod_{l=1}^{M-1} (h(2\pi/(M-1) + 2l\pi/M)^{k_l} \geq \frac{1}{M-1} \times \frac{(MN-M+1)!}{N!(N-1)(M-1)!} \left(1 - \frac{1}{M^2}\right)^{(N-1)(M-1)} \geq CM^N(1 + \frac{1}{M})^{(M-1)N} N^{-1/2}.$$

Therefore we get

$$R(\phi_N) \geq \left(\frac{1}{2} - \frac{(M-1)\ln(1 + \frac{1}{M})}{2\ln M}\right)N - \frac{\ln N}{4\ln M} - C$$

by an argument as in [D, p.217]. Observe that $2M\pi/(M-1) = 2\pi/(M-1) + 2\pi$. By an argument similar to that on p.220 of [D] we obtain

$$R(\phi_N) \leq \left(\frac{1}{2} - \frac{(M-1)\ln(1 + 1/M)}{2\ln M}\right)N - \frac{\ln N}{4\ln M} - C.$$

This completes the proof of the Theorem.

References


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