

AN EXTREMAL PROBLEM FOR TRIGONOMETRIC POLYNOMIALS

J. MARSHALL ASH AND MICHAEL GANZBURG

(Communicated by Christopher D. Sogge)

ABSTRACT. Let $T_n(x) = \sum_{k=0}^n (a_k \cos kx + b_k \sin kx)$ be a trigonometric polynomial of degree n . The problem of finding C_{np} , the largest value for C in the inequality $\max\{|a_0|, |a_1|, \dots, |a_n|, |b_1|, \dots, |b_n|\} \leq (1/C) \|T_n\|_p$ is studied. We find C_{np} exactly provided p is the conjugate of an even integer $2s$ and $n \geq 2s - 1$, $s = 1, 2, \dots$. For general p , $1 \leq p \leq \infty$, we get an interval estimate for C_{np} , where the interval length tends to 0 as n tends to ∞ .

1. INTRODUCTION

Let L_p be the Banach space of all 2π -periodic real-valued functions f with finite norm $\|f\|_p$ defined to be $\left(\int_{-\pi}^{\pi} |f(x)|^p dx\right)^{1/p}$ when $1 \leq p < \infty$, and $\operatorname{ess\,sup}_{x \in [-\pi, \pi]} |f(x)|$ when $p = \infty$. Let $p' = \frac{p}{p-1}$ be the conjugate exponent and for $n \geq 1$ let \mathcal{T}_n be the class of all trigonometric polynomials with real coefficients of degree n or less.

Inequalities between different norms of trigonometric polynomials play an important role in analysis. (See [Ti], 4.9, [N2], chapters 2 and 5, and [AWW].) In particular, Nikolskii [N1] obtained the inequality

$$(1.1) \quad \|T_n\|_q \leq Cn^{1/p-1/q} \|T_n\|_p, T_n \in \mathcal{T}_n, 1 \leq p \leq q \leq \infty, n \geq 1,$$

where $C < 2$. Let $C_n(p, q)$ denote the smallest possible value for the constant C in (1.1). Then $C_n(2, \infty) = \sqrt{\frac{2n+1}{2\pi n}}$ is known but the problem of finding $C_n(p, q)$ for any other (p, q) is open. (Cf. [Ti], p. 229.) Some estimates of $C_n(1, \infty)$ were obtained by Taykov [Ta].

In this paper we study inequalities like (1.1) where $\|T_n\|_q$ on the left-hand side is replaced by the ℓ_∞ norm of the Fourier coefficients of T_n .

Let $T_n(x) = \sum_{k=0}^n (a_k \cos kx + b_k \sin kx)$ be a polynomial in \mathcal{T}_n and let $\hat{T}_n = \{a_0, a_1, \dots, a_n, b_1, \dots, b_n\}$ be the vector of its coefficients. Set

$$\|\hat{T}_n\|_\infty = \max\{|a_0|, |a_1|, \dots, |a_n|, |b_1|, \dots, |b_n|\}$$

Received by the editors January 9, 1997 and, in revised form, May 12, 1997.

1991 *Mathematics Subject Classification*. Primary 42A05; Secondary 41A44.

Key words and phrases. Trigonometric polynomial, inequalities between different norms, best constants.

and put

$$C_{np} = \inf_{\substack{T_n \in \mathcal{T}_n \\ \|\hat{T}_n\|_\infty = 1}} \|T_n\|_p, \text{ for } 1 \leq p \leq \infty.$$

Equivalently, C_{np} is the largest possible value for C in the inequality

$$\|\hat{T}_n\|_\infty \leq \left(\frac{1}{C}\right) \|T_n\|_p.$$

When viewed in this way, the problem of finding C_{np} has much the same flavor as the problem of finding $C_n(p, q)$ mentioned above.

The problem of finding C_{np} was posed by Ash, Wang, and Weinberg [AWW]. In the final section of this paper, we explain what motivated the question and propose some related open problems.

Here we find C_{np} for some particular $p \in (1, 2]$ and establish an asymptotic estimate for C_{np} , $1 \leq p \leq \infty$, as $n \rightarrow \infty$.

2. RESULTS

Theorem 2.1. (A) For $1 \leq p \leq \infty$, $n \geq 1$,

$$(2.1) \quad \alpha_p \leq C_{np} \leq \alpha_p \left(1 + \frac{1}{n}\right),$$

where $\alpha_1 = \pi$ and for $1 < p \leq \infty$,

$$(2.2) \quad \alpha_p = \pi \left(\int_{-\pi}^{\pi} |\cos x|^{p'} dx \right)^{-1/p'} = \pi \left(\frac{2^{p'-1} \Gamma(p'/2 + 1)^2}{\pi \Gamma(p' + 1)} \right)^{1/p'}.$$

(B) If $p = \frac{2s}{2s-1}$ so that $p' = 2s$, $s = 1, 2, \dots$, then for $n \geq 2s - 1$, $C_{np} = \alpha_p$.

Proof. This consists of four steps. First we establish the lower estimate in (2.1). Then we study properties of a function f_p which is “extremal for $C_{\infty p}$ ”, $1 < p \leq \infty$ and use f_p to prove part **(B)** of Theorem 2.1. Next, we use the Fejér mean of f_p to obtain the upper estimate in (2.1) for $1 < p \leq \infty$. This idea for $p = \infty$ was first used by Bernstein ([B], pp. 29–31) for solving a different problem. Finally, using the Fejér kernel we prove the upper estimate in (2.1) for $p = 1$.

Step 1. Let $T_n(x) = \sum_{k=0}^n (a_k \cos kx + b_k \sin kx)$ satisfy the condition $\|\hat{T}_n\|_\infty = 1$. If $|a_0| = 1$, then for $1 \leq p \leq \infty$

$$(2.3) \quad \|T_n\|_p \geq (2\pi)^{-1/p'} \left| \int_{-\pi}^{\pi} T_n(x) dx \right| = (2\pi)^{1/p} |a_0| = (2\pi)^{1/p}.$$

If $|a_0| \neq 1$, then $|a_0| < 1$ and either $|a_\ell| = 1$ or $|b_\ell| = 1$ for some $\ell \geq 1$. First suppose that $|a_\ell| = 1$ for some $\ell \geq 1$. Fix $p \in (1, \infty]$. An application of Hölder’s inequality and (2.2) shows that

$$\|T_n\|_p \geq \alpha_p \pi^{-1} \left| \int_{-\pi}^{\pi} T_n(x) \cos \ell x dx \right| = \alpha_p.$$

Since a similar argument holds if $|b_\ell| = 1$ for some $\ell \geq 1$,

$$(2.4) \quad \|T_n\|_p \geq \alpha_p \text{ whenever } |a_\ell| = 1 \text{ or } |b_\ell| = 1 \text{ for some } \ell \geq 1.$$

It is clear that (2.4) holds for $p = 1$ as well. From the dichotomy that either $|a_0| = 1$ or $|a_0| \neq 1$ and inequalities (2.3) and (2.4) it follows that

$$(2.5) \quad \|T_n\|_p \geq \min \left\{ (2\pi)^{1/p}, \alpha_p \right\} = (2\pi)^{1/p} \min \{1, \gamma(p')/2\},$$

where $\gamma(t) = \left((2\pi)^{-1} \int_{-\pi}^{\pi} |\cos x|^t dx \right)^{-1/t}$, $1 \leq t \leq \infty$. But $1/\gamma$ is nondecreasing ([Z], Vol. I, 1.10.12(i)), so $\sup_{1 \leq t \leq \infty} \gamma(t) = \gamma(1) = \pi/2$, and the lower estimate in (2.1) follows from inequality (2.5).

Step 2. Since Step 1 basically consisted of applying Holder's inequality, to get the reverse inequality it is natural to work with a function which is extremal for Holder's inequality. (Cf., for example, the proof of Theorem I.9.6 on page 20 of [Z].) This motivates considering the function

$$f_p(x) = \pi (\alpha_p/\pi)^{p'} |\cos x|^{p'-1} \operatorname{sgn}(\cos x)$$

which has the following properties for $1 < p \leq \infty$.

(a) $\|f_p\|_p = \alpha_p$,

(b) f_p has the Fourier expansion $\sum_{k=0}^{\infty} c_{2k+1} \cos(2k+1)x$, where

$$c_{2k+1} = \frac{\pi (\alpha_p/\pi)^{p'} 2^{2-p'} \Gamma(p')}{\Gamma((p'-2k)/2) \Gamma((p'+2k+2)/2)}, \quad k = 0, 1, \dots,$$

(c) in particular, $c_1 = 1$,

(d) $|c_{2k+1}|$ is a decreasing sequence for $k \geq 0$, and

(e) $\max_{k \geq 0} |c_{2k+1}| = 1$.

Properties (a), (b), and (c) (and also the evaluation of α_p in equation (2.2) above) can be easily derived from formula 2.5.3.1 and the fourth formula from the end of Appendix II.1 in [P], property (d) is a consequence of the relations

$$\frac{|c_{2k+3}|}{|c_{2k+1}|} = \frac{|p'/2 - (k+1)|}{|p'/2 + (k+1)|} < 1,$$

and property (e) follows from properties (c) and (d).

If $p' = 2s$, $s \geq 1$, is an even integer, then $f_p \in \mathcal{T}_{2s-1}$. Therefore for $n \geq 2s-1$, $f_p \in \mathcal{T}_n$ and properties (a), (b), and (e) yield the estimate $C_{np} \leq \alpha_p$. Combining this with the lower estimate in (2.1) proves part (B) of Theorem 2.1.

If $p' \neq 2s$, then $f_p \notin \mathcal{T}_n$ for any n . In this case we are able to obtain only the asymptotic estimate $C_{np} = \alpha_p(1 + O(\frac{1}{n}))$.

Step 3. Let

$$(2.6) \quad K_n(t) = \frac{1}{\pi} \left(\frac{1}{2} + \sum_{s=1}^n \left(1 - \frac{s}{n+1}\right) \cos st \right)$$

be the Fejér kernel of degree n so that

$$(2.7) \quad K_n(t) \geq 0 \text{ and } \int_{-\pi}^{\pi} K_n(t) dt = 1.$$

Next, let

$$\begin{aligned} F_{np}(x) &= (K_n * f_p)(x) = \int_{-\pi}^{\pi} f_p(x-t) K_n(t) dt \\ &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(1 - \frac{2k+1}{n+1}\right) c_{2k+1} \cos(2k+1)x \end{aligned}$$

be the Fejér mean of f_p , $1 < p \leq \infty$, $n \geq 1$. Applying properties (2.7) and Young's inequality ([Z], Vol. I, 2.1.15), we obtain

$$(2.8) \quad \|F_{np}\|_p \leq \|f_p\|_p, 1 < p \leq \infty.$$

Denoting $Q_{np}(x) = \frac{n+1}{n}F_{np}$, $n \geq 1$, we have $Q_{np} \in \mathcal{T}_n$,

$$(2.9) \quad \|Q_{np}\|_p \leq \alpha_p(1 + \frac{1}{n}),$$

and

$$(2.10) \quad \|\hat{Q}_{np}\|_\infty = 1.$$

Inequality (2.9) follows from (2.8) and property **(a)**. The identity $\frac{n+1}{n} = (1 - \frac{1}{n+1})^{-1}$ allows us to write the coefficients of Q_{np} as

$$A_{2k+1} = \left(\frac{1 - \frac{2k+1}{n+1}}{1 - \frac{1}{n+1}} \right) c_{2k+1}, 0 \leq k \leq \frac{n-1}{2}.$$

To prove (2.10) it is sufficient to note that in view of properties **(c)** and **(d)** there hold the relations $|A_{2k+1}| \leq 1$, $0 \leq k \leq \frac{n-1}{2}$, and $A_1 = 1$. Relations (2.9) and (2.10) yield the upper estimate in (2.1) for $1 < p \leq \infty$.

Step 4. For $n \geq 1$ the polynomial $Q_{n1}(x) = \pi \frac{n+1}{n} K_n(x)$, where K_n is defined by equation (2.6), satisfies the conditions

$$\|Q_{n1}\|_1 \leq \pi(1 + \frac{1}{n}), \|\hat{Q}_{n1}\|_\infty = 1.$$

This yields the upper estimate in (2.1) for $p = 1$. □

Corollary 2.2. *The following asymptotic estimates hold*

$$\begin{aligned} C_{np} &= \alpha_p(1 + O(\frac{1}{n})), 1 \leq p \leq \infty \\ C_{n1} &= \pi(1 + O(\frac{1}{n})), \text{ and} \\ C_{n\infty} &= \frac{\pi}{4}(1 + O(\frac{1}{n})). \end{aligned}$$

Remark 2.1. We followed the custom of people working in approximation theory when we wrote

$$T_n(x) = \sum_{k=0}^n (a_k \cos kx + b_k \sin kx) \text{ and } \hat{T}_n = \{a_0, a_1, \dots, a_n, b_1, \dots, b_n\}.$$

(See, for example, [B] or [Ti].) But another quite prevalent tradition would have us write

$$(2.11) \quad T_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \text{ and } \hat{T}_n = \{a_0, a_1, \dots, a_n, b_1, \dots, b_n\}.$$

This leads to a different (and much easier) problem. Let T_n and \hat{T}_n be given by (2.11) and set

$$C_{np}^* = \inf_{\substack{T_n \in \mathcal{T}_n \\ \|\hat{T}_n\|_\infty = 1}} \|T_n\|_p, \text{ for } 1 \leq p \leq \infty.$$

Then we have the complete (and trivial) solution

$$C_{np}^* = \frac{1}{2} (2\pi)^{1/p}$$

with the extremal maximal polynomial $T_n(x) = \frac{1}{2}$. To prove the corresponding lower estimate, let T_n and \hat{T}_n be given by (2.11) with $\|\hat{T}_n\|_\infty = 1$. The following relations hold (cf. (2.5)):

$$\|T_n\|_p \geq \min \left\{ \frac{1}{2} (2\pi)^{1/p}, \alpha_p \right\} = (2\pi)^{1/p} \min \left\{ \frac{1}{2}, \frac{\gamma(p')}{2} \right\} = \frac{1}{2} (2\pi)^{1/p}.$$

Remark 2.2. Theorem 2.1 has the flavor of the Heisenberg uncertainty principle in the sense that in this paper we have looked for the smallest possible function on the “function side” (in the sense of L_p norm) for a function of fixed size (in the sense of ℓ_∞) on the “transform side.”

Remark 2.3. We note that the equality $C_{n2} = \pi^{1/2}$ following from part **(B)** of Theorem 2.1 is an easy consequence of Parseval’s formula.

Remark 2.4. Bernstein ([B], pp. 29–31) studied this problem: to find

$$M_{nk} = \inf_{\substack{T_n \in \mathcal{T}_n \\ a_k^2 + b_k^2 = 1}} \|T_n\|_\infty,$$

where k is a fixed integer, $1 \leq k \leq n - 1$. He obtained that for each given k , $\lim_{n \rightarrow \infty} M_{nk} = \frac{\pi}{4}$.

3. MOTIVATION AND FURTHER OPEN PROBLEMS

The first obvious generalization would involve replacing the pair $(\ell_\infty, L_p[-\pi, \pi])$ by $(\ell_q, L_p[-\pi, \pi])$.

To find other interesting generalizations, we will begin by looking at the questions that motivated this paper. Fix a small $\varepsilon > 0$. The trigonometric polynomials

$$S_n(x) = \frac{d}{2^n} (1 - \cos x)^n = d \sin^{2n} \left(\frac{x}{2} \right), \text{ where } d = d(n) = \frac{4^n}{\binom{2n}{n}},$$

when restricted to the interval $J_\varepsilon = [-\pi + \varepsilon, \pi - \varepsilon]$ have the property that they decay rapidly and uniformly in x to 0 as $n \rightarrow \infty$, although the largest Fourier coefficient of each S_n is 1. Having uniformly small polynomials with large coefficients facilitates the construction of many interesting counterexamples. For example, the series $S(x, y) = \sum_{n=1}^{\infty} n^{100} S_n(x) \cos ny$ is uniformly convergent on $J_\varepsilon \times [-\pi, \pi]$. Euler’s identity and the binomial theorem allow us to write $\sin^{2n} \left(\frac{x}{2} \right) = \sum_{k=0}^n c_k^{(n)} \cos kx$ for appropriate constants $c_k^{(n)}$. This shows that $S(x, y)$ may be thought of as a double trigonometric series which has its square partial sums converging uniformly on $J_\varepsilon \times [-\pi, \pi)$ despite having unbounded coefficients. This idea is carried much further in reference [AW]. Victor Shapiro recently called our attention to a paper of Walter Rudin in which the trigonometric polynomials $\{S_n(x)\}$ were used in a similar way to provide interesting examples in the area of spherical harmonics. (See page 302 of [R].)

The crucial property enjoyed by $S_n(x)$ is that

$$(3.1) \quad \frac{\sup_{x \in J_\varepsilon} |S_n(x)|}{\|\hat{S}\|_\infty}$$

is quite small. For some time Ash and Wang thought that $S_n(x)$ might well minimize the ratio (3.1), but Peter Borwein showed them that this is not the case. (See page 227 of [AWW].) It still appears that $S_n(x)$ comes close to minimizing (3.1), but finding the element of \mathcal{T}_n which minimizes (3.1) is our first unsolved problem.

To try to attack a similar, but easier, problem, we let $\varepsilon \rightarrow 0$, i.e., we replaced J_ε by $J_0 = [-\pi, \pi]$. This amounts to the problem of trying to find $C_{n\infty}$, the minimum of $\frac{\|S_n\|_\infty}{\|S\|_\infty}$. However, notice that $C_{n\infty}$ tends to $\frac{\pi}{4}$, rather than 0 as n tends to ∞ , and the optimizing trigonometric polynomials tend to the function $f_\infty(x) = \pi \operatorname{sgn}(\cos x)/4$ rather than looking anything at all like $S_n(x)$. Thus our case has a resolution which sheds very little light on the original problem.

If the original problem involving the minimization of the ratio (3.1) were to be solved, the next open problem would be to find the best constants, or at least good estimates for the pair $(\ell_\infty, L_p[J_\varepsilon])$ or even for the pair $(\ell_q, L_p[J_\varepsilon])$, where $0 < \varepsilon < \pi$.

REFERENCES

- [AW] J. M. Ash and G. Wang, *One and two dimensional Cantor-Lebesgue type theorems*, Trans. Amer. Math. Soc., **349** (1997), 1663–1674. MR **97h**:42006
- [AWW] J. M. Ash, G. Wang, and D. Weinberg, *A Cantor–Lebesgue theorem with variable “coefficients,”* Proc. Amer. Math. Soc., **125**(1997), 219–228. MR **97c**:42011
- [B] S. N. Bernstein, *Extremal Properties of Polynomials*, GROL, Leningrad, Moscow, 1937. (In Russian.)
- [N1] S. M. Nikolskii, *Inequalities for entire functions of finite degree and their applications in the theory of differentiable functions of many variables*, Proceedings of MIAN USSR, **38**(1951), 244–278. (In Russian.) MR **14**:32e
- [N2] S. M. Nikolskii, *Approximation of Functions in Many Variables and Embedding Theorems*, Nauka, Moscow, 1977. (In Russian.) MR **81f**:46046
- [P] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series*, Vol. I, Gordon and Breach, New York, 1986. MR **88f**:00013
- [R] W. Rudin, *Uniqueness theory for Laplace series*, Trans. Amer. Math. Soc., **68**(1950), 287–303. MR **11**:430a
- [Ti] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, New York, 1963. MR **33**:465
- [Ta] L. V. Taykov, *A circle of extremal problems for trigonometric polynomials*, Ukrainskii Mat. Zhurnal, **20**(1965), 205–211. (In Russian.)
- [Z] A. Zygmund, *Trigonometric Series*, Vol. I, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR **21**:6498

DEPARTMENT OF MATHEMATICS, DEPAUL UNIVERSITY, CHICAGO, ILLINOIS 60614
E-mail address: `mash@math.depaul.edu`

DEPARTMENT OF MATHEMATICS, HAMPTON UNIVERSITY, HAMPTON, VIRGINIA 23668
E-mail address: `ganzbrgm@fusion.hamptonu.edu`