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COMPLETE POSITIVITY OF ELEMENTARY OPERATORS

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Abstract. In this paper, we prove that if $\mathcal{S}$ is an n-dimensional subspace of $L(H)$, then $\mathcal{S}$ is $([\frac{n}{2}]+1)$-reflexive, where $[\frac{n}{2}]$ denotes the greatest integer not larger than $\frac{n}{2}$. By the result, we show that if $\Phi(\ \cdot\ ) = \sum\limits_{i=1}^{n} A_i(\ \cdot\ )B_i$ is an elementary operator on a C^*-algebra $\mathcal{A}$, then Φ is completely positive if and only if Φ is $([\frac{n-1}{2}]+1)$-positive.

In this paper, let H denote a complex Hilbert space. Let $H^{(n)}$ denote the direct sum of n copies of H. For $T \in L(H)$, we write $T^{(n)}$ for the operator on $H^{(n)}$ which is the direct sum of n copies of T; the notation is extended to a subset of $L(H)$ by $\mathcal{S}^{(n)} \equiv \{T^{(n)} \in L(H^{(n)}) :\ T \in \mathcal{S}\}$. If $\mathcal{S}$ is a subspace of $L(H)$, $\mathcal{S}$ is called n-reflexive if $\mathcal{S}^{(n)} = ref(\mathcal{S}^{(n)}) \equiv \{T^{(n)} \in L(H^{(n)}) : T^{(n)}x \in [\mathcal{S}^{(n)}x]$, for all $x \in H^{(n)}\}$, where $[\cdot]$ denotes norm closed linear span. By the definiton, we have that if $\mathcal{S}$ is m-reflexive, then $\mathcal{S}$ is n-reflexive for $n \geq m$. A separating vector for a subspace $\mathcal{S}$ of $L(H)$ is a vector $x \in H$ such that $T \mapsto Tx, T \in \mathcal{S}$, is an injective map. For $x, y \in H$, let $x \otimes y$ denote the rank-one operator $u \mapsto (u,x)y$.

Let $\mathcal{A}$ denote a C^*-algebra. Then $\mathcal{A}$ is called primitive, if $\mathcal{A}$ has a faithful irreducible representation on some Hilbert space. An elementary operator Ψ on $\mathcal{A}$ is a linear mapping of the form $\Psi : T \mapsto \sum\limits_{i=1}^{n} A_iTB_i$, where $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are subsets of $\mathcal{A}$. In this paper, we assume that all elementary operators are nonzero. A linear map Φ on $\mathcal{A}$ is positive (resp. hermitian-preserving) if $\Phi(T)$ is positive (resp. hermitian) for all positive (resp. hermitian) T in $\mathcal{A}$. We define $\Phi_n = \Phi \otimes I_n : M_n(\mathcal{A}) \to M_n(\mathcal{A})$ by $\Phi \otimes I_n((T_{ij})_{n\times n}) = (\Phi(T_{ij}))_{n\times n}$. Φ is said to be n-positive if $\Phi \otimes I_n$ is positive. If Φ is n-positive for all n, then Φ is said to be completely positive.

In [4], Magajna states the following problem:

For each positive integer r determine the smallest $k = k(r)$ such that all r-dimensional subspaces of $L(H)$ are k-reflexive.

In [4], Magajna proves $k \leq r$. In this paper, we prove that if $\mathcal{S}$ is an n-dimensional subspace of $L(H)$, then $\mathcal{S}$ is $([\frac{n}{2}]+1)$-reflexive. Also by this result, we study complete positivity of elementary operators on a C^*-algebra $\mathcal{A}$. We prove that if $\Phi(\ \cdot\ ) = \sum\limits_{i=1}^{n} A_i(\ \cdot\ )B_i$ is an elementary operator on a C^*-algebra $\mathcal{A}$, then Φ is

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completely positive if and only if Φ is $([\frac{n-1}{2}]+1)$-positive. This result improves Theorem 2 [6].

Lemma 1. *If the operators $A_j = a_j \otimes c$ $(j = 1, ..., n)$ form a basis of a subspace $\mathcal{S}$ of $L(H)$, then $\mathcal{S}$ is reflexive.*

Proof. Let $T \in L(H)$ be such that for any $x \in H, Tx \in [\mathcal{S}x]$, we have $T = b \otimes c$. If x is orthogonal to all a_j, then $A_j x = 0$ for all j. Thus $Tx = 0$ and x must be orthogonal to b. This implies that $b \in span\{a_1, ..., a_n\}$, hence $T \in \mathcal{S}$. □

The following result is given in [2].

Lemma 2. *Let $A, B \in L(H)$. Then Ax and Bx are linearly dependent for every x in $L(H)$ if and only if one of the following conditions holds:*
(i) A and B are linearly dependent;
(ii) there exist $x_0, x_1, x_2 \in H$ with $A = x_1 \otimes x_0$ and $B = x_2 \otimes x_0$.

Theorem 3. *If $\mathcal{S}$ is a subspace of $L(H)$ and $dim\,\mathcal{S} = n$, then $\mathcal{S}$ is $([\frac{n}{2}]+1)$-reflexive.*

Proof. If $n = 1, 2$, by Magajna's result [4], then the theorem is true.

Suppose now, inductively, that the theorem is true for any subspace of $L(H)$ of dimension at most $n-1$, where $n \geq 3$, and let $\mathcal{S}$ be an n-dimensional subspace of $L(H)$. Let $\{A_1, ..., A_n\}$ be a basis of $\mathcal{S}$ and define

$$k = max\{dim\,span\{A_1x, ..., A_nx\} : x \in H\},$$

then $1 \leq k \leq n$.

If $k = 1$, by Lemmas 1 and 2, we have that the theorem is true.

If $k = n$, then $span\{A_1, ..., A_n\}$ has a separating vector x_0. In the following, we prove that $span\{A_1, ..., A_n\}$ is 2-reflexive. Suppose

$$T^{(2)}\begin{pmatrix} x \\ y \end{pmatrix} \in [\mathcal{S}^{(2)}\begin{pmatrix} x \\ y \end{pmatrix}]$$

for every pair x, y of vectors in H. Then for each $y \in H$, there is a $T_y \in \mathcal{S}$ satisfying $T_y x_0 = Tx_0, T_y y = Ty$. Since x_0 is separating for $\mathcal{S}$, T_y must be independent of y, so $T \in \mathcal{S}$. By $n \geq 3$, we have $([\frac{n}{2}]+1) \geq 2$. Hence $\mathcal{S}$ is $([\frac{n}{2}]+1)$-reflexive.

If $2 \leq k \leq n-1$, we may assume, by reordering the A_i if necessary, that there exists a vector x_0 such that $\{A_i x_0\}_{i=1}^{k}$ is linearly independent. Hence there is a unique $k \times (n-k)$ complex matrix (a_{ij}) such that

$$A_{k+j}x_0 = \sum_{i=1}^{k} a_{ij} A_i x_0, \ j = 1, ..., n-k. \tag{1}$$

In the following, let $l = [\frac{n}{2}]+1$. If $A \in L(H)$, for any $x \in H^{(l)}$, satisfies $A^{(l)}x \in [\mathcal{S}^{(l)}x]$, then for any $x_1, ..., x_{l-1}$ in H, there exist scalars $t_1, ..., t_n$ such that

$$\begin{pmatrix} Ax_0 \\ \vdots \\ Ax_{l-1} \end{pmatrix} = t_1 \begin{pmatrix} A_1x_0 \\ \vdots \\ A_1x_{l-1} \end{pmatrix} + \ldots + t_n \begin{pmatrix} A_nx_0 \\ \vdots \\ A_nx_{l-1} \end{pmatrix}. \tag{2}$$

Since $Ax_0 \in span\{A_1x_0, ..., A_nx_0\} = span\{A_1x_0, ..., A_kx_0\}$, there exist scalars $v_1, ..., v_k$ such that

$$Ax_0 = \sum_{i=1}^{k} v_i A_i x_0. \tag{3}$$

By (1), (2) and (3), we have that

$$Ax_g = \sum_{i=1}^{k}(v_i - \sum_{j=1}^{n-k} t_{j+k} a_{ij}) A_i x_g + \sum_{j=1}^{n-k} t_{j+k} A_{k+j} x_g,\ g = 1, ..., l-1. \tag{4}$$

Let

$$C = A - \sum_{i=1}^{k} v_i A_i \text{ and } B_j = A_{k+j} - \sum_{i=1}^{k} a_{ij} A_i,\ j = 1, ..., n-k. \tag{5}$$

For any $x_1, ..., x_{l-1}$, by (2), (4) and (5), we have

$$\begin{pmatrix} Cx_1 \\ \vdots \\ Cx_{l-1} \end{pmatrix} = t_{k+1} \begin{pmatrix} B_1x_1 \\ \vdots \\ B_1x_{l-1} \end{pmatrix} + \ldots + t_n \begin{pmatrix} B_{n-k}x_1 \\ \vdots \\ B_{n-k}x_{l-1} \end{pmatrix}.$$

Since $2 \le k \le n-1$ and $n-k \le n-2$, we have $l-1 \ge [\frac{n-k}{2}]+1$. By the inductive hypothesis, we have $C \in span\{B_1, ..., B_{n-k}\}$, hence $A \in span\{A_1, ..., A_n\}$. □

Corollary 4. *Let $\mathcal{S}$ be as in Theorem 5. If $dim\, \mathcal{S}_F = m$, where $\mathcal{S}_F$ denotes all finite-rank operators in $\mathcal{S}$, then $\mathcal{S}$ is $([\frac{m}{2}]+1)$-reflexive.*

Proof. By Theorem 2.6 [3], we have that $ref(\mathcal{S}^{(n)}) = \mathcal{S}^{(n)} + ref(\mathcal{S}_F^{(n)})$. Hence $\mathcal{S}^{(n)}$ is reflexive if and only if $\mathcal{S}_F^{(n)}$ is reflexive. By Theorem 3, it follows that $\mathcal{S}$ is $([\frac{m}{2}]+1)$-reflexive. □

Remark. By Theorem 3, a routine modification of the proof of Proposition 4.3 [4], we may prove that for any r-dimensional subspace $\mathcal{S}$ of a countably generated von Neumann algebra $\mathcal{R}$ on a Hilbert space H the space $\overline{\mathcal{R}'\mathcal{S}}$ is $([\frac{r}{2}]+1)$-reflexive (relative to L(H)), and the space $\overline{\varepsilon\mathcal{S}}$ is $([\frac{r}{2}]+1)$-reflexive relative to $\mathcal{R}$ where ε is the center of $\mathcal{R}$.

Lemma 5. *If $\Phi(\ \cdot\ ) = \sum_{i=1}^{n} A_i(\ \cdot\ )B_i$ is an elementary operator on $L(H)$, then Φ is completely positive if and only if Φ is $([\frac{n-1}{2}]+1)$-positive.*

Proof. The necessity is obvious, we only need to prove the sufficiency.

We may assume that $\{A_1, ..., A_n\}$ and $\{B_1, ..., B_n\}$ are linearly independent. Since Φ is positive, it follows that Φ is a hermitian-preserving elementary operator. By Corollary 4.9 [5], we have

$$\Phi(\ \cdot\ ) = \sum_{i=1}^{k} D_i(\ \cdot\ )D_i^* - \sum_{i=k+1}^{n} D_i(\ \cdot\ )D_i^*, \tag{6}$$

where $\{D_i\}_{i=1}^{n}$ is linearly independent.

To prove that Φ is completely positive, it suffices to prove $k = n$. If $n = 1$, by (6), since Φ is positive, then the lemma is true. In the following, let $m = [\frac{n-1}{2}]+1$.

If $n > 1$ and $k \leq n-1$, since Φ is m-positive, for any vectors x and y in $H^{(m)}$, we have

$$\langle(\Phi_m(x\otimes x))y,y\rangle = \sum_{i=1}^{k} \mid \langle D_i^{(m)}x,y\rangle \mid^2 - \sum_{i=k+1}^{n} \mid \langle D_i^{(m)}x,y\rangle \mid^2 \geq 0. \tag{7}$$

By (7), we have that

$$D_j^{(m)}x \in span\{D_1^{(m)}x,...,D_k^{(m)}x\},\ k+1\leq j\leq n. \tag{8}$$

Since $1 \leq k \leq n-1$, by Theorem 3, we have that $span\{D_1,...,D_k\}$ is $([\frac{k}{2}]+1)$-reflexive. Since $[\frac{k}{2}]+1 \leq [\frac{n-1}{2}]+1$, we have that $span\{D_1,...,D_k\}$ is m-reflexive. By (8), we have that $D_{k+1},...,D_n$ belongs to $span\{D_1,...,D_k\}$. Since $\{D_i\}_{i=1}^n$ is linearly independent, this is a contradiction. □

Theorem 6. *If* $\Phi(\ \cdot\ ) = \sum\limits_{i=1}^{n} A_i(\ \cdot\ )B_i$ *is an elementary operator on a* C^**-algebra* $\mathcal{A}$*, then* Φ *is completely positive if and only if* Φ *is* $([\frac{n-1}{2}]+1)$*-positive.*

Lemma 7. *Theorem 6 holds when* $\mathcal{A}$ *is primitive.*

Proof. The necessity is trivial, so we have only to prove the sufficiency.

Since $\mathcal{A}$ is primitive, we may assume that $\mathcal{A}$ acts irreducibly on the Hilbert space H. By Φ, we may induce an elementary operator $\tilde{\Phi}$ on $L(H)$ defined by $\tilde{\Phi}(T) = \sum\limits_{i=1}^{n} A_iTB_i$ for any T in $L(H)$. Since $\mathcal{A}$ is irreducible, we have that $\mathcal{A}$ is strongly dense in $L(H)$. Hence Φ is m-positive on $\mathcal{A}$ if and only if $\tilde{\Phi}$ is m-positive on $L(H)$. Since Φ is $([\frac{n-1}{2}]+1)$-positive, we have that $\tilde{\Phi}$ is $([\frac{n-1}{2}]+1)$-positive. By Lemma 5, we have that $\tilde{\Phi}$ is completely positive. Hence Φ is completely positive. □

Proof of Theorem 6. The forward implication is obvious.

Conversely, let π be an irreducible representation of $\mathcal{A}$ on H. Then Φ induces an elementary operator $\pi\Phi(\ \cdot\ ) = \sum\limits_{i=1}^{n} \pi(A_i)(\ \cdot\ )\pi(B_i)$ on $L(H)$. By Lemma 7, we have that $\pi\Phi$ is completely positive. Let $\pi_a = \bigoplus\limits_{t\in\hat{A}} \pi_t$ be the reduced atomic representation of $\mathcal{A}$ on $H_a = \bigoplus\limits_{t\in\hat{A}} H_t$. Then π_a is a faithful representation of $\mathcal{A}$ on H_a. Since $\pi_t\Phi$ is completely positive on $L(H_t)$, we have $\pi_a\Phi$ is completely positive on $\prod\limits_{t\in\hat{A}} L(H_t)$. Since π_a is a faithful representation of $\mathcal{A}$ on $L(H_a)$, we have that Φ is completely positive. □

Remark. Theorem 6 improves Theorem 2 [6] which gives that if Φ and $\mathcal{A}$ are as in Theorem 6, then Φ is n-positive if and only if Φ is completely positive.

Note added in proof

Recently Z. Pan and the author resolved Magajna's problem [4]. We prove that if $\mathcal{S}$ is an n-dimensional subspace of $L(H)$, then $\mathcal{S}$ is $[\sqrt{2n}]$-reflexive and $k(n) = [\sqrt{2n}]$.

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