LUSIN SETS

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(Communicated by Andreas R. Blass)

Abstract. We show that a set of real numbers is a Lusin set if, and only if, it has a covering property similar to the familiar property of Rothberger.

In [7] Rothberger defined the property $C''$: A set $X$ of real numbers has property $C''$ if: For every sequence $(U_n : n \in \mathbb{N})$ of open covers of $X$, there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n$ $U_n \in U_n$ and $\{U_n : n \in \mathbb{N}\}$ is a cover for $X$. This is an example of the following selection hypothesis: Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set. Then $S_1(\mathcal{A}, \mathcal{B})$ denotes: For every sequence $(O_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(T_n : n \in \mathbb{N})$ such that $T_n \in O_n$ for each $n$, and $\{T_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The game $G_1(\mathcal{A}, \mathcal{B})$ associated with this hypothesis is played as follows: ONE and TWO play an inning per positive integer. In the $n$–th inning ONE first chooses a set $O_n \in \mathcal{A}$, and TWO responds with a $T_n \in O_n$. TWO wins a play $O_1,T_1,\ldots,O_n,T_n,\ldots$ if $\{T_n : n \in \mathbb{N}\}$ is in $\mathcal{B}$; otherwise, ONE wins.

Let $(X,\tau)$ be a topological space which is at least $T_3$. Define:

- $\mathcal{K}$ is the set of those $\mathcal{U} \subset \tau$ such that $X = \bigcup\{\overline{U} : U \in \mathcal{U}\}$.
- $\mathcal{K}_1$ is the set of $\mathcal{U}$ in $\mathcal{K}$ such that no element of $\mathcal{U}$ is dense in $X$, and for each finite set $F \subset X$, there is a $U \in \mathcal{U}$ such that $F \subset \overline{U}$.
- $\mathcal{O}$ is the collection of all open covers of $X$.

In this notation property $C''$ is $S_1(\mathcal{O}, \mathcal{O})$. A $T_3$–space has property $S_1(\mathcal{O}, \mathcal{K})$ if, and only if, it has property $S_1(\mathcal{O}, \mathcal{O})$. If a $T_3$–space has property $S_1(\mathcal{K}, \mathcal{K})$, then it has property $S_1(\mathcal{O}, \mathcal{O})$ (since $\mathcal{O} \subset \mathcal{K}$).

A set of real numbers is a Lusin set if it is uncountable but its intersection with every first category set is countable. In [5] Lusin showed that the Continuum Hypothesis implies the existence of a Lusin set. In Theorem 3.18 of [4] Kunen shows that for each regular uncountable cardinal number $\kappa$ it is consistent that the real line has cardinality $\kappa$, and there is a Lusin set of cardinality $\kappa$. Lusin sets have property $S_1(\mathcal{O}, \mathcal{O})$. We shall show that a set of real numbers is a Lusin set if, and only if, it has property $S_1(\mathcal{K}, \mathcal{K})$ (Theorem 2). It will follow that not all subsets of $\mathbb{R}$ having property $S_1(\mathcal{O}, \mathcal{O})$ have property $S_1(\mathcal{K}, \mathcal{K})$.

A related selection hypothesis, denoted by $S_{\text{fin}}(\mathcal{A}, \mathcal{B})$, is defined as follows: For every sequence $(O_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is a sequence $(T_n : n \in \mathbb{N})$ such that for each $n$ $T_n$ is a finite subset of $O_n$ and $\bigcup_{n=1}^{\infty} T_n$ is in $\mathcal{B}$. The game $G_{\text{fin}}(\mathcal{A}, \mathcal{B})$
associated with this hypothesis is played as follows: ONE and TWO play an inning
per positive integer. In the $n$–th inning ONE first chooses an $O_n \in A$, after which
TWO chooses a finite set $T_n \subseteq O_n$. TWO wins a play $O_1, T_1, \ldots, O_n, T_n, \ldots$ if
$\bigcup_{n=1}^{\infty} T_n \in B$: otherwise, ONE wins. It is well–known that the property $S_1(O, O)$
is stronger than the property $S_{fin}(O, O)$. We shall see that for subsets of the real
line $S_1(K, K)$ and $S_{fin}(K, K)$ are equivalent.

We shall need a connection with the following notion, due to Reclaw: A subset
$X$ of $\mathbb{R}$ is said to be an $R^M$–set if for every Borel subset $B$ of $\mathbb{R} \times \mathbb{N}$ such that
for each $x \in X$ the set $B_x := \{ f : (x, f) \in B \}$ is of the first category, $\bigcup_{x \in X} B_x$ is
not all of $\mathbb{N}$. Since a Borel subset of a subspace is always the intersection of the
subspace with a Borel subset of the superspace, we may in this definition restrict
attention to Borel subsets of $X \times \mathbb{N}$ instead of $\mathbb{R} \times \mathbb{N}$.

**Theorem 1** (Reclaw, [6]). Lusin sets are $R^M$–sets.

**A characterization of Lusin sets**

A space is $K$–Lindelöf if each element of $K$ has a countable subset in $K$.

**Theorem 2.** If $X \subseteq \mathbb{R}$ is uncountable, then the following are equivalent:

1. $X$ has property $S_1(K, K)$.
2. $X$ has property $S_{fin}(K, K)$.
3. $X$ is $K$–Lindelöf.
4. $X$ is a Lusin set.

5. ONE has no winning strategy in the game $G_1(K, K)$.
6. ONE has no winning strategy in the game $G_{fin}(K, K)$.

**Proof.** The proofs of $1 \Rightarrow 2$, $2 \Rightarrow 3$, $5 \Rightarrow 6$, $5 \Rightarrow 1$ and $6 \Rightarrow 2$ are standard. We show that $3 \iff 4$ and $3 \Rightarrow 5$.

$3 \Rightarrow 4$: Assume that $X$ is uncountable, but not a Lusin set. Let $C$ be an uncountable
closed, nowhere dense set such that $X \cap C$ is uncountable. Define an element
of $K$ for $X$ as follows: For $y \in X \setminus C$, let $V_y$ be a neighborhood of $y$ with $\overline{V_y}$ disjoint
from $C$. For $y \in X \cap C$, choose for each $n$ an open interval $I_n(y)$ in $(y - \frac{1}{2^n}, y + \frac{1}{2^n})$
such that $\overline{I_n(y)} \cap C = \emptyset$. Then put $V_y = \bigcup_{n<\infty} I_n(y)$. The set $U = \{ V_y : y \in X \}$
is in $K$. Also, for each $y$ such that $\overline{V_y} \cap C \neq \emptyset$ we have $\overline{V_y} \cap C = \{ y \}$. Since $X \cap C$ is
uncountable, no countable subset of $U$ is in $K$ for $X$. Thus, $X$ is not $K$–Lindelöf.

$4 \Rightarrow 3$: Let $X \subseteq \mathbb{R}$ be a Lusin set and let $U$ be an element of $K$ for $X$. We may
assume that $X$ is dense in $\mathbb{R}$. Let $D \subset X$ be a countable dense subset of $X$ which
is contained in $\bigcup U$. For each $d \in D$ pick an element $U_d$ of $U$ which contains it.
Since $X \setminus \bigcup_{d \in D} U_d$ is nowhere dense, it is countable. For each element $d$ of this
countable set, choose an element $U_d \in U$ with $d \in U_d$. Then the collection of all
$U_d$‘s we selected during these two stages is a countable set in $K$ for $X$.

$3 \Rightarrow 5$: Let $F$ be a strategy for ONE. By $3$ $X$ is $K$–Lindelöf; thus we may assume
that in each inning $F$ requires that ONE plays a countable element of $K$. Using $F$
construct the following array $(U_\sigma : \sigma \in \omega^\omega)\) of open subsets of $X$:
$(U_n : n \in \mathbb{N})$ enumerates ONE’s first move, $F(\emptyset)$; $(U_{n_1} : n \in \mathbb{N})$ enumerates $F(U_{n_1})$; $(U_{n_1, n_2} : n \in \mathbb{N})$ enumerates $F(U_{n_1, U_{n_1}, n_2})$, and so on. The array has the property that for
each $\sigma$, $X = \bigcup_{n \in \mathbb{N}} U_\sigma \cap n$.

First, define a subset $C$ of $\mathbb{R} \times \mathbb{N}$ by: $C = \{ (x, f) : (\forall n)(x \notin U_{f[n+1]}^f) \}$. Then
$C$ is a $G_\delta$–subset of $\mathbb{R} \times \mathbb{N}$. Moreover, for each $x \in X$ the set $C_x$ is closed and
nowhere dense. Now apply the fact that $X$ is an $R^M$-set (from 3 $\Rightarrow$ 4, already established, and Theorem 1): pick $f \in \mathbb{N} \setminus \bigcup x \in X C_x$. Then the play

$$F(\emptyset), U_f(1), F(U_f(1)), U_f(1), f(2), F(U_f(1), U_f(1), f(2)), \ldots$$

is lost by ONE. \qed

A set of real numbers in $S_1(\mathcal{O}, \mathcal{O})$ need not be in $S_1(\mathcal{K}, \mathcal{K})$. The reason for this is that in [1] Galvin and Miller show that Martin’s Axiom implies the existence of an uncountable first category set which has property $S_1(\mathcal{O}, \mathcal{O})$, while Lusin sets are of the second category.

$\mathcal{K}_\Omega$–Lindelöf sets of real numbers

A space is $\mathcal{K}_\Omega$–Lindelöf if each element of $\mathcal{K}_\Omega$ has a countable subset in $\mathcal{K}_\Omega$. If a set of real numbers is $\mathcal{K}_\Omega$–Lindelöf, then it is easily seen to be $\mathcal{K}$–Lindelöf, and thus a Lusin set. Answering two questions from an earlier version of this paper, Winfried Just proved in [2] that if there is any Lusin set at all, then there is a Lusin set which is not $\mathcal{K}_\Omega$–Lindelöf.

Problem 1. Could it be that there are Lusin sets, but none is $\mathcal{K}_\Omega$–Lindelöf?

Given some special axioms, one can show that there are uncountable $\mathcal{K}_\Omega$–Lindelöf sets of real numbers. In particular: The axiom $\diamond$ asserts that there is a sequence $(A_\alpha : \alpha < \omega_1)$ such that

$\diamond.1$ For each $\alpha$, $A_\alpha \subset \alpha$, and

$\diamond.2$ For every subset $S$ of $\omega_1$, the set $\{\alpha < \omega_1 : S \cap \alpha = A_\alpha\}$ is stationary.

It is well known that the axiom $\diamond$ is consistent relative to the consistency of classical mathematics and implies but is not equivalent to the Continuum Hypothesis.

Theorem 3 ($\diamond$). There exists an uncountable $\mathcal{K}_\Omega$–Lindelöf set of real numbers.

Proof. Let $(A_\alpha : \alpha < \omega_1)$ be a sequence as in $\diamond$. Let $(O_\alpha : \alpha < \omega_1)$ bijectively enumerate all the nonempty open subsets of $\mathbb{R}$. Let $(G_\alpha : \alpha < \omega_1)$ bijectively enumerate all the dense $G_\delta$-subsets of $\mathbb{R}$.

For each $\alpha < \omega_1$ put $S_\alpha = \{O_\gamma : \gamma \in A_\alpha\}$. Then $(S_\alpha : \alpha < \omega_1)$ is a $\diamond$ sequence for the family of open subsets of $\mathbb{R}$, in the following sense: For $U$ a collection of open subsets of $\mathbb{R}$ and for $\alpha < \omega_1$, write $U[\alpha]$ to denote the set $\{O_\gamma \in U : \gamma < \alpha\}$. Then for every family $U$ of nonempty open subsets of $\mathbb{R}$, $\{\alpha < \omega_1 : U[\alpha] = S_\alpha\}$ is a stationary set.

We shall recursively choose irrational numbers $x_\alpha$, $\alpha < \omega_1$, for which the set $L = \mathbb{Q} \cup \{x_\alpha : \alpha < \omega_1\}$ is the desired Lusin set by recursively choosing for each $\gamma < \omega_1$ $(S_\gamma(\delta) : \delta < \omega_1)$, $(U_n^\gamma : n < \omega)$ and $x_\gamma$ such that:

a) If $\delta < \gamma < \omega_1$, then $S_\gamma(\delta) = \omega$;

b) If $\gamma \leq \alpha < \nu < \omega_1$, then $S_\gamma(\alpha), S_\gamma(\nu) \subseteq S_\gamma(\gamma)$ are infinite subsets of $\omega$ such that $S_\gamma(\nu) \subseteq^* S_\gamma(\alpha)$;

c) If for the set $\mathbb{Q} \cup \{x_\delta : \delta < \gamma\}$ we have $S_\gamma$ in $\mathcal{K}_\Omega$, then $U_n^\gamma$, $n < \omega$ are elements of $S_\gamma$ such that:

1. For each finite subset $G$ of $\mathbb{Q} \cup \{x_\delta : \delta < \gamma\}$, for all but finitely many $n$, $G \subseteq \overline{U_n^\gamma}$, and

2. for all $\alpha \geq \gamma$, $\{n : x_\alpha \in U_n^\gamma\} \supseteq S_\gamma(\alpha)$;
d. If \( S_\gamma \) is not in \( K_\Omega \) for \( Q \cup \{ x_\delta : \delta < \gamma \} \), then for each \( n \) \( U_n^\gamma = \mathbb{R} \), and for all \( \delta < \omega_1 \), \( S_\gamma(\delta) = \omega \).

e. For \( \gamma \leq \delta < \omega_1 \), \( x_\delta \) is a member of \( G_\gamma \setminus \{ x_\nu : \nu < \delta \} \).

Assuming that this recursive construction can be carried out, \( L \) would have the desired properties: By e it would be a Lusin set. To see that it would also be \( K_\Omega \)-Lindelöf, let \( \mathcal{U} \) be an uncountable element of \( K_\Omega \) for \( L \). Define \( f : \omega_1 \rightarrow \omega_1 \) so that for each \( \alpha \), \( f(\alpha) \geq \alpha \) is such that \( \mathcal{U}[f(\alpha)] \) is in \( K_\Omega \) for \( Q \cup \{ x_\delta : \delta < \alpha \} \). Then let \( C \) be a closed, unbounded subset of \( \omega_1 \) such that for all \( \alpha \in C \), if \( \gamma < \alpha \), then \( f(\gamma) < \alpha \). Since \( S = \{ \alpha < \omega_1 : \mathcal{U}[\alpha] = S_\alpha \} \) is a stationary set, let \( \rho \) be a limit ordinal in \( C \cap S \). By the definition of \( C \) we see that \( \mathcal{U}[\rho] \) is in \( K_\Omega \) for \( Q \cup \{ x_\delta : \delta < \rho \} \), and so \( S_\rho \) is in \( K_\Omega \) for this set. By c above, \( U_n^\rho \), \( n < \omega \) are members of \( S_\rho \) such that each finite subset of \( Q \cup \{ x_\gamma : \gamma < \rho \} \) is in all but finitely many of the sets \( U_n^\rho \). The remaining clause of c implies that \( S_\rho \) is a countable subset of \( \mathcal{U} \) which is in \( K_\Omega \) for \( L \).

It remains to show that the recursive construction can be carried out. First, set \( S_\alpha(\delta) = \omega \) whenever \( \delta < \gamma < \omega_1 \).

Now begin the construction by first considering \( S_0 \). It is an empty set and so is not in \( K_\Omega \) for \( Q \). Set \( U_n^0 = \mathbb{R} \) for each \( n \), \( S_0(\delta) = \omega \) for each \( \delta \), and choose \( x_0 \in G_0 \setminus Q \). This defines \( S_0(0) \), \( U_n^0 : n < \omega \) and \( x_0 \) so that all the relevant requirements for the recursive construction are met.

Let \( 0 < \alpha < \omega_1 \) be given, and assume that for each \( \beta < \alpha \) we have defined \( (S_\beta(\gamma) : \gamma < \alpha) \), \( U_n^\beta : n < \omega \) and \( x_\beta \) such that all the relevant requirements of the recursive construction have been met. Now consider \( S_\alpha \). For each \( \beta < \alpha \) choose an infinite set \( T_\beta \) such that for all \( \gamma < \alpha T_\beta \subseteq S_\beta(\gamma) \), and when possible, \( T_\beta = \omega \).

If \( S_\alpha \) is not in \( K_\Omega \) for \( Q \cup \{ x_\gamma : \gamma < \alpha \} \), then for each \( n \) put \( U_n^\alpha = \mathbb{R} \), and for each \( \delta \) put \( S_\alpha(\delta) = \omega = T_\alpha \). For each \( \beta \leq \alpha \), define

\[
H_\beta = \bigcap_{m<\omega} \left( \bigcup_{n \geq m, n \in T_\beta} U_n^\beta \right).
\]

Each \( H_\beta \) is a dense \( G_\delta \)-set. Choose

\[
x_\alpha \in \left( \bigcap_{\gamma \leq \alpha} H_\gamma \cap G_\gamma \right) \setminus (Q \cup \{ x_\delta : \delta < \alpha \}),
\]

and after this define for each \( \gamma < \alpha \), \( S_\gamma(\alpha) = \{ n \in T_\gamma : x_\alpha \in U_n^\gamma \} \). This specifies \( x_\alpha, S_\gamma(\alpha), \gamma \leq \alpha \), and \( U_n^\alpha \), \( n < \omega \), such that all the requirements of the recursive construction are met.

If \( S_\alpha \) is in \( K_\Omega \) for \( Q \cup \{ x_\gamma : \gamma < \alpha \} \), then write this latter set as \( \bigcup_{\mu<\omega} F_\mu \) where for each \( n \) \( \emptyset \neq F_n \subset F_{n+1} \) and \( F_n \) is finite. Then choose for each \( n \) a \( U_n^\alpha \in S_\alpha \) such that \( F_n \subseteq U_n^\alpha \). Also set \( T_\alpha = \omega \). For each \( \beta \leq \alpha \) the set \( H_\beta = \bigcap_{m<\omega} (\bigcup_{n \geq m, n \in T_\beta} U_n^\beta) \) is a dense \( G_\delta \)-subset of \( \mathbb{R} \). Then choose

\[
x_\alpha \in \left( \bigcap_{\beta \leq \alpha} H_\beta \cap G_\beta \right) \setminus (Q \cup \{ x_\beta : \beta < \alpha \}).
\]

After this, define for each \( \beta \leq \alpha \), \( S_\beta(\alpha) = \{ n \in T_\beta : x_\alpha \in U_n^\beta \} \).

This specifies \( S_\beta(\alpha), \beta \leq \alpha, U_n^\alpha, n < \omega \) and \( x_\alpha \) such that all the prescriptions of the recursive construction are met. \( \square \)
Proof. Observe that
\[
\omega
\]

Could it be that some Lusin set is
\[
\text{Theorem 5}
\]

by clopen sets and each finite subset of \(L\). We know that a Lusin set need not have the selection property \(S_1(\mathcal{K}_L, \mathcal{K}_L)\):

**Theorem 4 (CH).** There exists a Lusin set which does not satisfy \(S_1(\mathcal{K}_L, \mathcal{K}_L)\).

**Proof.** In [8] we constructed, using the Continuum Hypothesis, a Lusin set
\[
\mathcal{S}
\]

\[\] bijectively list all the nonempty open subsets of \(L\) and each finite subset of \(L\) is contained in an element of \(\mathcal{S}\), and yet for every sequence \((U_n : n \in \mathbb{N})\) with for each \(n \ U_n \in \mathcal{S}\), there is a two-element subset \(F \) of \(L\) such that for each \(n, F \not\subseteq U_n\).

**Problem 2.** Could it be that there are Lusin sets, but none satisfy \(S_1(\mathcal{K}_L, \mathcal{K}_L)\)?

**Problem 3.** Could it be that some Lusin set is \(\mathcal{K}_L\)-Lindelöf, but none has property \(S_1(\mathcal{K}_L, \mathcal{K}_L)\)?

Using \(\diamond\) one can construct Lusin sets which have property \(S_1(\mathcal{K}_L, \mathcal{K}_L)\):

**Theorem 5 (\(\diamond\)).** There exists a Lusin set which has property \(S_1(\mathcal{K}_L, \mathcal{K}_L)\).

**Proof.** Observe that \(\diamond\) implies: There is a sequence \((A^n_\alpha : n < \omega) : \alpha < \omega_1\) such that:

1. For each \(\alpha\) and for each \(n\), \(A^n_\alpha \subseteq \alpha\);
2. Whenever \((A^n : n < \omega)\) is a sequence of subsets of \(\omega_1\), then \(\alpha < \omega_1 : (\forall n)(A^n \cap \alpha = A^n_\alpha)\) is a stationary set.

To see this, let \(\Psi\) be a bijection from \(\omega_1\) to \(\omega_1 \times \omega\) such that for all \(\alpha\), if \(\Psi(\alpha) = (\gamma, n)\), then \(\gamma \leq \alpha\). Let \((B_\alpha : \alpha < \omega_1)\) be a \(\diamond\)-sequence for \(\omega_1\). For each \(\alpha\) put
\[
A_\alpha = \{\Psi(\gamma) : \gamma \in B_\alpha\}.
\]
Then for each \(\alpha\) and each \(n\) put
\[
A^n_\alpha = \{\gamma : (\gamma, n) \in A_\alpha\}.
\]

From now on, let \((A^n_\alpha : n < \omega) : \alpha < \omega_1)\) be as above. Also, let \((U_\alpha : \alpha < \omega_1)\) bijectively list all the nonempty open subsets of \(\mathbb{R}\). For each \(n\) and \(\alpha\) define
\[
S^n_n = \{U_\gamma : \gamma \in A^n_\alpha\}.
\]
Then \((S^n_n : n < \omega) : \alpha < \omega_1\) has the property that if \((U_\alpha : n < \omega)\) is any sequence of families of open subsets of \(\mathbb{R}\), and if as before we write for each \(n\) and \(\alpha\), \(U_\alpha[\alpha] = \{U_\gamma : \gamma < \alpha \text{ and } U_\gamma \in U_\alpha\}\), then the set \(\alpha < \omega_1 : (\forall n)(U_\alpha[\alpha] = S^n_n)\) is a stationary subset of \(\omega_1\).

Let \((G_\alpha : \alpha < \omega_1)\) be a bijective listing of all the dense \(G_\delta\)-subsets of \(\mathbb{R}\).

Since the construction is analogous to that of a \(\mathcal{K}_L\)-Lindelöf set, we just state the prescriptions for the recursion, trusting that the interested reader will check details as needed. We recursively choose \(x_\alpha, \alpha < \omega_1\), such that the set \(L = \mathbb{Q} \cup \{x_\alpha : \alpha < \omega_1\}\) has the required properties, by recursively choosing for each \(\gamma < \omega_1\)
\[
(S_\gamma(\delta) : \delta < \omega_1), \ (U^n_\gamma : n < \omega) \text{ and } x_\gamma \text{ such that:}
\]
a If \(\delta < \gamma < \omega_1\), then \(S_\gamma(\delta) = \omega_1\);
b If \(\gamma < \alpha < \omega_1\), then \(S_\gamma(\alpha), \ S_\gamma(\nu) \subseteq S_\gamma(\gamma) \text{ and } S_\gamma(\nu) \subseteq S_\gamma(\alpha)\) are infinite;
c If \((S^n_n : n < \omega)\) is a sequence from \(\mathcal{K}_L\) for \(\mathbb{Q} \cup \{x_\delta : \delta < \gamma\}\), then
1. for each \(n\), \(U^n_n \in S^n_n\),
2. for every finite set \( G \subseteq \mathbb{Q} \cup \{ x_\delta : \delta < \gamma \} \), for all but finitely many \( n \), \( G \subseteq U_n^\gamma \), and
3. for all \( \alpha \geq \gamma \), \( \{ n : x_n \in U_\alpha^\gamma \} \supseteq S_\gamma(\alpha) \); 
   d If some \( S_\gamma^o \) is not in \( K_\Omega \) for \( \mathbb{Q} \cup \{ x_\delta : \delta < \gamma \} \), then for each \( n U_n^\gamma = \mathbb{R} \) and for all \( \delta \), \( S_\gamma(\delta) = \omega \); 
   e If \( \gamma \leq \delta < \omega_1 \), then \( x_\delta \in G_\gamma \setminus (\mathbb{Q} \cup \{ x_\nu : \nu < \delta \}) \).

\[
\text{Ramsey theory}
\]

In the following theorem, the symbol \( \mathcal{A} \to (\mathcal{B})^m \) means: For each \( A \in \mathcal{A} \) and for each function \( f : [A]^m \to \{ 1, \ldots, m \} \), there is a \( B \in \mathcal{B} \) such that \( B \subseteq A \), and \( f \) is constant on \( [B]^m \).

\textbf{Theorem 6.} If \( X \subseteq \mathbb{R} \) is uncountable and \( K_\Omega \to (\mathcal{K})^2_2 \), then \( X \) is a Lusin set.

\textbf{Proof.} Let \( X \) be as in the hypotheses. If we can show that every element of \( K_\Omega \) has a countable subset in \( K \), then it follows that every element of \( K \) has a countable subset in \( K_\Omega \), and this would imply that \( X \) is a Lusin set. Since \( X \) is second countable, the elements of \( K_\Omega \) have cardinality at most \( 2^{\aleph_0} \). The well-known negative partition relation \( 2^{\aleph_0} \not\rightarrow (\aleph_1)^2_2 \) of Kurepa and Sierpiński provides for each element \( \mathcal{U} \) of \( K_\Omega \) a function \( f : [\mathcal{U}]^2 \to \{ 1, 2 \} \) which has no uncountable homogeneous set. Thus, the partition relation \( K_\Omega \to (\mathcal{K})^2_2 \) implies that every element of \( K_\Omega \) has a countable subset which is in \( K \).

\textbf{Theorem 7.} If \( X \subseteq \mathbb{R} \) is a \( K_\Omega \)-Lindelöf set, then for each \( n K_\Omega \to (\mathcal{K})^2_n \).

\textbf{Proof.} Since \( X \) is \( K_\Omega \)-Lindelöf, it is a Lusin set. Let \( \mathcal{U} \in K_\Omega \) as well as a function \( f : [\mathcal{U}]^2 \to \{ 1, \ldots, n \} \) be given. We may assume that \( \mathcal{U} \) is countable, and enumerate it bijectively as \( (U_n : n < \omega) \). Recursively construct sequences \( (i_n : n < \omega) \) and \( (U_n : n < \omega) \) so that for each \( m \):

1. \( i_m \in \{ 1, \ldots, n \} \);
2. \( U_{m+1} = \{ U_j \in U_m : j > m + 1 \text{ and } f(\{ U_{m+1}, U_j \}) = i_{m+1} \} \) is in \( K_\Omega \);
3. \( i_0 = \{ U_j \in \mathcal{U} : j > 1 \text{ and } f(\{ U_0, U_j \}) = i_0 \} \) is in \( K_\Omega \).

For \( j \leq n \), put \( V_j = \{ U_n : i_n = j \} \). Then the \( V_j \)'s partition each \( U_m \) into finitely many classes. As each \( U_m \) is in \( K_\Omega \), for each \( m \) there is a \( j_m \) with \( V_{j_m} \cap U_m \in K_\Omega \). Since for each \( m \) we have \( U_{m+1} \subseteq U_m \), we may fix a specific value \( j \) such that for each \( m \), \( U_m \cap V_j \in K_\Omega \).

Assign a strategy \( \sigma \) to ONE of the game \( G_\Omega(K,K) \) as follows: ONE's first move is \( \sigma(\emptyset) = U_0 \cap V_j \). If TWO responds with \( U_{n_0} \in U_0 \cap V_j \), then ONE plays \( \sigma(U_{n_0}) = \{ U_m \in U_{n_0} \cap V_j : m > n_0 \} \). If TWO now responds with \( U_{n_1} \), then ONE plays \( \sigma(U_{n_0}, U_{n_1}) = \{ U_m \in U_{n_1} \cap V_j : m > n_1 \} \), and so on.

Since \( X \) is a Lusin set, this strategy is not winning for ONE. Consider a play lost by ONE, say \( \sigma(\emptyset), U_{n_0}, \sigma(U_{n_0}), U_{n_1}, \sigma(U_{n_0}, U_{n_1}), \ldots \). Then \( \{ U_{n_0}, U_{n_1}, \ldots \} \) is in \( K \), and \( f \) has value \( j \) for each pair from this set.

\textbf{Problem 4.} Could there be a Lusin set which does not satisfy the partition relation \( K_\Omega \to (\mathcal{K})^2_2 \)?

\textbf{Problem 5.} Could there be a Lusin set which satisfies \( K_\Omega \to (\mathcal{K})^2_2 \), but which is not \( K_\Omega \)-Lindelöf?
REFERENCES


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